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Topological approach to random differential inclusions

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Dedicated to the memory of Professor Andrzej Granas (1929–2019)

Abstract

In the present paper random multivalued admissible operators are considered. First for such operators we shall formulate the following topological results: Schauder-type Fixed Point Theorems, Leray–Schauder Alternative, Granas Continuation Method and Topological Degree.

Next these problems will be transformed to the existence problems, periodic problems and implicit problems for random differentuial inclusions.

Let us remark that this paper constitute a summary and complement of the following earlier papers: [2], [3], [5], [6], [10], [11], [14] and [15]. This work can be considered as an advanced survey with some new results: mainly concerning the theory of random differential inclusions. We believe that this paper will be useful for mathematiciants and students intrested in topological methods of nonconvex analysis.

Keywords: random operators, multivalued mappings, fixed points, topological essentiality, topological degree, differential inclusions.
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1. Topological background

In this paper all topological spaces are assumed to be metric. We shall deal with Čech homology functor $H = \{H_n\}_{n\geq 0}$ with compact carriers and coefficients in the field of rational numbers Q. A space X is called *acyclic* provided we have:

$$H_n(X) = \begin{cases} 0 & \text{for } n \geq 1, \\ Q & \text{for } n = 0. \end{cases}$$

Note that any contractible space or any R_δ-space is acyclic.

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Definition 1.1. A space X is called *absolute retract* or *extension space* (written $X \in \mathrm{AR}$) provided for every space Y and for every closed subset $B \subset Y$ any continuation map $f\colon B \to X$ has extension over Y, i.e. there exists a continuation map $\widetilde{f}\colon Y \to X$ such that

$$\widetilde{f}(y) = f(y), \quad \text{for every } y \in B.$$

Note that any absolute retract X is contractible and hence acycylic.

Definition 1.2. A continuous map $p\colon Y \to X$ is called a *Vietoris map* provided the following conditions are satisfied:

(a) p is onto, i.e. $p(Y) = X$,

(b) p is proper, i.e. for every compact $K \subset X$ the counter image $p^{-1}(K)$ of K under p is compact too,

(c) for every point $x \in X$ the set $p^{-1}(x)$ is acyclic.

In what follows we shall deal with multivalued mappings. We shall say that for a multivalued map $\varphi\colon X \multimap Y$ is called upper semicontinuous (u.s.c.) provided for every open $U \subset Y$ the set $\varphi^{-1}(U) = \{x \in X;\ \varphi(x) \subset U\}$; φ is called lower semicontinuous map (l.s.c.) provided for every closed $B \subset Y$ the set $\varphi^{-1}(B) = \{x \in X;\ \varphi(x) \subset B\}$ φ is called continuous provided φ is both u.s.c. and l.s.c.; finally φ is compact if the closure of $\varphi(X)$ in Y is a compact set. An u.s.c. mapping $\varphi\colon X \multimap Y$ is called acyclic provided $\varphi(x)$ is an acyclic set for every $x \in X$.

A multivalused map $\varphi\colon X \multimap Y$ is called admissible provided there exists space Z and two continuous maps $p\colon Z \to X$ and $q\colon Z \to Y$ such that

(1) p is a Vietoris map, and

(2) $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$.

It is well known that admissible mappings have some important properties, namely:

(3) any admissible map is u.s.c.,

(4) any acyclic map is admissible,

(5) if $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Y_1$ are admissible, then the composition $\psi \circ \varphi\colon X \multimap Y_1$ defined by the formula:

$$(\psi \circ \varphi)(x) = \psi(\varphi(x)) = \bigcup_{y \in \varphi(x)} \psi(y) \quad \text{for every } x \in X,$$

is admissible too.

(6) furthermore the Cartesian product $\varphi \times \psi$; the sum $\varphi + \psi$ and $f \cdot \varphi$, are admissible provided φ and ψ are admissible.

Assume that $A \subset X$ and $\varphi\colon A \multimap X$ is a multivalued map. Then we let

$$\mathrm{Fix}(\varphi) = \{x \in A;\ x \in \varphi(x)\}.$$

We shall use the following version of Schauder Fixed Point Theorem:

Theorem 1.3. *Assume that $X \in \mathrm{AR}$ and $\varphi\colon X \multimap X$ is a compact admissible map, then φ has a fixed point, i.e.*

$$\mathrm{Fix}(\varphi) \neq \emptyset.$$

Theorem 1.4. *Let E be a normal space nad $\varphi\colon E \multimap E$ be an admissible map such that, for every bounded subset $B \subset E$, the set $\overline{\varphi(B)}$ is compact. Let*

$$\mathcal{E}(\varphi) = \{x \in E;\ x \in \lambda\varphi(x) \text{ for some } \lambda \in (0,1)\}.$$

Then $\mathcal{E}(\varphi)$ is bounded or φ has a fixed point.

For details concerning this section see [13], [1], [14].

