Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753 /mathenot.816576
10 (1) 27-44 (2022) - Research Article
ISSN: 2147-6268

Absolute Lucas Spaces with Matrix and Compact Operators

Fadime Gökçe

Abstract
The main purpose of this study is to introduce the absolute Lucas series spaces and to investigate their some algebraic and topological structure such as some inclusion relations, $BK-$ to this space, duals and Schauder basis. Also, the characterizations of matrix operators related to these space with their norms are given. Finally, by using Hausdorff measure of noncompactness, the necessary and sufficient conditions for a matrix operator on them to be compact are obtained.

Keywords: *Absolute summability; Lucas numbers; matrix transformations; sequence spaces; bounded operators; operator norm; Hausdorff measures of noncompactness.*

AMS Subject Classification (2020): *Primary: 40C05; 11B39; 40D25; 40F05; 46A45.*

1. Introduction

Let ω be the set of all sequences of complex numbers. A vector subspace of ω is called *a sequence space*. The spaces l_∞, c, c_0, Ψ, bs, cs, l and l_p $(p > 1)$ stand for the classes of all bounded, convergent, null and finite sequences and the classes of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively.

Let X and Y be two sequence spaces and $A = (a_{nv})$ be an arbitrary infinite matrix with complex components for all $n, v \in \mathbb{N} = \{0, 1, 2, ...\}$. If the series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

converges for all $n \in \mathbb{N}$, then, by $A(x) = (A_n(x))$, we indicate the A-transform of the sequence $x = (x_v)$. Also, if $Ax = (A_n(x)) \in Y$ for every $x \in X$, then, A is called a matrix transformation from the sequence space X into the sequence space Y, and the class of all infinite matrices from X into Y is denoted by (X, Y).

A summability method is denoted by the matrix A if the transform sequence $A(x)$ converges to a real number.

The multiplier space of X and Y is identified by

$$S(X, Y) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in Y \text{ for all } x \in X\}.$$

Received : *26-10-2020,* ***Accepted :*** *17-02-2021*

According to this notation, duals of the space X are described as

$$X^{\alpha}=S(X,l),\ \ X^{\beta}=S(X,cs),\ \ X^{\gamma}=S(X,bs).$$

If $a_{nn}\neq 0$ for all n and $a_{nv}=0$ for $n<v$, then it is said that A is a triangle.

The concept of the domain of an infinite matrix A in the sequence space X is described as

$$X_A=\{x=(x_n)\in\omega : A(x)\in X\}$$

which is a new sequence space. In this connection, by means of the concept of the matrix domain, different new sequence spaces have been presented and their topological, algebraic structure and matrix transformations have been studied in literature. For example, one can see some of these spaces in references ([1, 2], [10–12], [23]).

A sequence space X is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : X\rightarrow\mathbb{C}$ defined by $p_n(x)=x_n$ for all $n\in\mathbb{N}$. Further, an FK-space X whose metric is given by a norm is said to be a BK-space. The theory of FK- and BK-spaces has an important role in summability theory. For example, the operators between BK-spaces are continuous and the matrix domain of a triangle A in the BK-space X is also a BK-space and its norm is given by

$$\|x\|_{X_A}=\|A(x)\|_X\,,$$

[4]. Let X be a normed sequence space and (b_k) be a sequence in X. If there exists a unique sequence of coefficients (x_k) such that, for each $x\in X$,

$$\left\|x-\sum_{k=0}^{n}x_kb_k\right\|\rightarrow 0,\ \ n\rightarrow\infty$$

then, the sequence (b_k) is called the Schauder basis (or briefly basis) for X, and in this case it is written that $x=\sum\limits_{k=0}^{\infty}x_kb_k$. It is said that an FK-space X, consisting all finite sequences, has AK property if every sequence $x=(x_k)\in X$ has a unique representation $x=\sum\limits_{j=0}^{\infty}x_je^{(j)}$, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in the jth place for each $j\in\mathbb{N}$. This means that the sequence $(e^{(j)})$ is a Schauder basis for any $FK-$ space with AK. For example, $(e^{(j)})$ is the Schauder basis of the space l_p, but the space l_∞ doesn't have the Schauder basis [20].

For arbitrary two Banach spaces X and Y, $\mathcal{B}(X,Y)$ denotes the set of all continuous linear operators from the space X into the space Y, and the operator norm of $A\in\mathcal{B}(X,Y)$ is stated by

$$\|A\|=\sup_{x\neq 0}\frac{\|A(x)\|_Y}{\|x\|_X}.$$

In the special case $Y=\mathbb{C}$, it is written that $X^*=\mathcal{B}(X,\mathbb{C})$, the set of all continuous linear functional on X.

If $a\in\omega$ and $X\supset\Psi$ is a BK-space, then

$$\|a\|_X^*=\sup_{x\in S_X}\left|\sum_{k=0}^{\infty}a_kx_k\right|$$

provided the right hand side of the equation exists, where S_X is the unit sphere in X, and it is finite for $a\in X^{\beta}$.

Throughout the whole paper, we suppose that $\phi=(\phi_n)$ is a sequence of positive numbers and p^* is conjugate of p, that is, $1/p+1/p^*=1$, $p>1$, and $1/p^*=0$ for $p=1$.

Let take $\sum x_v$ as an infinite series with nth partial sum s_n. Then, the series $\sum x_v$ is said to be summable $|A,\phi_n|_p$, if (see[29])

$$\sum_{n=0}^{\infty}\phi_n^{p-1}|A_n(s)-A_{n-1}(s))|^p<\infty,\ \ A_{-1}(s)=0.$$

This method includes some well known methods. For instance, if A is the matrix of weighted mean $\left(\bar{N},p_n\right)$ (resp. $\phi_n=P_n/p_n$), then it is reduced to the summability $\left|\bar{N},p_n,\phi_n\right|_p$ [31] (the summability $\left|\bar{N},p_n\right|_p$ [3]). Also if we take A as the matrix of Cesàro mean of order $\alpha>-1$ and $\phi_n=n$, then we get the summability $|C,\alpha|_p$ in Flett's notation [5]. The choice of the Fibonacci matrix instead of A leads to the absolute Fibonacci summability method [7]. In addition to the aforementioned spaces, several absolute series spaces have also taken place in the literature (see [6, 8, 19, 25, 27–29]).

The Lucas sequence (L_n) is one of the most interesting number sequences in mathematics and is named after the mathematician François Edouard Anatole Lucas (1842-1891). It is given by the Fibonacci recurrence relation with different initial condition such that

$$L_0 = 2,\, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,$$

also, the terms of the Lucas sequence have the following important properties

$$\sum_{k=1}^{n} L_k = L_{n+2} - 3,\ \sum_{k=1}^{n} L_{2k-1} = L_{2n} - 2,$$
$$\sum_{k=1}^{n} L_{2k} = L_{2n+1} - 1,\ \sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2,$$
$$L_{n-1}^2 + L_n L_{n-1} - L_n^2 = 5(-1)^{n+1}, n \geq 1,$$
$$L_{n-1} L_{n+1} - L_n^2 = 5(-1)^{n+1}, n \geq 1.$$

We refer reader to [13] for other properties. Additionally, just like the Fibonacci numbers, the rates of successive Lucas numbers converges to the golden ratio which is one of the most interesting irrationals playing an important role in number theory, algorithms, network theory, etc. Using Lucas numbers, the Lucas matrix $\hat{E}(r,s) = (\hat{e}_{nk}(r,s))$ has recently been defined [12] as

$$\hat{e}_{nk}(r,s) = \begin{cases} s\frac{L_n}{L_{n-1}}, & k = n-1 \\ r\frac{L_{n-1}}{L_n}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

where L_n be the nth Lucas number for every $n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$.

