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Abstract

In this article, we give the corrected version of Theorem 3.5 and Theorem 3.7 (a) in [α—Topological vector spaces, Hacet. J. Math. Stat. 47 (5), 1102-1107, 2018] by T. Al-Hawary.

Mathematics Subject Classification (2020). 46A19, 54C05, 46A99

Keywords. α—open sets, α—topological vector spaces, nearly topological vector spaces

This article concerns two results, namely Theorem 3.5 and Theorem 3.7 (a), in the original paper “α—Topological Vector Spaces by T. Al-Hawary”, Hacet. J. Math. Stat. 47 (5), 1102-1107, 2018. Actually, an incorrect detail is used by Al-Hawary in the argument of the proofs of Theorem 3.5 and Theorem 3.7 (a) in [1]. We identify this error and provide a counterexample to it before stating the amended version of these theorems.

Recall that a subset $A$ of a topological space $(X, \mathcal{I})$ is called $\alpha$—open [2] if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$. Njastad [2] showed that $\alpha$—open sets in $X$ form a topology on $X$ which is called the $\alpha$—topology on $X$.

We now state Theorem 3.5 and Theorem 3.7 (a) in [1].

**Theorem 1.** Let $(X, \alpha O(X))$ be an $\alpha$—topological vector space. Then every $U \in N_0(X)$, the collection of all $\alpha$—open sets in $X$ containing the zero vector, is absorbing.

**Theorem 2.** Let $(X, \alpha O(X))$ be an $\alpha$—topological vector space. For every $U \in N_0(X)$, there exists a balanced set $V \in N_0(X)$ s.t. $V \subseteq U$.

In the proof of these theorems in [1], we claim that the set $V_1$ need not contain any open interval of the form $(-\epsilon, \epsilon)$, for any $\epsilon > 0$. To see why this claim is correct in general, take e.g. the set $V_1 = \{\eta \in \mathbb{R}: \eta \neq \frac{1}{n}, \ n \in \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ denotes the set of positive integers. Then $\text{Int}(\text{Cl}(\text{Int}(V_1))) = \mathbb{R} \supseteq V_1$, i.e., $V_1$ is an $\alpha$—open set of $\mathbb{R}$. Obviously, $V_1$ does not contain any set of the form $(-\epsilon, \epsilon)$. We can also observe this by considering the following sets in $\mathbb{R}$:

$V_1 = \{\eta \in \mathbb{R}: -1 < \eta < 1, \ \eta \neq \frac{1}{n}, \ n \in \mathbb{Z}_+\}$

or

$V_1 = \{\eta \in \mathbb{R}: -1 < \eta < 1, \ \eta \notin \sigma\}$,
where \( \sigma = \{-\frac{1}{n} : n \in \mathbb{Z}_+\} \cup \{\frac{k}{n} : k \in \mathbb{Z}_+\} \).

From here on, we use the following notations:

- \( \mathfrak{S}^\alpha, X^\alpha \) and \( N_x(X) \) denote respectively the class of \( \alpha \)-open sets in a given topological space \( (X, \mathfrak{S}) \), the corresponding topological space \( (X, \mathfrak{S}^\alpha) \) and the collection of all \( \alpha \)-open sets in \( X \) (with respect to \( \mathfrak{S} \)) containing \( x \).

- All vector spaces are over the field \( F \in \{\mathbb{R}, \mathbb{C}\} \), and when we treat \( F \) as a topological space, we mean \( F \) is endowed with its ordinary topology.

Corrected version

**Definition 3.** A nearly topological vector space is a pair \((L, \mathfrak{S})\), where

- \( L \) is a vector space, and
- \( \mathfrak{S} \) is a topology on \( L \), with
  1. the vector addition mapping, \( L^\alpha \times L^\alpha \ni (x, y) \mapsto x + y \in L^\alpha \), and
  2. the scalar multiplication mapping, \( F \times L^\alpha \ni (\lambda, x) \mapsto \lambda x \in L^\alpha \)

are continuous.

It is obvious from the definition that every real nearly topological vector space is an \( \alpha \)-topological vector space, but the converse need not be true in general.

**Theorem 4.** Let \((L, \mathfrak{S})\) be a nearly topological vector space, and \( 0 \in L \) the zero vector. Then the following statements

1. Every \( \sigma \in N_o(L) \) is absorbing.
2. For every \( \nabla \in N_o(L) \), there exists a balanced set \( \sigma \in N_o(L) \) s.t. \( \sigma \subseteq \nabla \).

hold true.

**Proof.** (1) Fix any \( x \in L \), then the mapping

\[ F \ni \lambda \mapsto \lambda x \in L^\alpha \]

is continuous. Therefore for any \( \sigma \in N_o(L) \), there exists an open set \( \mathfrak{U} \) in \( F \) such that \( \mathfrak{U} \sigma \subseteq \sigma \) where \( \mathfrak{U} = \{x\} \). Without loss of generality, we can take \( \epsilon > 0 \) such that \((-\epsilon, \epsilon) \subseteq \mathfrak{U} \). Then \((-\epsilon, \epsilon) \mathfrak{U} \subseteq \sigma \), showing that \( \sigma \) is absorbing.

(2) Take any \( \nabla \in N_o(L) \), then there exists an open set \( \mathfrak{U} \) in \( F \), and \( \mathfrak{U} \in N_o(L) \) satisfying

\[ \mathfrak{U} \mathfrak{U} \subseteq \nabla, 0 \in \mathfrak{U}. \]

Choose any sufficiently small positive number \( \epsilon \) such that \( D = \{\lambda \in F : |\lambda| \leq \epsilon\} \subseteq \mathfrak{U} \), and set

\[ \sigma = \cup_{\lambda \in D} \lambda \mathfrak{U}. \]

Since the function \( L^\alpha \ni x \mapsto \lambda x \in L^\alpha \) (for any fixed \( 0 \neq \lambda \in F \)) is a homeomorphism, \( \sigma \in N_o(L) \). Now, for \( \mu \in F \) with \( |\mu| \leq 1 \), we have

\[ \mu \sigma = \cup_{\lambda \in D} \mu (\lambda \mathfrak{U}) = \cup_{\nu \in D} \nu \mathfrak{U}, \]

which ends the proof.

**Acknowledgment.** The author wishes to thank the editor for her valuable comments/suggestions concerning the improvement of the manuscript.

**References**