In Section 6 we shall use the notion of essential fixed point. Let X be an AR-space and $\varphi\colon X \multimap X$ a compact admissible map.

Definition 1.5. Let $x \in \mathrm{Fix}(\varphi)$ be a fixed point of φ. We shall say that x is *essential* provided for every open set $U \subset X$ such that $x \in U$ there exists an open $V \subset U$ for which the following conditions are satisfied:

(a) $x \in U$,

(b) $\partial U \cap \mathrm{Fix}(\varphi) = \emptyset$,

(c) fixed point index $\mathrm{Ind}(\varphi, \nu) \neq 0$ is different as 0, (for the definition of $\mathrm{Ind}(\varphi, \nu)$ see [3], [13]).

In [6] it is proved the following proposition:

Proposition 1.6. *If* $\dim \mathrm{Fix}(\varphi) = 0$, *then there exists an essential fixed point* $x \in \mathrm{Fix}(\varphi)$, *where* $\dim \mathrm{Fix}(\varphi)$ *denotes the topological dimension of* $\mathrm{Fix}(\varphi)$.

2. Random operators

By measurable space we shall mean the pair (Ω, Σ), where a set Ω is equipped with σ-algebra Σ of subsets of Ω. If X is a metric space then by $B(X)$ we shall denote the Borel σ-algebra of subsets of X. The symbol $\Sigma \otimes B(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all sets $A \times B$, where $A \in \Sigma$ and $B \in B(X)$.

Definition 2.1 ([2], [13]). A multivaluded map $\varphi\colon \Omega \times X \multimap Y$ is called a *measurable* provided, for every open $B \subset Y$, the set

$$\varphi^{-1}(B) \in \Sigma \otimes B(X),$$

where Y is a metric space.

Definition 2.2 ([2], [13]). A multivalued map $\varphi\colon \Omega \times X \multimap Y$ is called a *random operator* provided:

(a) φ is measurable,

(b) the map $\varphi(\omega, \cdot)(x) = \varphi(\omega, x)$ is u.s.c. with closed values for every $\omega \in \Omega$.

A random operator $\varphi\colon \Omega \times X \multimap Y$ is called *bounded* provided, for every $\omega \in \Omega$, the map $\varphi(\omega, \cdot)$ is bounded.

Assume that X is closed subset of Y.

Definition 2.3. A measurable map $\eta\colon \Omega \to X$ is called a *random fixed point* of the random operator $\varphi\colon \Omega \times X \multimap Y$ provided $\eta(\omega) \in \varphi(\omega, \eta(\omega))$ for every $\omega \in \Omega$.

The following theorem is crucial in this section.

Theorem 2.4. *Let Y be a separable space. Assume further that $\varphi\colon \Omega \times X \multimap Y$ is a random operator. Then φ has a random fixed point if and only if the map $\varphi(\omega, \cdot)\colon X \multimap Y$ has a fixed point for every $\omega \in \Omega$.*

For the proof of Theorem 2.4 see: [2], [4], [14], [13].

Definition 2.5. A random operator $\varphi\colon \Omega \times X \multimap Y$ is called *random admissible operator* provided the map $\varphi(\omega, \cdot)$ is admissible for every $\omega \in \Omega$.

From Theorems 1.3 and 2.4 and [2, (1.10)] immediately follows:

Theorem 2.6. *Let Y be a separable* AR*-space and $\varphi\colon \Omega \times Y \multimap Y$ be a compact random admissible operator, then φ has a random fixed point.*

We shall end this section by proving the Leray–Schauder Alternative for random operators.

Theorem 2.7 (Leray–Schauder Alternative)**.** *Let E be a separable normal space and $\varphi\colon \Omega \times E \multimap E$ be a random admissible operator such that the map $\varphi(\omega, \cdot)$ satisfies all assumptions of Theorem 1.4 for every $\omega \in \Omega$. Let $\mathcal{E}(\varphi', \omega) = \{x \in E;\ x \in \varphi(\omega, x)\}$. If $\mathcal{E}(\varphi, \omega)$ is an unbounded set for every $\omega \in \Omega$ the φ has a random fixed point.*

Proof. Since $\mathcal{E}(\varphi, \omega)$ is unbounded for every $\omega \in \Omega$ in view of Theorem 1.4 we deduce that $\varphi(\omega, \cdot)$ has a fixed point for every $\omega \in \Omega$. Hence our claim follows from Theorem 2.4. □

For more information about random operators see: [1], [2], [4], [14], [13], [18], [20].

3. Topological essentiality and degree for random admissible operators

First we shall recall the notion of topological essentiality , called also continuation method, and its properties for random admissible operators. Note that this notion was introduced un 1962 by A. Granas (see [16], [17] for singlevalued maps). next it was studied for multivalued mappings iun [11], [13]–[15]. The random case was presented in [2], [14].