The aim of this paper is to define the absolute sumability space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and investigate its some inclusion relations, $\alpha-, \beta-, \gamma-$ duals and basis. Also, some matrix and compact operators on this space are characterized and their operator norms and Hausdorff measures of noncompactness are determined.

It is required the following lemmas in proving theorems.

Lemma 1.1. *[18] Let T be a triangle, X and Y be two arbitrary subsets of ω. Then, we have*

(a) $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.

(b) Further, if X and Y are BK-spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Lemma 1.2. *[30] Let $1 < p < \infty$. Then,*

1. *$A \in (l, c) \Leftrightarrow$ (i) $\lim_n a_{nv}$ exists for $v \geq 0$, (ii) $\sup_{n,v} |a_{nv}| < \infty$,*

2. *$A \in (l, l_\infty) \Leftrightarrow$ (ii) holds,*

3. *$A \in (l, c_0) \Leftrightarrow$ (iii) $\lim_n a_{nv} = 0$ for all $v \geq 0$ and (ii) hold,*

4. *$A \in (l_p, c) \Leftrightarrow (i)$ holds, $(iv) \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{p^*} < \infty$,*

5. *$A \in (l_p, l_\infty) \Leftrightarrow (iv)$ holds,*

6. $A \in (l_p, c_0) \Leftrightarrow$ *(iii) and (iv) hold.*

Lemma 1.3. *[14] Let $1 \le p < \infty$. Then, $A \in (l, l_p)$ if and only if*

$$\|A\|_{(l,l_p)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^p \right\}^{\frac{1}{p}}.$$

Lemma 1.4. *[30] Let $1 < p < \infty$. Then, $A \in (l_p, l)$ if and only if*

$$\|A\|_{(l_p,l)} = \sup_{N \in \mathfrak{F}} \left\{ \sum_{v=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{nv} \right|^{p^*} \right\}^{1/p^*}$$

where $\mathfrak{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.5. *[27] Let $1 < p < \infty$. Then, $A \in (l_p, l)$ if and only if*

$$\|A\|'_{(l_p,l)} = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{p^*} \right\}^{1/p^*} < \infty.$$

Moreover since $\|A\|_{(l_p,l)} \le \|A\|'_{(l_p,l)} \le 4 \|A\|_{(l_p,l)}$, there exists $1 \le \xi \le 4$ such that $\|A\|'_{(l_p,l)} = \xi \|A\|_{(l_p,l)}$.

Lemma 1.6. *[18] Let $1 < p < \infty$ and p^* denote the conjugate of p. Then, $l_p^\beta = l_{p^*}$ and $l_\infty^\beta = c^\beta = c_0^\beta = l$, $l^\beta = l_\infty$. Also, let X denote any of the spaces l_∞, c, c_0, l and l_p. Then, we have*

$$\|a\|_X^* = \|a\|_{X^\beta}$$

for all $a \in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the X^β.

Lemma 1.7. *[15] Let X and Y be BK-spaces. Then, we have*

(a) *$(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = A(x)$ for all $x \in X$.*

(b) *If X has AK, then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, for every operator $L \in \mathcal{B}(X, Y)$ there exists a matrix $A \in (X, Y)$ such that by $L(x) = A(x)$ for all $x \in X$.*

Lemma 1.8. *[4] Let $X \supset \Psi$ be a BK-space and Y be any of the spaces ℓ_∞, c, c_0. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,l_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

2. Hausdorff Measure of Noncompactness

If S and R are subsets of a metric space (X, d) and, for every $r \in R$, there exists an $s \in S$ such that $d(r, s) < \varepsilon$ then, S is called an ε-net of R; if S is finite, then the ε-net S of R is called a finite ε-net of R. Let X, Y be two Banach spaces. It is said that a linear operator $L : X \to Y$ is compact if its domain is all of X and the sequence $(L(x_n))$ has a convergent subsequence in Y, for every bounded sequence (x_n) in X. The class of such operators is denoted by $\mathcal{C}(X, Y)$. If Q is any bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is given by

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } X\},$$

and χ is named the Hausdorff measure of noncompactness. Using the Hausdorff measure of noncompactness, some compact operators on various sequence spaces are characterized by many authors. For example, Mursaleen and Noman in [21, 22], Malkowsky and Rakocevic in [17] have used the Hausdorff measure of noncompactness method to characterize the class of compact operators on some known spaces, (see also [7, 8, 15, 26]).

The following lemma is very important to calculate the Hausdorff measure of noncompactness of any bounded subset of the space l_p.

Lemma 2.1. *([24]) Let Q be a bounded subset of the normed space X where $X = l_p$ for $1 \leq p < \infty$ or $X = c_0$. If $P_r : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ...x_r, 0, 0, ...)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I - P_r)(x)\|\right).$$

Let X and Y be two Banach spaces, χ_1 and χ_2 be Hausdorff measures on X and Y, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Y and there exists a constant $M > 0$ such that $\chi_2(L(Q)) \leq M\chi_1(Q)$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded set } Q \subset X\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$ then it is written that $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

There is a significant relation between compact operators and Hausdorff measure of noncompactness. The following lemma gives this relation.

Lemma 2.2. *[18] Let X and Y be two Banach spaces and $L \in \mathcal{B}(X, Y)$. Also, let the set $S_x = \{x \in X : \|x\| \leq 1\}$ be the unit sphere in X. Then,*

$$\|L\|_\chi = \chi(L(S_x))$$

and

$$L \in \mathcal{C}(X, Y) \Leftrightarrow \|L\|_\chi = 0.$$

Lemma 2.3. *[16] Let X be a normed sequence space, $T = (t_{nv})$ be an infinite triangle matrix, χ_T and χ define the Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.*

Lemma 2.4. *[22] Let $X \supset \Psi$ be a BK-space with AK or $X = l_\infty$. If $A \in (X, c)$, then, we have*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \text{exists for all } k,$$

$$\alpha = (\alpha_k) \in X^\beta,$$

$$\sup_n \|A_n - \alpha\|^*_X < \infty,$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty}\alpha_k x_k \text{ for every } x = (x_k) \in X.$$

Lemma 2.5. *[22] Let $X \supset \Psi$ be a BK-space. Then,*

(a) If $A \in (X, c_0)$, then

$$\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*\right).$$

(b) If X has AK or $X = l_\infty$ and $A \in (X, c)$, then

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n - a\|^*\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n - a\|^*\right)$$

where $a = (a_k)$ defined by $a_k = \lim_{n\to\infty} a_{nk}$, for all $n \in \mathbb{N}$.

(c) If $A \in (X, l_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n>r}\|A_n\|^*\right).$$

3. Absolute Lucas summability spaces

In this section, firstly, the summability space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ as the set of all series summable by absolute Lucas method is introduced, and it is proved that this space is a BK-space which is linearly isomorphic to l_p for $1 \le p < \infty$. Also, giving some inclusion relations, $\alpha-$, $\beta-$ and $\gamma-$ duals and Schauder basis of this space are investigated.

