Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (4) (2021), 1170 – 1171
DOI : 10.15672/hujms.816050

ADDENDUM

Addendum to "$\alpha-$Topological Vector Spaces"[Hacet. J. Math. Stat. 47 (5), 1102-1107, 2018]

Madhu Ram

Department of Mathematics, University of Jammu, Jammu-180006, J& K, India

Abstract

In this article, we give the corrected version of Theorem 3.5 and Theorem 3.7 (a) in [$\alpha-$Topological vector spaces, Hacet. J. Math. Stat. **47** (5), 1102-1107, 2018] by T. Al-Hawary.

Mathematics Subject Classification (2020). 46A19, 54C05, 46A99

Keywords. $\alpha-$open sets, $\alpha-$topological vector spaces, nearly topological vector spaces

This article concerns two results, namely Theorem 3.5 and Theorem 3.7 (a), in the original paper "*$\alpha-$Topological Vector Spaces by T. Al-Hawary*", Hacet. J. Math. Stat. **47** (5), 1102-1107, 2018. Actually, an incorrect detail is used by Al-Hawary in the argument of the proofs of Theorem 3.5 and Theorem 3.7 (a) in [1]. We identify this error and provide a counterexample to it before stating the amended version of these theorems.

Recall that a subset A of a topological space $(X, \Im)$ is called $\alpha-$open [2] if $A \subseteq Int(Cl(Int(A)))$. Njastad [**2**] showed that $\alpha-$open sets in X form a topology on X which is called the $\alpha-$topology on X.

We now state Theorem 3.5 and Theorem 3.7 (a) in [1].

Theorem 1. *Let $(X, \alpha O(X))$ be an $\alpha-$topological vector space. Then every $U \in N_0(X)$, the collection of all $\alpha-$open sets in X containing the zero vector, is absorbing.*

Theorem 2. *Let $(X, \alpha O(X))$ be an $\alpha-$topological vector space. For every $U \in N_0(X)$, there exists a balanced set $V \in N_0(X)$ s.t. $V \subseteq U$.*

In the proof of these theorems in [1], we claim that the set V_1 need not contain any open interval of the form $(-\epsilon, \epsilon)$, for any $\epsilon > 0$. To see why this claim is correct in general, take e.g. the set $V_1 = \{\eta \in \mathbb{R} \colon \eta \neq \frac{1}{n}, \; n \in \mathbb{Z}_+\}$, where $\mathbb{Z}_+$ denotes the set of positive integers. Then $Int(Cl(Int(V_1))) = \mathbb{R} \supseteq V_1$, i.e., V_1 is an $\alpha-$open set of $\mathbb{R}$. Obviously, V_1 does not contain any set of the form $(-\epsilon, \epsilon)$. We can aslo observe this by considering the following sets in $\mathbb{R}$:

$$V_1 = \{\eta \in \mathbb{R} \colon \; -1 < \eta < 1, \; \eta \neq \frac{1}{n}, \; n \in \mathbb{Z}_+\}$$

or

$$V_1 = \{\eta \in \mathbb{R} \colon \; -1 < \eta < 1, \; \eta \notin \sigma\},$$

Email address: madhuram0502@gmail.com
Received: 25.10.2020; Accepted: 12.03.2021

where $\sigma = \{-\frac{1}{n} \colon n \in \mathbb{Z}_+\} \cup \{\frac{1}{k} \colon k \in \mathbb{Z}_+\}$.

From here on, we use the following notations:

• $\Im^\alpha$, X^α and $N_x(X)$ denote respectively the class of $\alpha-$open sets in a given topological space $(X,\ \Im)$, the corresponding topological space $(X,\ \Im^\alpha)$ and the collection of all $\alpha-$open sets in X (with respect to $\Im$) containing x.

• All vector spaces are over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$, and when we treat $\mathbb{F}$ as a topological space, we mean $\mathbb{F}$ is endowed with its ordinary topology.

Corrected version

Definition 3. A nearly topological vector space is a pair $(L,\ \Im)$, where

- L is a vector space, and
- $\Im$ is a topology on L, with

(1) the vector addition mapping, $L^\alpha \times L^\alpha \ni (x,\ y) \ \mapsto\ x + y \in L^\alpha$, and

(2) the scalar multiplication mapping, $\mathbb{F} \times L^\alpha \ni (\lambda,\ x) \ \mapsto\ \lambda x \in L^\alpha$

are continuous.

It is obvious from the definition that every real nearly topological vector space is an $\alpha-$topological vector space, but the converse need not be true in general.

Theorem 4. *Let $(L,\ \Im)$ be a nearly topological vector space, and $o \in L$ the zero vector. Then the following statements*

(1) *Every $\sigma \in N_o(L)$ is absorbing.*

(2) *For every $\nabla \in N_o(L)$, there exists a balanced set $\sigma \in N_o(L)$ s.t. $\sigma \subseteq \nabla$.*

hold true.

Proof. (1) Fix any $x \in L$, then the mapping

$$\mathbb{F} \ni \lambda \ \mapsto\ \lambda x \in L^\alpha$$

is continuous. Therefore for any $\sigma \in N_o(L)$, there exists an open set $\mathfrak{U}$ in $\mathbb{F}$ such that $\mathfrak{U}\mathfrak{V} \subseteq \sigma$ where $\mathfrak{V} = \{x\}$. Without loss of generality, we can take $\epsilon > 0$ such that $(-\epsilon,\ \epsilon) \subseteq \mathfrak{U}$. Then $(-\epsilon,\ \epsilon)\mathfrak{V} \subseteq \sigma$, showing that σ is absorbing.

(2) Take any $\nabla \in N_o(L)$, then there exists an open set $\mathfrak{U}$ in $\mathbb{F}$, and $\mathfrak{V} \in N_o(L)$ satisfying

$$\mathfrak{U}\mathfrak{V} \subseteq \nabla, 0 \in \mathfrak{U}.$$

Choose any sufficiently small positive number ϵ such that $D = \{\lambda \in \mathbb{F} \colon |\lambda| \leq \epsilon\} \subseteq \mathfrak{U}$, and set

$$\sigma = \cup_{\lambda \in D} \lambda \mathfrak{V}.$$

Since the function $L^\alpha \ni x \ \mapsto\ \lambda x \in L^\alpha$ (for any fixed $0 \neq \lambda \in \mathbb{F}$) is a homeomorphism, $\sigma \in N_o(L)$. Now, for $\mu \in \mathbb{F}$ with $|\mu| \leq 1$, we have

$$\mu\sigma = \cup_{\lambda \in D} \mu(\lambda \mathfrak{V}) = \cup_{\nu \in D} \nu \mathfrak{V},$$

which ends the proof. □

Acknowledgment. The author wishes to thank the editor for her valuable comments/suggestions concerning the improvement of the manuscript.

References

[1] T. Al-Hawary, *α-Topological Vector Spaces*, Hacet. J. Math. Stat. **47** (5), 1102-1107, 2018.

[2] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math. **15**, 961-970, 1965.