We shall start from some notations. Let E, F be two Banach spaces and $U \subset E$ be an open set. By ∂U we shall denote the boundary of U in E. We shall consider the following classes of random multivalued mappings:

$$A_{\partial U}(\Omega \times U, F) = \{\varphi\colon \Omega \times \overline{U} \multimap F;\ \varphi \text{ is a random admissible map}\},$$
$$A_c(\Omega \times U, F) = \{\varphi\colon \Omega \times \overline{U} \multimap F;\ \varphi \text{ is a random admissible and compact map}\},$$
$$A_0(\Omega \times U, F) = \{\varphi \in A_c(\Omega \times U, F);\ \varphi(\omega, x) = 0 \text{ for every } \partial U \text{and } \omega \in \Omega\},$$

Definition 3.1. A map $\varphi \in A_{\partial U}(\Omega \times U, F)$ is called *essential* provided for every $\psi \in A_0(\Omega \times U, F)$ there exists $x \in U$ such that $\varphi(\omega, x) \cap \psi(\omega, x) \neq \emptyset$ for every $\omega \in \Omega$.

Below we shall list properties of the above notion.

Properties 3.2.

(1) (Existence) If φ is essential, then there exists $x \in U$ such that $0 \in \varphi(\omega, x)$ for every $\omega \in \Omega$.

(2) (Perturbation) If φ is essential and $\psi \in A_0(\Omega \times U, F)$, then $(\varphi + \psi) \in A_{\partial U}(\Omega \times U, F)$ and it is essential.

(3) (Coincidence) Let φ be essential and $\psi \in A_0(\Omega \times U, F)$. Assume further that the set

$$B = \{x \in \overline{U};\ \varphi(\omega, x) \cap (t \cdot \psi(\omega, x)) \neq \emptyset \text{ for some } t \in [0, 1] \text{ and every } \omega \in \Omega\}$$

is a closed subset of $\overline{U}$ and $B \subset U$. Then there exists $x \in U$ such that $x \in \varphi(\omega, x) \cap \psi(\omega, x)$ for every $\omega \in \Omega$.

(4) (Normalization) Assume that $\overline{U} \in \mathrm{AR}$ and $i\colon \Omega \times \overline{U} \to E$ is the random map defined as follows:

$$i(\omega, x) = x \quad \text{for every } x \in \overline{U} \text{ and } \omega \in \Omega.$$

The map i is essential if and only if $0 \in U$.

(5) (Localization) Let φ be an essential map. Assume further that:

(i) V is an open subset of U such that $\overline{V} \in \mathrm{AR}$,

(ii) $\{x \in \overline{U};\ 0 \in \varphi(\omega, x)\} \subset V$.

Then the map $\widetilde{\varphi}\colon \Omega \times \overline{V} \multimap F$, $\widetilde{\varphi}(\omega, x) = \varphi(\omega, x)$ for every $x \in \overline{V}$ and $\omega \in \Omega$ is essential.

(6) (Homotopy) Let φ be an assential map. If $\eta\colon \Omega \times \overline{U} \times [0,1] \multimap F$ is a random admissible and compact map such that:

(i) $\eta(\omega, x, 0) = \{0\}$ for every $\omega \in \Omega$ and $x \in \partial U$, and

(ii) the set $\{x \in \overline{U};\ \varphi(\omega, x) \cap \eta(\omega, x, t) \neq \emptyset$ for some $t \in [0,1]$ and every $\omega \in \Omega\}$ is a closed subset of U.

Then the map $(\varphi - \eta)(\,\cdot\,, \cdot\,, 1)$ is essential.

(7) (Continuation) Let φ be essential such that for every comact $K \subset E$ the set $\{x \in \overline{U};\ \varphi(\omega, x) \cap K \neq \emptyset\}$ is compact for every $\omega \in \Omega$. Assume further that $\eta\colon \Omega \times \overline{U} \times [0,1] \multimap F$ is a random admissible compact map such that $\eta(\omega, x, 0) = \{0\}$ for every $\omega \in \Omega$ and $x \in \partial U$. Then there exists $\varepsilon > 0$ such that the map $(\varphi - \eta)(\,\cdot\,, \cdot\,, \lambda)$ is essential for every $\lambda \in [0, \varepsilon]$.

Now we shall recall the notion of topological degree for random operators. For details see [2] and [13], [14]. By $\mathbb{R}^n$ we shall denote n-dimensional euclidean space. We let:

$$\begin{aligned} K^n(r) &= \{x \in \mathbb{R}^n;\ \|x\| \le r\}, \qquad S^{n-1}(r) = \{x \in \mathbb{R}^n;\ \|x\| = r\},\\ A(\Omega \times K^n(r), \mathbb{R}^n) &= \{\varphi\colon \Omega \times K^n(r) \multimap \mathbb{R}^n,\ 0 \notin \varphi(\omega, x) \text{ for every } \omega \in \Omega \text{ and } x \in S^{n-1}(r),\\ &\qquad \operatorname{Deg}\varphi(\omega, \cdot\,) \text{ is defined for every } \omega \in \Omega\}. \end{aligned}$$

Example 3.3. (i) If $\varphi\colon K^n \multimap \mathbb{R}^n$ is an u.s.c. map with R_δ-values, then $\operatorname{Deg}(\varphi)$ is well defined (see [13]).