If the Lucas matrix is taken instead of A, then $|A, \phi_n|_p$ summability is reduced to the absolute Lucas summability. Then, since (s_n) is a sequence of partial sum of the series $\sum x_k$, it follows that

$$\begin{aligned}
\hat{E}_n(r,s)(s) = \sum_{k=1}^{n} \hat{e}_{nk}(r,s)s_k &= \sum_{k=1}^{n} x_k \sum_{v=k}^{n} \hat{e}_{nv}(r,s) \\
&= x_n \hat{e}_{nn}(r,s) + \sum_{k=1}^{n-1} (\hat{e}_{nn}(r,s) + \hat{e}_{n,n-1}(r,s)) x_k \\
&= x_n r \tfrac{L_{n-1}}{L_n} + \sum_{k=1}^{n-1} \left(s \tfrac{L_n}{L_{n-1}} + r \tfrac{L_{n-1}}{L_n} \right) x_k \\
&= \sum_{k=1}^{n} l_{nk} x_k
\end{aligned}$$

where the matrix $\mathcal{L}(r,s) = (l_{nk})$ is given by

$$l_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k = n \\ s\frac{L_n}{L_{n-1}} + r\frac{L_{n-1}}{L_n}, & 1 \le k \le n-1 \\ 0, & k > n. \end{cases} \tag{3.1}$$

So, we get

$$\begin{aligned}
\hat{E}_n(s) - \hat{E}_{n-1}(s) &= x_n r \tfrac{L_{n-1}}{L_n} + x_{n-1} \left(s \tfrac{L_n}{L_{n-1}} + r \tfrac{5(-1)^{n+1}}{L_n L_{n-1}} \right) + \sum_{k=1}^{n-2} \tfrac{5(-1)^n}{L_{n-1}} \left(\tfrac{s}{L_{n-2}} - \tfrac{r}{L_n} \right) x_k \\
&= \sum_{k=1}^{n} \xi_{nk} x_k
\end{aligned}$$

where

$$\xi_{nk} = \begin{cases} r\frac{L_{n-1}}{L_n}, & k = n \\ s\frac{L_n}{L_{n-1}} + r\frac{5(-1)^{n+1}}{L_n L_{n-1}}, & k = n-1 \\ \frac{5(-1)^n}{L_{n-1}} \left(\frac{s}{L_{n-2}} - \frac{r}{L_n} \right), & 1 \le k \le n-2 \\ 0, & k > n. \end{cases} \tag{3.2}$$

Hence, the space $|\mathcal{L}(r,s)|_p$ can be stated by

$$|\mathcal{L}(r,s)|_p = \left\{ x \in \omega : \left(\phi_n^{1/p^*} \sum_{k=1}^{n} \xi_{nk} x_k \right) \in l_p \right\}.$$

On the other hand, according to the matrix domain, this space is redefined by

$$\left|\mathcal{L}^{\phi}(r,s)\right|_p = (l_p)_{E^{(p)} \circ \mathcal{L}(r,s)}$$

where

$$e_{nk}^{(p)} = \begin{cases} \phi_n^{1/p^*}, & k = n \\ -\phi_n^{1/p^*}, & k = n-1 \\ 0, & k \ne n, n-1. \end{cases} \tag{3.3}$$

Also, we note

$$(E^{(p)} \circ \mathcal{L}(r,s))_n(x) = \phi_n^{1/p^*} (\mathcal{L}(r,s)_n(x) - \mathcal{L}_{n-1}(r,s)(x)).$$

Moreover, since every triangle matrix has a unique triangle inverse [32], the matrices $\mathcal{L}(r,s)$ and $E^{(p)}$ have unique inverses $\tilde{\mathcal{L}}(r,s) = (\tilde{l}_{nk})$ and $\tilde{E}^{(p)} = (\tilde{e}_{nk})$ whose terms are given by

$$\tilde{l}_{nk} = \begin{cases} \frac{1}{r}\frac{L_n}{L_{n-1}}, & k = n \\ \frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_k L_{k-1}}\left(\frac{s}{r}L_n^2 + L_{n-1}^2\right), & 1 \leq k \leq n-1 \\ 0, & k > n \end{cases} \tag{3.4}$$

$$\tilde{e}_{nv}^{(p)} = \begin{cases} \phi_v^{-1/p^*}, & 1 \leq v \leq n \\ 0, & v > n \end{cases} \tag{3.5}$$

respectively.

First, to understand the space better, we exibit some relations between the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and l_p.

Theorem 3.1. *Let $\phi \in l_\infty$ and $1 \leq p < \infty$. Then, $l_p \subset \left|\mathcal{L}^{\phi}(r,s)\right|_p$.*

Proof. For $p = 1$, it is clear, it is omitted. Let $p > 1$. By the properties of Lucas numbers, the series $\sum_n \frac{1}{L_n}$ is convergent and also $(\frac{1}{L_n})$ is a decreasing sequence. So, it follows from Abel's Theorem that $\frac{n}{L_n} \to 0$ as $n \to \infty$. This gives $\sum_{k=0}^{n} |\xi_{nk}| = O(1)$ and $\sum_{n=k}^{\infty} |\xi_{nk}| = O(1)$. Hence, by Hölder's inequality, it is obtained that

$$\begin{aligned} \|x\|_{|\mathcal{L}^{\phi}(r,s)|_p} &= \left\{\sum_{n=1}^{\infty}\left|\phi_n^{1/p^*}\sum_{k=1}^{n}\xi_{nk}x_k\right|^p\right\}^{1/p} \\ &\leq \left\{\sum_{n=1}^{\infty}\phi_n^{p-1}\sum_{k=1}^{n}|\xi_{nk}|\,|x_k|^p\left(\sum_{k=1}^{n}|\xi_{nk}|\right)^{p/p^*}\right\}^{1/p} \\ &= O(1)\left\{\sum_{k=1}^{\infty}|x_k|^p\sum_{n=k}^{\infty}|\xi_{nk}|\right\}^{1/p} \\ &= O(1)\left\{\sum_{k=1}^{\infty}|x_k|^p\right\}^{1/p} = O(1)\,\|x\|_{l_p}, \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $1 \leq p \leq q < \infty$. If there is a constant $M > 0$ such that $\phi_n \leq M$ for all $n \in \mathbb{N}$, then $\left|\mathcal{L}^{\phi}(r,s)\right|_p \subset \left|\mathcal{L}^{\phi}(r,s)\right|_q$.*

Proof. To prove the inclusion, take $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$. Since $l_p \subset l_q$ for $1 \leq p \leq q < \infty$, it is clear that $\left(\phi_n^{1/p^*}\sum_{j=0}^{n}\xi_{nj}x_j\right) \in l_q$. Also, by considering $\phi_n \leq M$ for all $n \in \mathbb{N}$, it can be written that

$$M^{\frac{q}{p^*}-\frac{q}{q^*}}\left|\phi_n^{\frac{1}{q^*}}\sum_{j=1}^{n}\xi_{nj}x_j\right|^q \leq \left|\phi_n^{1/p^*}\sum_{j=1}^{n}\xi_{nj}x_j\right|^q$$

which implies that $x \in |\mathcal{L}(r,s)|_q$. □

The following result is useful to determine a Schauder basis for the matrix domain of a special triangular matrix in a linear metric space.

Lemma 3.1. *([9]). Let T be a triangular matrix and S be its inverse. If (b_k) is a Schauder basis of the metric space (X,d), then $(S(b_k))$ is a basis of X_T with respect to the metric d_T given by $d_T(z_1,z_2) = d(Tz_1,Tz_2)$ for all $z_1,z_2 \in X_T$.*

Theorem 3.3. *Let $1 \leq p < \infty$. Then, the set $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ is a linear space with coordinate-wise addition and scalar multiplication. Also, it is a BK-space with respect to the norm*

$$\|x\|_{|\mathcal{L}^{\phi}(r,s)|_p} = \left\|E^{(p)} \circ \mathcal{L}(r,s)(x)\right\|_{l_p}.$$

Moreover, the sequence $b^{(j)}=(b_n^{(j)})$ defined by

$$b_n^{(j)}=\begin{cases}\phi_j^{-1/p^*}\left(\frac{1}{r}\frac{L_n}{L_n-1}+\sum\limits_{k=j}^{n-1}\frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2+L_{n-1}^2\right)\right), & 1\leq j\leq n-1\\ \phi_n^{-1/p^*}\frac{1}{r}\frac{L_n}{L_n-1}, & j=n\\ 0, & j>n,\end{cases}$$

is a Schauder basis for the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$.

Proof. Since the space l_p is a BK-space for $1\leq p<\infty$ and $E^{(p)}\circ\mathcal{L}^{\phi}(r,s)$ is a triangle matrix, it follows from Theorem 4.3.2 of [32], $\left|\mathcal{L}^{\phi}(r,s)\right|_p=(l_p)_{E^{(p)}\circ\mathcal{L}(r,s)}$ is a BK-space. On the other hand, it is known that the sequence $(e^{(j)})$ is the Schauder basis of the space l_p. So, it can be obtained by Lemma 3.1 that $b^{(j)}=((\tilde{L}(r,s)\circ\tilde{E}^{(p)})_n(e^{(j)}))$ is a Schauder basis of the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$. □

Theorem 3.4. *Let $1\leq p<\infty$. Then, there exists a linear isomorphism between the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and l_p i.e., $\left|\mathcal{L}^{\phi}(r,s)\right|_p\cong l_p$.*

Proof. To prove this, it should be shown that the existence of a linear bijection between the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and l_p where $1\leq p<\infty$. Let consider the maps $\mathcal{L}(r,s):\left|\mathcal{L}^{\phi}(r,s)\right|_p\to(l_p)_{E^{(p)}}$, $E^{(p)}:(l_p)_{E^{(p)}}\to l_p$ given by (3.1) and (3.3). Since the matrices corresponding to these maps are triangles, these are linear bijections. So, the composite function $E^{(p)}\circ\mathcal{L}(r,s)$ has the same property. Further, one can see that the norm is preserved. This completes the proof. □

We use the following notations in the rest of the paper.

$$\eta_{nj}=\frac{1}{r}\frac{L_n}{L_{n-1}}+\sum_{k=j}^{n-1}\frac{(-1)^{n-k}}{r}\left(\frac{s}{r}\right)^{n-1-k}\frac{1}{L_kL_{k-1}}\left(\frac{s}{r}L_n^2+L_{n-1}^2\right),$$

$$D_1=\left\{\epsilon\in\omega:\sum_{n=j+1}^{\infty}\eta_{nj}\epsilon_n\text{ exists for all }j\right\},$$

$$D_2=\left\{\epsilon\in\omega:\sup_m\left\{\phi_m^{-1}\left|\frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m\right|^{p^*}+\sum_{j=1}^{m-1}\phi_j^{-1}\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j+\sum_{n=j+1}^{m}\eta_{nj}\epsilon_n\right|^{p^*}\right\}<\infty\right\},$$

$$D_3=\left\{\epsilon\in\omega:\sup_{m,j}\left\{\left|\frac{1}{r}\frac{L_m}{L_{m-1}}\epsilon_m\right|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j+\sum_{n=j+1}^{m}\eta_{nj}\epsilon_n\right|\right\}<\infty\right\},$$

$$D_4=\left\{\epsilon\in\omega:\sum_{j=1}^{\infty}\frac{1}{\phi_j}\left\{\sum_{n=j+1}^{\infty}|\eta_{nj}\epsilon_n|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j\right|\right\}^{p^*}<\infty\right\},$$

$$D_5=\left\{\epsilon\in\omega:\sup_j\left\{\sum_{n=j+1}^{\infty}|\eta_{nj}\epsilon_n|+\left|\frac{1}{r}\frac{L_j}{L_{j-1}}\epsilon_j\right|\right\}<\infty\right\}.$$

Theorem 3.5. *Let $1<p<\infty$ and $\phi=(\phi_n)$ be a sequence of positive numbers. Then,*
(i) $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|\right\}^{\alpha}=D_5,\quad\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\alpha}=D_4$,
(ii) $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|\right\}^{\beta}=D_1\cap D_3,\quad\left\{\left|\mathcal{L}(r,s)\right|_p\right\}^{\beta}=D_1\cap D_2$,
(iii) $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|\right\}^{\gamma}=D_3,\quad\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\gamma}=D_2$.

Proof. Since the proofs of the other parts are similar, we just calculate the β-dual of the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$. Recall that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ iff $\epsilon x = (\epsilon_n x_n) \in cs$ for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$. Take $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$, $\mathcal{L}(r,s)(x) = y$ and $z = E^{(p)}(y)$. Then, $z \in l_p$. It follows from (3.4) and (3.5) that

$$\begin{aligned}
\sum_{n=1}^{m} \epsilon_n x_n &= \epsilon_1 x_1 + \sum_{n=2}^{m} \epsilon_n \left(\tfrac{1}{r} \tfrac{L_n}{L_{n-1}} y_n + \sum_{k=1}^{n-1} \tfrac{(-1)^{n-k}}{r} \left(\tfrac{s}{r}\right)^{n-1-k} \tfrac{1}{L_k L_{k-1}} \left(\tfrac{s}{r} L_n^2 + L_{n-1}^2\right) y_k \right) \\
&= \sum_{j=1}^{m} \phi_j^{-1/p^*} \sum_{n=j}^{m} \epsilon_n \tfrac{1}{r} \tfrac{L_n}{L_{n-1}} z_n \\
&+ \sum_{j=1}^{m-1} \phi_j^{-1/p^*} \left(\sum_{n=j+1}^{m} \sum_{k=j}^{n-1} \epsilon_n \tfrac{(-1)^{n-k}}{r} \left(\tfrac{s}{r}\right)^{n-1-k} \tfrac{1}{L_k L_{k-1}} \left(\tfrac{s}{r} L_n^2 + L_{n-1}^2\right) \right) z_j \\
&= \phi_m^{-1/p^*} \epsilon_m \tfrac{1}{r} \tfrac{L_m}{L_{m-1}} z_m + \sum_{j=1}^{m-1} \phi_j^{-1/p^*} \left(\epsilon_j \tfrac{1}{r} \tfrac{L_j}{L_{j-1}} + \sum_{n=j+1}^{m} \epsilon_n \eta_{nj} \right) z_j \\
&= \sum_{j=1}^{m} h_{mj} z_j
\end{aligned}$$

where the matrix $H = (h_{mj})$ is defined by

$$h_{mj} = \begin{cases} \phi_j^{-1/p^*} \left(\epsilon_j \tfrac{1}{r} \tfrac{L_j}{L_{j-1}} + \sum\limits_{n=j+1}^{m} \epsilon_n \eta_{nj} \right), & 1 \le j \le m-1 \\ \phi_m^{-1/p^*} \epsilon_m \tfrac{1}{r} \tfrac{L_m}{L_{m-1}}, & j = m \\ 0, & j > m. \end{cases}$$

This means that $\epsilon \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ iff $H \in (l_p, c)$. Thus, by applying Lemma 1.2 to the matrix H, we obtain $\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta} = D_1 \cap D_2$. This completes the proof. □

4. Matrix transformations on space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$

In this section, we characterize some classes of matrix operators on that space and compute their norms.