(ii) $f\colon K^n(r) \to X$ is a continuous single valued mapping and $\varphi\colon X \multimap \mathbb{R}^n$ is an u.s.c. map with R_δ-values, then $\operatorname{Deg}(\varphi \circ f)$ is well defined (see [13]).

Of course both in (i) and (ii) we have assumed the $0 \notin \varphi(x)$ $(0 \notin \varphi(f(x))$ for every $x \in S^{n-1}$.

By homotopy in $A(\Omega \times K^n(r), \mathbb{R}^n)$ we shall understand a random homotopy, i.e. a random map $\mathfrak{X}\colon \Omega \times K^n(r) \times [0,1] \multimap \mathbb{R}^n$ such that

$$0 \notin \varphi\big(\Omega \times S^{n-1}(r) \times [0,1]\big) \quad \text{and} \quad \mathfrak{X}(\,\cdot\,, \cdot\,, t) \in A(\Omega \times K^n(r), \mathbb{R}^n), \quad \text{for every } t \in [0,1].$$

We define a multivalued map $\operatorname{Deg}_r\colon A(\Omega \times K^n(r), \mathbb{R}^n) \to \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, by putting

$$\operatorname{Deg}_r(\varphi) = \{\operatorname{Deg}(\omega, \cdot\,);\ \omega \in \Omega\}. \tag{3.1}$$

Then we have the following properties:

Properties 3.4.

(8) (Existence) If $0 \notin \operatorname{Deg}_r(\varphi)$, then there exists a measurable function $\xi\colon \Omega \to B^n(r)$ such that $0 \in \varphi(\omega, \xi(\omega))$ for $\omega \in \Omega$, where $B^n(r) = \{x \in K^n(r);\ \|x\| < r\}$.

(9) (Localization) If $\varphi \in A\big(\Omega \times K^n(r), \mathbb{R}^n\big)$, then

$$\operatorname{Deg}_r\big(\mathfrak{X}(\,\cdot\,, \cdot\,, 0)\big) = \operatorname{Deg}_r\big(\mathfrak{X}(\,\cdot\,, \cdot\,, 1)\big).$$

For more information see [14], [2], [13].

4. Existence results

Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a bounded random operator with convex values and let $\psi\colon \Omega \multimap \mathbb{R}^n$ be a measurable map with compact convex values. We shall consider the following boundary value problem for random differential inclusions:

$$\begin{cases} x'(\omega,t) \in \varphi(\omega,t,x(\omega,t)),\\ x(\omega,0) \in \psi(\omega), \end{cases} \tag{4.1}$$

where the solution $x\colon \Omega \times [0,1] \to \mathbb{R}^n$ is a map such that $x(\cdot,t)$ is measurable and $x(\omega,\cdot)$ is absolutely continuous for almost all $\omega \in \Omega$ and $t \in [0,1]$. By $S(\varphi,\psi)$ we shall denote the set of all solutions of 4.1.

Under the above assumptions we have:

Theorem 4.1. *$S(\varphi,\psi) \neq \emptyset$.*

Proof. First we define the map $F\colon \Omega \times C\big([0,1],\mathbb{R}^n\big) \multimap C\big([0,1],\mathbb{R}^n\big)$ by the formula

$$f(\omega,x) = \left\{ v;\ v(t) = \int_0^t n(\tau)\,d\tau \text{ and } n(\tau) \in \varphi(\omega,\tau,x(\tau)),\ \text{where } n \text{ is Lebesque integrable selector of } \varphi \right\}.$$

It is proved in [2] (see Theorem 4.2) that F is a bounded random operator with compact convex values.

Consequently the map $\widetilde{F}\colon \Omega \times C([0,1],\mathbb{R}^n) \multimap C([0,1],\mathbb{R}^n)$ defined by the formula

$$\widetilde{F}(\omega,x) = \psi(\omega) + F(\omega,x) = \big\{y + v,\ y \in \psi(\omega) \text{ and } v \in F(\omega,x)\big\}.$$

Hence $\widetilde{F}$ is a bounded random operator with compact index values suth that $\widetilde{F}(\omega,\cdot)$ is a compact admissible map for all $\omega \in \Omega$. So, in view of Theorem 1.3, $\widetilde{F}$ has a fixed point for every $\omega \in \Omega$. Therefore it follows form Theorem 2.4 that $\widetilde{F}$ has a random fixed point. Finally, it is easy to see that every random fixed point of $\widetilde{F}$ is a solution of (4.1). The proof of Theorem 4.1 is completed. □

Remark 4.2. The case when $\psi(\omega) = A$ for every $\omega \in \Omega$ and A is a convex compact set (4.1) is called a generalized Cauchy problem for random of differential inclusions.

Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a measurable map with closed values. If for every $\omega \in \Omega$ the map $\varphi(\omega,\cdot,\cdot)$ is l.s.c. (lower semi continuous) then we shall say that φ is a l-random operator.

In Section 6 we shall use the following result:

Theorem 4.3. *If φ is l-random operator, then there exists a solution of the following problem:*

$$x'(\omega,l) \in \varphi(\omega,t,x(\omega,l))$$

The proof of Theorem 4.3 is analogous to the respective result in deterministic case (see [1], [15], [13]).

Now we shall consider some boundary problems for for k-order random differential inclusions, $k \geq 1$. We shall start from some notations.

Let X, Y be two metric spaces. By (X,Y) we shall denote the set of all continuous functions from X to Y. By $C^k\big([0,1],\mathbb{R}^n\big)$, $k = 0,1,\ldots$ we shall denote Banach space of all C^k-functions from the interval $[0,1]$ to $\mathbb{R}^n$ with usual norm defined by

$$\|x\| = \max\big\{\|x(t)\|;\ t \in [0,1]\big\} + \max\big\{\|x'(t)\|;\ t \in [0,1]\big\} + \max\big\{\|x^{(2)}(t)\|;\ t \in [0,1]\big\} + \ldots + \max\big\{\|x^{(k)}(t)\|;\ t \in [0,1]\big\},$$

where $x^{(k)}$ denotes k-th derivative of x; we also put $x' = x^{(1)}$, $x^{(0)} = x$, $C\big([0,1],\mathbb{R}^n\big) = C^0\big([0,1],\mathbb{R}^n\big)$. Finally, we let

$$\mathbb{R}^{n\cdot k} = \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n_t\text{-times}}.$$

Let $\varphi\colon \Omega\times[0,1]\times\mathbb{R}^{n\cdot k}\multimap\mathbb{R}^n$ be a random bounded operator with convex compact values. Let

$$\psi_i, l_i\colon \Omega\times C^{k-1}\big([0,1],\mathbb{R}^n\big)\multimap\mathbb{R}^n,\quad i=0,1,\ldots,k-1,$$

where ψ_i are random bounded admissible operators and l_i be (single valued) bounded random operators for every $i=0,1,\ldots,k-1$. Assume further that

$$l_{k-1}(\omega,x)=(\omega,x(0))\quad\text{for every } \omega\in\Omega \text{ and } x\in C^{k-1}\big([0,1],\mathbb{R}^n\big).$$

For any $\mu\in[0,1]$ we shall consider the following value problem:

$$\begin{cases} x^{(k)}(\omega,t)\in\mu\varphi(\omega,x(\omega,t),x'(\omega,t),\ldots,x^{(k)}(\omega,t)),\\ l_0(\omega,x)\in\mu\psi_0(\omega,x),\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ l_{k-1}(\omega,x)=(\omega,x(0))\in\mu\psi_{k-1}(\omega,x),\end{cases}\tag{4.2$_\mu$}$$

where the solution $x\colon\Omega\times[0,1]\to\mathbb{R}^n$ is a random map such that $x(\omega,\cdot\,)$ is C^{k-1}-function such that its $(k-1)$-derivative is absolutely continuous for every $\omega\in\Omega$.

We shall associate with $(4.2)_\mu$ the random mapping:

$$\begin{cases} g\colon\Omega\times C^{k-1}\big([0,1],\mathbb{R}^n\big)\to C\big([0,1],\mathbb{R}^n\big)\times\mathbb{R}^{n(k-1)}\\ \text{given by}\\ g(\omega,x)=\big(x^{(k-1)},l_0(\omega,x),\ldots,l_{k-2}(\omega,x)\big).\end{cases}\tag{4.3}$$

Let $S_\mu(\varphi,l_i,\psi_i)$ denote the set of all solutions. As some application of the topological essentiality we obtain:

Theorem 4.4. *Under the above assumptions and the following two assumptions of* $(4.2)_\mu$*:*

(a) *there exists $r\geq 0$ such that for every $\mu\in[0,1]$ the set $S_\mu(\varphi,l_i,\psi_i)$ is bounded, i.e. $\|x(\omega,\cdot\,)\|<r$ for every $\omega\in\Omega$ and $x\in S_\mu(\varphi,l_i,\psi_i)$,*

(b) *the map g restricted the set $\Omega\times M$ is essential, where $M=\big\{x\in C^{k-1}\big([0,1],\mathbb{R}^n\big);\ \|x\|\leq r\big\}$ we have that $S_1(\varphi,l_i,\psi_i)\neq\emptyset$.*