Lemma 4.1. *Let $1 < p < \infty$. If $a = (a_k) \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$, then, for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$, $\tilde{a}^{(p)} = (\tilde{a}_k^{(p)}) \in l_{p^*}$, $\tilde{a}^{(1)} \in l_{\infty}$ and*

$$\sum_k a_k x_k = \sum_k \tilde{a}_k^{(p)} z_k$$

holds, where $z = E^{(p)}(\mathcal{L}(r,s)(x)) \in l_p$ and

$$\tilde{a}_k^{(p)} = \phi_k^{-1/p^*} \left(a_k \frac{1}{r} \frac{L_k}{L_{k-1}} + \sum_{n=k+1}^{\infty} a_n \eta_{nk} \right).$$

Lemma 4.2. *Assume that $1 < p < \infty$. Then, we have $\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|_p} = \left\|\tilde{a}^{(p)}\right\|_{l_{p^*}}$ for all $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ and $\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|} = \left\|\tilde{a}^{(1)}\right\|_{l_{\infty}}$ for all $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|\right\}^{\beta}$ where $\tilde{a}^{(p)}$ as in Lemma 4.1.*

Proof. Let $a \in \left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$. It can be immediately seen from Lemma 4.1, $\tilde{a}^{(p)} \in l_{p^*}$ and $\tilde{a}^{(1)} \in l_{\infty}$. So, using Lemma 1.6 and Lemma 4.1 , we get

$$\|a\|^*_{|\mathcal{L}^{\phi}(r,s)|_p} = \sup_{x \in S_{|\mathcal{L}^{\phi}(r,s)|_p}} \left| \sum_{v=0}^{\infty} a_v x_v \right| = \sup_{z \in S_{l_p}} \left| \sum_{v=0}^{\infty} \tilde{a}_v^{(p)} z_v \right| = \left\|\tilde{a}^{(p)}\right\|^*_{l_p} = \left\|\tilde{a}^{(p)}\right\|_{l_{p^*}}.$$

The proof for the case $k = 1$ is quite easy, so it is omitted. □

Theorem 4.1. *Let $1<p<\infty$, $A=(a_{nk})$ be an infinite matrix of complex numbers for each $n,k\in\mathbb{N}$ and define the matrix $B^{(n)}=\left(b_{mk}^{(n)}\right)$, $\bar{B}=(\bar{b}_{nk})$ and $\hat{B}=(\hat{b}_{nk})$ as follows:*

$$b_{mk}^{(n)}=\begin{cases}\phi_k^{-1/p^*}\left(a_{nk}\frac{1}{r}\frac{L_k}{L_{k-1}}+\sum\limits_{j=k+1}^{m}a_{nj}\eta_{jk}\right), & 0\le k\le m-1\\ \phi_m^{-1/p^*}a_{nm}\frac{1}{r}\frac{L_m}{L_{m-1}}, & k=m\\ 0, & j>m,\end{cases}$$

$$\bar{b}_{nk}=\lim_{m\to\infty}b_{mk}^{(n)},$$

$$\hat{B}=E^{(1)}\circ\mathcal{L}(r,s)\circ\bar{B}.$$

Then, $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$ if and only if

$$\sum_{j=k+1}^{\infty}\eta_{jk}a_{nj}\ \textit{exists for all } k, \tag{4.1}$$

$$\sup_m\left\{\frac{1}{\phi_m}\left|\frac{1}{r}\frac{L_m}{L_{m-1}}a_{nm}\right|^{p^*}+\sum_{k=1}^{m-1}\frac{1}{\phi_k}\left|\frac{1}{r}\frac{L_k}{L_{k-1}}a_{nk}+\sum_{j=k+1}^{m}\eta_{jk}a_{nj}\right|^{p^*}\right\}, \tag{4.2}$$

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\left|\hat{b}_{nk}\right|\right)^{p^*}<\infty. \tag{4.3}$$

If $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$, then A defines a bounded linear operator L_A such that $L_A(x)=A(x)$ and

$$\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)}=\left\|\hat{B}\right\|_{(l_p,l)}.$$

Proof. $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$ if and only if $A(x)$ is well defined and belongs to the space $\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$. By Theorem 3.5, $A(x)$ is well defined, or, $(a_{nk})_{k=0}^{\infty}\in\left\{\left|\mathcal{L}^{\phi}(r,s)\right|_p\right\}^{\beta}$ if and only if (4.1) and (4.2) hold.

Beside, for any matrix $R=(r_{nv})\in(l_p,c)$, the remaining term of the series tends to zero uniformly in n, that is

$$\left|\sum_{v=m}^{\infty}r_{nv}x_v\right|\le\left(\sum_{v=m}^{\infty}|r_{nv}|^{p^*}\right)^{\frac{1}{p^*}}\left(\sum_{v=m}^{\infty}|x_v|^k\right)^{\frac{1}{p}}\to 0,(m\to\infty)$$

which gives the series $R_n(x)=\sum\limits_{v=0}^{\infty}r_{nv}x_v$ converges uniformly in n. So we have

$$\lim_n R_n\left(x\right)=\sum_{v=0}^{\infty}\lim_n r_{nv}x_v. \tag{4.4}$$

It follows from (3.4), (3.5) and (4.4)

$$A_n(x)=\lim_m\sum_{k=0}^{m}a_{nk}x_k=\lim_m\sum_{r=0}^{m}b_{mr}^{(n)}z_r=\sum_{r=0}^{\infty}\bar{b}_{nr}z_r.$$

Taking into $\left|\mathcal{L}^{\phi}(r,s)\right|_p\cong l_p$ for $1\le p<\infty$, it follows that $A(x)\in\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $\bar{B}\in\left(l_p,\left|\mathcal{L}^{\phi}(r,s)\right|\right)$. In other words, since $\left|\mathcal{L}^{\phi}(r,s)\right|_p=(l_p)_{E^{(p)}\circ\mathcal{L}(r,s)}$, $A(x)\in\left|\mathcal{L}^{\phi}(r,s)\right|$ for all $x\in\left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $\hat{B}\in(l_p,l)$. Also, a few calculations show that the matrix $\hat{B}$ is expressed as

$$b_{nk}^{*}=\sum_{v=0}^{n}l_{nv}(r,s)\bar{b}_{vk}=r\frac{L_{n-1}}{L_n}\bar{b}_{nk}+\sum_{v=0}^{n-1}\left(s\frac{L_n}{L_{n-1}}+r\frac{L_{n-1}}{L_n}\right)\bar{b}_{vk},$$

$$\hat{b}_{nr} = \phi_k^{1/p^*}\left(b^*_{nk} - b^*_{n-1,k}\right),\ \ n \geq 1 \text{ and } \hat{b}_{0k} = b^*_{0k}.$$

Now, if we apply Lemma 1.3 to the matrix $\hat{B}$, we get the condition (4.3). So, the first part of the proof is completed.

On the other hand, since the spaces $\left|\mathcal{L}^\phi(r,s)\right|_p$ and $\left|\mathcal{L}^\phi(r,s)\right|$ are BK-spaces, if $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, \left|\mathcal{L}^\phi(r,s)\right|\right)$, then, by Theorem 4.2.8 of [32], L_A defines a bounded operator such that $L_A(x) = A(x)$. To calculate the operator norm of A, we consider the isomorphisms $\mathcal{L}(r,s) : \left|\mathcal{L}^\phi(r,s)\right|_p \to (l_p)_{E^{(p)}}, E^{(p)} : (l_p)_{E^{(p)}} \to l_p$. Now, it is clear to see that $A = \tilde{\mathcal{L}}(r,s) \circ \tilde{E}^{(1)} \circ \hat{B} \circ E^{(p)} \circ \mathcal{L}(r,s)$ and so

$$\begin{aligned}
\|A\|_{\left(\left|\mathcal{L}^\phi(r,s)\right|_p, \left|\mathcal{L}^\phi(r,s)\right|\right)} &= \sup_{x \neq 0} \frac{\|A(x)\|_{\left|\mathcal{L}^\phi(r,s)\right|}}{\|x\|_{\left|\mathcal{L}^\phi(r,s)\right|_p}} \\
&= \sup_{x \neq 0} \frac{\left\|\tilde{\mathcal{L}}(r,s) \circ \tilde{E}^{(1)} \circ \hat{B} \circ E^{(p)} \circ \mathcal{L}(r,s)(x)\right\|_{\left|\mathcal{L}^\phi(r,s)\right|}}{\|x\|_{\left|\mathcal{L}^\phi(r,s)\right|_p}} \\
&= \sup_{z \neq 0} \frac{\left\|\hat{B}(z)\right\|_l}{\|z\|_{l_p}} = \left\|\hat{B}\right\|_{(l_p,l)} \ (z = E^{(p)} \circ \mathcal{L}(r,s)(x))
\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let $1 \leq p < \infty$, $A = (a_{nk})$ be an infinite matrix with complex components for all $n, k \in \mathbb{N}$, $B^{(n)} = \left(b^{(n)}_{mk}\right)$ and $\bar{B} = (b_{nk})$ be as in Theorem 4.1 with $1/p^* = 0$. Besides, define $\tilde{B} = E^{(p)} \circ \mathcal{L} \circ \bar{B}$. Then, $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, \left|\mathcal{L}^\phi(r,s)\right|_p\right)$ if and only if*

$$\sum_{v=j+1}^{\infty} \eta_{vj} a_{nv} \ \textit{exists for all } j \tag{4.5}$$

$$\sup_{m,j} \left\{ \left| \frac{1}{r} \frac{L_m}{L_{m-1}} a_{nm} \right| + \left| \frac{1}{r} \frac{L_j}{L_{j-1}} a_{nj} + \sum_{k=j+1}^{m} \eta_{kj} a_{nk} \right| \right\} < \infty, \tag{4.6}$$

$$\sup_j \sum_{n=1}^{\infty} \left|\tilde{b}_{nj}\right|^p < \infty. \tag{4.7}$$

Moreover, if $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, \left|\mathcal{L}^\phi(r,s)\right|_p\right)$, then A denotes a bounded linear operator L_A such that $L_A(x) = A(x)$ and

$$\|L_A\|_{\left(\left|\mathcal{L}^\phi(r,s)\right|, \left|\mathcal{L}^\phi(r,s)\right|_p\right)} = \left\|\tilde{B}\right\|_{(l,l_p)}.$$

Proof. $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|, \left|\mathcal{L}^\phi(r,s)\right|_p\right)$ if and only if $A_n = (a_{nv})_{v=0}^\infty \in \left\{\left|\mathcal{L}^\phi(r,s)\right|\right\}^\beta$ and $A(x) \in \left|\mathcal{L}^\phi(r,s)\right|_p$ where $x \in \left|\mathcal{L}^\phi(r,s)\right|$. By Theorem 3.5, it is clear that $A_n \in \left\{\left|\mathcal{L}^\phi(r,s)\right|\right\}^\beta$ iff (4.5) and (4.6) hold. Also, if any matrix $R = (r_{nv}) \in (l,c)$, then the series $R_n(x) = \sum_{v=0}^{\infty} r_{nv} x_v$ converges uniformly in n. Because, the remaining term of the series tends to zero uniformly in n, since

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \sup_v |r_{nv}| \sum_{v=m}^{\infty} |x_v| \to 0 \ \ (m \to \infty)$$

and so

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{4.8}$$

Considering the equation (4.8), it can be written

$$A_n(x) = \lim_m \sum_{k=0}^{m} a_{nk} x_k = \lim_m \sum_{r=0}^{m} b^{(n)}_{mr} z_r = \sum_{r=0}^{\infty} \bar{b}_{nr} z_r.$$

Since $\left|\mathcal{L}^{\phi}(r,s)\right| \cong l$, then, it is obtained $A(x) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ whenever $x \in \left|\mathcal{L}^{\phi}(r,s)\right|$ iff $\bar{B}(z) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ i.e., $\tilde{B}(z) = E^{(p)} \circ \mathcal{L}(r,s) \circ \bar{B}(z) \in l_p$ for all $z \in l$, where $z = E^{(p)} \circ \mathcal{L}(r,s)(x)$, or, equivalently, $\tilde{B} \in (l, l_p)$. So, if we apply Lemma 1.5 to the matrix $\tilde{B}$, the last condition is immediately obtained, which completes the first part of the proof.

Since the spaces $\left|\mathcal{L}^{\phi}(r,s)\right|_p$, $1 \le p < \infty$, are BK-space, by Theorem 4.2.8 of [32], L_A defines a bounded operator such that $L_A(x) = A(x)$.

Moreover, from Theorem 3.4, it can be seen that $A = \tilde{\mathcal{L}}(r,s) \circ \tilde{E}^{(p)} \circ \tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)$ and so,

$$\begin{aligned} \|L_A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|, \left|\mathcal{L}^{\phi}(r,s)\right|_p\right)} &= \sup_{x \neq 0} \frac{\|A(x)\|_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}}{\|x\|_{\left|\mathcal{L}^{\phi}(r,s)\right|}} = \sup_{x \neq 0} \frac{\left\|\tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)(x)\right\|_{l_p}}{\left\|E^{(p)} \circ \mathcal{L}(r,s)(x)\right\|_l} \\ &= \sup_{z \neq 0} \frac{\left\|\tilde{B}(z)\right\|_{l_p}}{\|z\|_l} = \left\|\tilde{B}\right\|_{(l,l_p)}, (z = E^{(1)} \circ \mathcal{L}(r,s)(x)). \end{aligned}$$

□

Theorem 4.3. *Let $1 \le p < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$ and $B = (b_{nv})$ be a matrix satisfying the following relation*

$$b_{nk} = \phi_n^{1/p^*} \sum_{v=0}^{n} \xi_{nv} a_{vk}. \tag{4.9}$$

Then, for any sequence spaces λ, $A \in \left(\lambda, \left|\mathcal{L}^{\phi}(r,s)\right|_p\right)$ if and only if $B \in (\lambda, l_p)$.

Proof. Take $x \in \lambda$. It follows from (4.9) that

$$\sum_{k=0}^{\infty} b_{nk} x_k = \phi_n^{1/p^*} \sum_{v=0}^{n} \xi_{nv} \sum_{k=0}^{\infty} a_{vk} x_k.$$

By (3.2), it is seen immediately that $B_n(x) = \left(E^{(p)} \circ \mathcal{L}(r,s)\right)_n (A(x))$ for all $x \in \lambda$. So, it is obtained that $A_n(x) \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ whenever $x \in \lambda$ if and only if $B(x) \in l_p$ whenever $x \in \lambda$, which completes the proof of the theorem. □

Theorem 4.4. *Let $1 \le p < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$. Then, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X\right)$ if and only if*

$$V^{(n)} \in (l_p, c) \text{ for all } n \in \mathbb{N},$$

$$\tilde{A}^{(p)} \in (l_p, X),$$

where the matrices $V^{(n)}$ and $\tilde{A}$ are defined as

$$\tilde{a}_{nk}^{(p)} = \phi_k^{-1/p^*} \left(\frac{1}{r} \frac{L_k}{L_{k-1}} a_{nk} + \sum_{j=k+1}^{\infty} a_{nj} \eta_{jk} \right)$$

and

$$v_{mk}^{(n)} = \begin{cases} \phi_k^{-1/p^*} \left(a_{nk} \frac{1}{r} \frac{L_k}{L_{k-1}} + \sum\limits_{j=k+1}^{m} a_{nj} \eta_{jk} \right), & 0 \le k \le m-1 \\ \phi_m^{-1/p^*} a_{nm} \frac{1}{r} \frac{L_m}{L_{m-1}}, & k = m \\ 0, & k > m. \end{cases}$$

Proof. Let $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X\right)$ and $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$. Note that $\left|\mathcal{L}^{\phi}(r,s)\right|_p = (l_p)_{E^{(p)} \circ \mathcal{L}(r,s)}$. Considering (3.4) and (3.5), we get

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} v_{mk}^{(n)} z_k \tag{4.10}$$

for all $n, m \in \mathbb{N}$. It can be seen immediately that Ax is well defined for all $x \in \left|\mathcal{L}^{\phi}(r,s)\right|_p$ iff $V^{(n)} \in (l_p, c)$. Further, letting $m \to \infty$, it is seen from (4.10) that $Ax = \tilde{A}^{(p)} z$. Since $Ax \in X$, then $\tilde{A}^{(p)} z \in X$, that is $\tilde{A} \in (l_p, X)$.