Proof. Let $F\colon\Omega\times M\multimap C\big([0,1],\mathbb{R}^n\big)$ be defined by the same formula as F in the proof of Theorem 4.1. Evidently F is compact random operator with convex values and hence it is random compact admissible map. Consider the map $\widehat{\varphi}\colon\Omega\times M\multimap C\big([0,1],\mathbb{R}^n\big)\times\mathbb{R}^{n(k-2)}$ given by the formula

$$\widehat{\varphi}(\omega,x)=g(\omega,x)-f(\omega,x)\times\psi_0(\omega,x)\times\ldots\times\varphi_{k-2}(\omega,x).$$

In view of (6) is sectional we deduce that $\widehat{\varphi}$ is compact admissible random operator. Moreover, from (a) we deduce that $0\notin\widehat{\varphi}(\omega,x)$, for every $\omega\in\Omega$ and $x\in\partial M$.

For the proof it is sufficient to show that $0\in\widehat{\varphi}(\omega,x)$, for every $\omega\in\Omega$ and some $x\in M$, i.e. $\widehat{\varphi}$ is essential. To do it we define the following homotopy

$$\mathcal{X}\colon\Omega\times[0,1]\times M\multimap C\big([0,1],\mathbb{R}^n\big)\times\mathbb{R}^{n(k-2)}$$

by the formula

$$\mathcal{X}(\omega,\mu,x)=g(\omega,x)-\mu\cdot F(\omega,x)\times\psi(\omega,x)\times\ldots\times\psi_{k-2}(\omega,x).$$

Observe that $\mathcal{X}(\,\cdot\,,0,\,\cdot\,)=g(\,\cdot\,,\,\cdot\,)$ and $\mathcal{X}(\,\cdot\,,1,\,\cdot\,)=\widehat{\varphi}(\,\cdot\,,\,\cdot\,)$ and g is essential so from the homotopy property 3.2 (6) we infer that $\widehat{\varphi}$ is essential and it implies that there exists a solution of $(4.2)_1$; the proof is completed. □

Remarks 4.5. (a) Observe that, for $k=1$ problem $(4.2)_1$ reduces to (4.1). But in (4.1) is stronger assumption about ψ than about ψ_0 in $(4.2)_1$.

(b) Note that Examples 5.4 and 5.5 in [15] can be reformulated onto the random case.

(c) Evidently interval $[0,1]$ can be replaced by an arbitrary interval $[a,b]$.

5. Periodic problem

Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random bounded operator with convex values. In this section we shall consider the following periodic problem for random differential inclusions:

$$\begin{cases} x'(\omega,t) \in \varphi(\omega,t,x(\omega,t)), \\ x(\omega,0) = x(\omega,1). \end{cases} \tag{5.1}$$

To study the above problem we shall define the random operator $P\colon \Omega \times \mathbb{R}^n \multimap C\big([0,1]\mathbb{R}^n\big)$ by the formula:

$$P(\omega,y) = S(\varphi(\omega,\,\cdot\,,\,\cdot\,,y)) \tag{5.2}$$

where $S(\varphi(\omega,\,\cdot\,,\,\cdot\,,y)$ is the set of all solutions of the deterministic Cauchy problem:

$$\begin{cases} x'(t) \in \varphi(\omega,c(t,x(t))), \\ x(0) = y. \end{cases}$$

It is well known (comp. [9]) that under the above assumptions $S(\omega,\,\cdot\,,\,\cdot\,,y)$ is non empty acyclic set.

Theorem 5.1. *P is the random admissible operator.*

For the proof see 4.2 in [2].

Let $l_1\colon C\big([0,a],\mathbb{R}^n\big) \to \mathbb{R}^n$, $l_1(x) = x(1)$ be the evaluation map. The map $P_1 = e_1 \circ P$ is called the random Poincaré operator along the trajectoria of (5.1). The following proposition is self evident.

Proposition 5.2. *Problem (5.1) has a solution if and only if the random Poincaré operator $P_1\colon \Omega\times\mathbb{R}^n \multimap \mathbb{R}^n$ has a random fixed point.*

To obtain a random fixed point of P_1 we shall follow an approach based on random topological degree theory (for the deterministic case see e.g. [1], [7], [8], [13]).

To find a fixed point of P_1 we associate with P_1 the random vector field $\widetilde{P}_1\colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined as follows:

$$\widetilde{P}_1(\omega,x) = x - P_1(\omega,x).$$

The following proposition is self evident.

Proposition 5.3. *If $0 \notin \mathrm{Deg}(\widetilde{\varphi}_1)$, then problem (5.1) has solution, where $\mathrm{Deg}(\widetilde{\varphi}_1)$ is considered for $\widetilde{\varphi}_1$ on some ball $K^n(r)$.*

In order to show that $0 \notin \mathrm{Deg}\big(\widetilde{P}_1\big)$ we shall adopt to random case the guiding potential method (comp. [2]).