Conversely, let $V^{(n)} \in (l_p, c)$ and $\tilde{A}^{(p)} \in (l_p, X)$. Since $V^{(n)} \in (l_p, c)$ with (4.10), we get $A_n \in \left\{ \left|\mathcal{L}^{\phi}(r,s)\right|_p \right\}^{\beta}$, for all n, which gives that Ax exists. Besides, we deduced from $\tilde{A}^{(p)} \in (l_p, X)$ and (4.10) as $m \to \infty$, $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X \right)$.

□

Now, we list the following notations:

1. $\lim\limits_{n\to\infty} \tilde{a}_{nv}^{(p)}$ exists for all $v \in \mathbb{N}$
2. $\lim\limits_{n\to\infty} \tilde{a}_{nv}^{(p)} = 0$ for all $v \in \mathbb{N}$
3. $\sup\limits_{n} \sum\limits_{v=0}^{\infty} \left|\tilde{a}_{nv}^{(p)}\right|^{p^*} < \infty$
4. $\sup\limits_{n,v} \left|\tilde{a}_{nv}^{(p)}\right| < \infty$
5. $\sup\limits_{N} \sum\limits_{v} \left| \sum\limits_{n\in N} \tilde{a}_{nv}^{(p)} \right|^{p^*} < \infty$
6. $\sup\limits_{v} \sum\limits_{n} \left|\tilde{a}_{nv}^{(p)}\right| < \infty$
7. $\sup\limits_{m} \sum\limits_{v=0}^{\infty} \left|v_{mv}^{(n)}\right|^{p^*} < \infty$
8. $\sup\limits_{m,v} \left|v_{mv}^{(n)}\right| < \infty$
9. $\lim\limits_{m\to\infty} v_{mv}^{(n)}$ exists for all $v, n \in \mathbb{N}$

By Theorem 4.4, we obtain following results giving the necessary and sufficient conditions for $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|(\mu), X\right)$ with $X \in \{l_\infty, c_0, c, l, cs, bs\}$.

Theorem 4.5. *Let $1 < p < \infty$. The following statements hold:*

(i) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, c \right) \Leftrightarrow (1), (3), (7)$ *and* (9) *hold.*
(ii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, c_0 \right) \Leftrightarrow (2), (3), (7)$ *and* (9) *hold.*
(iii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l_\infty \right) \Leftrightarrow (3), (7)$ *and* (9) *hold.*
(iv) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l \right) \Leftrightarrow (5), (7)$ *and* (9) *hold.*
(v) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, c \right) \Leftrightarrow (1), (4), (8)$ *and* (9) *hold.*
(vi) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, c_0 \right) \Leftrightarrow (2), (4), (8)$ *and* (9) *hold.*
(vii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l_\infty \right) \Leftrightarrow (4), (8)$ *and* (9) *hold.*
(viii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l \right) \Leftrightarrow (6), (8)$ *and* (9) *hold.*

Corollary 4.1. *Put* $a(n,k) = \sum\limits_{j=0}^{n} a_{jk}$ *instead of* a_{nk} *for all* n, k *in* $V^{(n)} = (v_{mv}^{(n)})$ *and* $\tilde{A}^{(p)} = (\tilde{a}_{nv}^{(p)})$. *Then,*

(i) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, cs \right) \Leftrightarrow (1), (3), (7)$ *and* (9) *hold.*
(ii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, bs \right) \Leftrightarrow (3), (7)$ *and* (9) *hold.*
(iii) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, cs \right) \Leftrightarrow (1), (4), (8)$ *and* (9) *hold.*
(iv) $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, bs \right) \Leftrightarrow (4), (8)$ *and* (9) *hold.*

Theorem 4.6. (i) *Let $1<p<\infty$ and X be one of the sequence spaces c_0, c, l_∞.*

$$A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, X\right)\Rightarrow\|L_A\|=\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,l_\infty\right)}=\sup_n\left\|\tilde{A}_n^{(p)}\right\|_{l_{p^*}},$$

$$A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|, X\right)\Rightarrow\|L_A\|=\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|,l_\infty\right)}=\sup_n\left\|\tilde{A}_n^{(1)}\right\|_{l_\infty}.$$

(ii) *Let $1<p<\infty$. There exists $1\le\xi\le 4$ such that*

$$A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l\right)\Rightarrow\|L_A\|=\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p,l\right)}=\left\|\tilde{A}^{(p)}\right\|_{(l_p,l)}=\frac{1}{\xi}\left\|\tilde{A}^{(p)}\right\|'_{(l_p,l)},$$

$$A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|, l_p\right)\Rightarrow\|L_A\|=\|A\|_{\left(\left|\mathcal{L}^{\phi}(r,s)\right|,l_p\right)}=\left\|\tilde{A}_n^{(1)}\right\|_{(l,l_p)}.$$

Proof. The proof of the theorem is obtained together with Lemma 1.8, Lemma 1.3 and Lemma 1.5. □

5. Compact Operators on absolute Lucas series spaces

The aim of this section is to establish some identities or estimates for the Hausdorff measures of noncompactness of the matrix operators on the $\left|\mathcal{L}^{\phi}(r,s)\right|_p$ and also to characterize certain classes of compact operators by using the Hausdorff measure of noncompactness.

Theorem 5.1. *Under the hypothesis of Theorem 4.1, if $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, \left|\mathcal{L}^{\phi}(r,s)\right|\right)$, then*

$$\|L_A\|_\chi=\frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}}$$

and

$$L_A \text{ is compact iff } \lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}}=0.$$

Proof. To determine the Hausdorff measure of noncompactness of L_A, take $S_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}$ as a unique ball in the space $\left|\mathcal{L}^{\phi}(r,s)\right|_p$. By using Lemma 2.1, Lemma 2.3 and Lemma 1.3, it is obtained that

$$\begin{aligned}
\|A\|_\chi &= \chi\left(A\left(S_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}\right)\right)\\
&= \chi\left(E^{(1)}\circ\mathcal{L}(r,s)\circ A\left(S_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}\right)\right)\\
&= \chi\left(\hat{B}\circ E^{(p)}\circ\mathcal{L}(r,s)\left(S_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}\right)\right)\\
&= \lim_{v\to\infty}\left(\sup_{z\in E^{(p)}(\mathcal{L}(r,s)(S_{\left|\mathcal{L}^{\phi}(r,s)\right|_p}))}\left\|(I-P_v)\left(\hat{B}(z)\right)\right\|\right)\\
&= \frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\hat{b}_{nr}\right|\right)^{p^*}\right\}^{\frac{1}{p^*}}.
\end{aligned}$$

Finally, by using Lemma 2.2, the compact operators in this class can be immediately characterized. □

Theorem 5.2. *Under the hypothesis of Theorem 4.2, if $A\in\left(\left|\mathcal{L}^{\phi}(r,s)\right|, \left|\mathcal{L}^{\phi}(r,s)\right|_p\right)$, then*

$$\|A\|_\chi=\lim_{v\to\infty}\left\{\sup_j\sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}}$$

and

$$L_A \text{ is compact iff } \lim_{v\to\infty}\left\{\sup_j \sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}} = 0.$$