A map $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is called a random potential provided the following two conditions are satisfied:

(a) $V(\,\cdot\,,x)$ is measurable for every $x \in \mathbb{R}^n$;

(b) $V(\omega,\,\cdot\,)$ is a C^1-map for every $\omega \in \Omega$.

Every random potential map V induces a random vector field $\partial V\colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ as follows:

$$\partial V(\omega,x) = \left(\frac{\partial V}{\partial x_1}(\omega,x),\ldots,\frac{\partial V}{\partial x_n}(\omega,x)\right)$$

for any $\omega \in \Omega$ and $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$.

If, for some $r > 0$, V satisfies the following condition: $0 \notin \partial V(\Omega \times S^{n-1}(r))$, then V is called direct potential. If V is a direct potential then, for every $s \geq r$, $0 \notin \partial V(\Omega \times S^{n-1}(s))$ and $\deg(V,K^n(s))$ is independent on s. Consequently we can put:

$$I(V) = \deg(\partial V), \tag{5.3}$$

where the topological degree $\deg(\partial V)$ or ∂V is condedered on arbitrary $K^n(s)$, $s \geq r$. Then $I(V)$ is called index of random direct potential V.

Definition 5.4. Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random bounded operator with convex values and let $V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a random direct potential. We shall say that V is a random guiding function for φ provided the following condition is satisfied:

$$\exists\, r_0 > 0 \quad \forall\, x, t, \omega \text{ with } \|x\| \geq r_0 \quad \exists\, y \in \varphi(\omega, t, x) \quad \text{such that} \quad \langle y, \partial V(\omega, x)\rangle \geq 0.$$

We have:

Theorem 5.5. *If φ posses a random guaiding function V such that $I(\varphi) \neq 0$ then problem (5.1) has a solution.*

Remark 5.6. Note that the assumptions about φ can be weaken (comp. [2] or [8], [7] in deterministic case).

6. Implicit problem

In this section the result obtained in [3], [5], [6] will be generalized to the random case.

Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random bounded admissible operator. We shall study the following random implicit problem for differential inclusions

$$x'(\omega, t) \in \varphi(\omega, t, x(\omega, t), x'(\omega, t)). \tag{6.1}$$

Since $\varphi(\omega, \cdot, \cdot, \cdot)$ is admissible we can define random map $\Psi\colon \Omega \times [0, 1 \; \times \mathbb{R}^n \multimap \mathbb{R}^n$ as follows

$$\Psi(\omega, t, x) = \operatorname{Fix} \varphi(\omega, t, x, \cdot).$$

Note that Ψ is a random operator with compact values but not convex in general.

From the other hand, if we will consider the following problem:

$$x'(\omega, t) \in \Psi(\omega, l, x(\omega, l)) \tag{6.2}$$

then any solution of (6.2) is a solution of (6.1). Unfortunately we can't apply Theorem 4.1 to obtain solution of (6.2).

To get solution of (6.1) we need one more assumption on φ, namely:

Assumption 6.1. *Assume that the topological dimension of the set* $\operatorname{Fix}(\varphi(\omega, t, x, \cdot)$ *is equal zero, i.e.*

$$\dim \operatorname{Fix}(\varphi(\omega, t, x, \cdot) = 0 \quad \textit{for every } \omega,\ t,\ x.$$

Observe that assumption 6.1 guaranties that, for every ω, t, x, there exists an assentail fixed point of the map $\varphi(\omega, t, x, \cdot)$ (see [3], [5], [6]). So we are able to define the following random operator

$$\eta(\omega, t, x) = \overline{\{y \in \mathbb{R}^n;\, y \text{ is an essential fixed point of the map } \varphi(\omega, t, x, \cdot)\}}$$

(see again [3], [5], [6]). Evidently every solution of the problem

$$x'(\omega, t) \in \eta(\omega, t, x(\omega, t, x(\omega, t)) \tag{6.3}$$

is a solution of (6.2) and hence it is solution of (6.1). Consequently in view of Theorem 4.3 we get:

Theorem 6.2. *Under the above assumptions (6.1) has a random solution.*

Proof. From Theorem 4.3 we obtain that the problem (6.3) has a random solution. But every solution of (6.3) is a random solution of (6.1) and the proof is complete. □

Remark 6.3. Let us remark that the methods used in this section remain true for the following random differenctial inclusions:

(a) random ordinary differential inclusions of higher order;

(b) hyperbolic random differential inclusions;

(c) elliptic random differential inclusions;

in deterministic case see [3], [6].

Remark 6.4. It is possible to consider implicit problem for random differential inclusions on proximate retracts (comp. [9]], in the deterministic case see [10].

7. Final remarks and comments

We already pointed out in Abstract the main novelity of this paper consists in presented applications.

Remark 7.1. Let us add that in Sections 4–6 all results remains true if we will assume about the random operator φ only that with respect to the variables t and x it is a Carathéodory map or it is linearly bounded map.

Note that Theorems 4.4 and 4.1 stand some new generalizations of the respective results in deterministic case. Let us remark also that the class of admissible random operators was introduced for the first time in the fixed point theory and the theory idfferential inclusions.

For better understanding random differential inclusions it is useful to present the following randomization scheme connected with periodic problem (comp. [2]).

Proposition 7.2. *Let $\varphi\colon \Omega \times [0,1] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a random operator. If for every $\omega \in \Omega$ the following one parameter family of deterministic periodic problems*

$$\begin{cases} x'(t) \in \varphi_\omega(t, x(t)) := \varphi(\omega, t, x(t)), \\ x(0) = x(1), \end{cases} \tag{7.1}$$

is solvable, then problem (5.1) has a (random) solution.

Proof. Let us consider the random Poincaré operator $P_1\colon \Omega \times \mathbb{R}^n \multimap \mathbb{R}^n$ defined in Section 5. We define $P_\omega\colon \mathbb{R}^n \multimap \mathbb{R}^n$ the deterministic Poincaré operator given by the formula:

$$P_\omega = P_1(\omega, \cdot), \quad \text{for every } \omega \in \Omega.$$

By assumption, for every $\omega \in \Omega$ there exist a deterministic fixed point. Now, in view Theorem 2.6 we infer that P_1 has a random fixed point and hence we obtain a random solution of (5.1). The proof is completed □

Remark 7.3. Observe that considering the map F defined in the proof of Theorem 4.1 and the maps F_ω, $F_\omega = F(\omega, \cdot, \cdot\}$, for every $\omega \in \Omega$, we can formulate randomization scheme for problem (4.1).

Remark 7.4. It is an open question to formulate randomization scheme for the problem (6.1).

References

[1] J. Andres, L. Gewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, 2003.

[2] J. Andres, L. Gewicz, *Random topological degree and random differential inclusions*, Topol. Methods Nonlinear Anal. **40** (2012), 337–358.

[3] J. Andres , L. Gewicz, *Implicit differential inclusions with acyclic right-hand sides an essential fixed point approach*, Dyn. Syst. **26** (2017), 237–258.

[4] T.D. Benavides, G.L. Acedo, H.K. Xu, *Random fixed points of set valued mappings*, Proc. Amer. Math. Soc. **124** (1996), 431–438.

[5] R. Bielawski, L. Gewicz, *Some applications of the Leray–Schauder alternative to differential equations*, NATO ASI Series Ser. C Math. Phys. Ser., Vol. 173, edited by S.P. Singh, 187–194.
[6] R. Bielawski, L. Gewicz, *A fixed point approach to differential equations*, Lecture Notes in Math., Vol. 1411, Springer, Berlin, 1989, 9–14.
[7] R. Bielawski, L. Gewicz, S. Plaskacz, *Topological approach to differential inclusions on closed subsets of* $\mathbb{R}^n$, Dynamics Reported, 1 New Series, Springer, 1991, 225–250.
[8] F.S. De Blasi, L. Gewicz, G. Pianigiani, *Topological degree and periodic solutions of differential inclusions*, Nonlinear Anal. **3** (1999), 217–245.
[9] F.S. De Blasi, J. Myjak, *On the solution sets for differential inclusions*, Bull. Polon. Math. Soc. **12** (1985), 17–23.
[10] M. Frigon, L. Gewicz, T. Kaczyński, *Differential inclusions and implicit equations on closed subset of euclidean n-dimensional space*, Proceedings of the First WCNA, Tampa, Florida, W. de Gruyter, Berlin 1996, 1197–1206.
[11] G. Gabor, L. Gewicz, M. Ślosarski, *Generalized topological essentiality and coincidence points of multivalued mappings*, Set-Valued Anal. **17** (2009), 1–19.
[12] L. Gewicz, *Topological structure of solution sets: current results*, Arch. Math. **36** (2000), 343–382.
[13] L. Gewicz, *Topological Fixed Point Theory for Multivalued Mappings*, Springer, 2006 (second edition).
[14] L. Gewicz, O. Gewicz, *Topological essentiality of random w-admissible operators*, Discuss. Math. **39** (2019), 123–134.
[15] L. Gewicz, M. Ślosarski, *Topological essentiality and differential inclusions*, Bull. Aust. Math. Soc. **45** (1992), 177–193.
[16] A. Granas, *The theory of compact vector fields and some of its applications*, Dissertationes Math. **30** (1962), 1–93.
[17] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer, 2003.
[18] N.S. Papageorgiou, *Random fixed points and random differential inclusions*, Int. J. Math. Sci. **3** (1988), 551–560.
[19] L.E. Rybiński, *Random fixed points and viable random solutions of functional differential equations*, J. Math. Anal. **142** (1989), 53–61.
[20] J.L. Strand, *Random ordinary differential equations*, J. Differential Equations **7** (1970), 538–553.