Proof. Let $S_{|\mathcal{L}^\phi(r,s)|}$ be a unit sphere in $\left|\mathcal{L}^\phi(r,s)\right|$. Since $E^{(p)} \circ \mathcal{L}(r,s) \circ AS_{|\mathcal{L}^\phi(r,s)|} = \tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)S_{|\mathcal{L}^\phi(r,s)|}$, it follows by Lemma 2.1, Lemma 2.3 and Lemma 1.5 that

$$\begin{aligned}
\|A\|_\chi &= \chi\left(AS_{|\mathcal{L}^\phi(r,s)|}\right) \\
&= \chi\left(E^{(p)} \circ \mathcal{L}(r,s) \circ AS_{|\mathcal{L}^\phi(r,s)|}\right) \\
&= \chi\left(\tilde{B} \circ E^{(1)} \circ \mathcal{L}(r,s)S_{|\mathcal{L}^\phi(r,s)|}\right) \\
&= \lim_{v\to\infty}\left(\sup_{z\in E^{(1)}\circ\mathcal{L}(r,s)\left(S_{|\mathcal{L}^\phi(r,s)|}\right)}\left\|(I-P_v)(\tilde{B}(z))\right\|_{l_p}\right) \\
&= \lim_{v\to\infty}\left\{\sup_j \sum_{n=v+1}^{\infty}\left|\tilde{b}_{nj}\right|^p\right\}^{\frac{1}{p}}.
\end{aligned}$$

Using Lemma 2.2, the last part of the proof is completed. □

Theorem 5.3. *Let $1 < p < \infty$. Then,*

(a) If $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, c_0\right)$, then

$$\|L_A\|_\chi = \lim_{j\to\infty}\sup_{n>j}\left\|\tilde{A}_n^{(p)}\right\|_{l_{p^*}} = \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*},$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\sup\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*} = 0.$$

(b) If $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, c\right)$, then

$$\frac{1}{2}\lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)} - \tilde{a}_k\right|^{p^*} \le \|L_A\|_\chi \le \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)} - \tilde{a}_k\right|^{p^*}$$

and

$$L_A \text{ is compact iff } \lim_{n\to\infty}\sup\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)} - \tilde{a}_k\right|^{p^*} = 0$$

where $\tilde{a} = (\tilde{a}_k)$ is defined by $\tilde{a}_k = \lim_{n\to\infty}\tilde{a}_{nk}$, for all $n \in \mathbb{N}$.

(c) If $A \in \left(\left|\mathcal{L}^\phi(r,s)\right|_p, l_\infty\right)$, then

$$0 \le \|L_A\|_\chi \le \lim_{j\to\infty}\sup_{n>j}\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*},$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty}\sup\sum_{k=1}^{\infty}\left|\tilde{a}_{nk}^{(p)}\right|^{p^*} = 0.$$

(*d*) *If* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|, l_p\right)$, $1 \leq p < \infty$, *then*

$$\|L_A\|_{\chi} = \lim_{j\to\infty}\left(\sup_v\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(1)}\right|^p\right)^{1/p}\right),$$

and

$$L_A \text{ is compact iff } \lim_{j\to\infty}\sup_v\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(1)}\right|^p = 0.$$

(*e*) *If* $A \in \left(\left|\mathcal{L}^{\phi}(r,s)\right|_p, l\right)$, $1 < p < \infty$, *then there exists* $1 \leq \xi \leq 4$ *such that*

$$\|L_A\|_{\chi} = \frac{1}{\xi}\lim_{j\to\infty}\left(\sum_{v=1}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(p)}\right|\right)^{p^*}\right)^{1/p},$$

and

$$L_A \text{ is compact iff } \lim_{j\to\infty}\sum_{v=1}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{a}_{nv}^{(p)}\right|\right)^{p^*} = 0.$$

Proof. The proof of the theorem can be obtained by combining Lemma 4.2 and Lemma 2.5, so it has been left to reader. □

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Altay,B., Başar, F. and Mursaleen, M.: *On the Euler sequence spaces which include the spaces l_p and l_∞ I*. Inform. Sci. **176** (10), 1450-1462, (2005)

[2] Başarır, M., Başar, F. and Kara, E. E.: *On the spaces of Fibonacci difference absolutely p-summable, null and convergent sequences*. Sarajevo J. Math. **12** (25), 167-182,(2016)

[3] Bor, H.: *On summability factors of infinite series*, Tamkang J. Math. **16** (1), 13-20, (1985)

[4] Djolovic, I. and Malkowsky, E.: *Matrix transformations and compact operators on some new mth-order difference sequences*. Appl. Math. Comput. **198** (2), 700-714, (2008)

[5] FLett, T.M.: *On an extension of absolute summability and some theorems of Littlewood and Paley*. Proc. Lond. Math. Soc. **7**, 113-141, (1957)

[6] Gökçe, F. and Sarıgöl, M.A.: *On absolute Euler spaces and related matrix operators*. Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. *90*(5), 769-775, (2020)

[7] Gökçe, F. and Sarıgöl, M.A.: *Some matrix and compact operators of the absolute Fibonacci series spaces*. Kragujevac J. Math. **44** (2), 273–286, (2020)

[8] Hazar Güleç, G. C. : *Characterization of some classes of compact and matrix operators on the sequence spaces of Cesaro matrices*. Operators and Matrices, **13** (3), 809-822, (2019)

[9] Jarrah, A.M. and Malkowsky, E.: *Ordinary absolute and strong summability and matrix transformations*. Filomat, **17**, 59-78, (2003)

[10] Kara, E. E. and Ilkhan, M.: *Some properties of generalized Fibonacci sequence spaces*. Linear Multilinear Algebra, **64** (11), 2208-2223, (2016)

[11] Kara, E.E.: *Some topological and geometrical properties of new Banach sequence spaces*. J. Inequal. Appl. **2013** (1), 38, (2013)

[12] Karakaş, M. and Karakaş, A.M.: *A study on Lucas difference sequence spaces $l_p(\hat{E}(r,s))$ and $l_\infty(\hat{E}(r,s))$*. Maejo International Journal of Science and Technology, **12**, 70-78, (2018)

[13] Koshy, T.: *Fibonacci and Lucas numbers with applications* (Vol. 51). John Wiley and Sons, (2011)

[14] Maddox, I.J.: *Elements of functinal analysis*, Cambridge University Press, London, New York, (1970)

[15] Malkowsky, E.: *Compact matrix operators between some $BK-$ spaces, in: M. Mursaleen (Ed.)*. Modern Methods of Analysis and Its Applications, Anamaya Publ., New Delhi, (2010), 86-120.

[16] Malkowsky, E. and Rakocevic, V.: *On matrix domains of triangles*. Appl. Math. Comput., **189** (2), 1146-1163, (2007)

[17] Malkowsky E. and Rakocevic V.: *Measure of noncompactness of Linear operators between spaces of sequences that are $(\bar{N},q)$ summable or bounded*. Czechoslovak Math. J. ,**51**(3):505-522, (2001)

[18] Malkowsky, E. and Rakocevic, V.: *An introduction into the theory of sequence space and measures of noncompactness*. Zb. Rad.(Beogr), **9** (17), 143-234, (2000)

[19] Mohapatra, R.N. and Sarıgöl, M.A.: *On matrix operators on the series spaces $\left|\bar{N}_p^\theta\right|_k$*. Ukrain. Mat. Zh., **69** (11), 1524-1533, (2017)

[20] Mursaleen, M.: *Applied Summability Methods*. Springer, Heidelberg, (2013)

[21] Mursaleen M., Noman A.K.: *Compactness of matrix operators on some new difference sequence spaces*. Linear Algebra App. 436: 41-52, (2012)

[22] Mursaleen, M. and Noman, A.K.: *Compactness by the Hausdorff measure of noncompactness*. Nonlinear Anal. **73** (8), 2541-2557, (2010)

[23] Mursaleen, M., Başar, F. and Altay, B.: *On the Euler sequence spaces which include the spaces l_p and l_∞ II*. Nonlinear Anal. **65** (3), 707–717, (2006)

[24] Rakocevic, V.: *Measures of noncompactness and some applications*. Filomat. **12** (2), 87-120, (1998)

[25] Sarıgöl, M.A.: *Spaces of Series Summable by Absolute Cesàro and Matrix Operators*. Comm. Math Appl. **7** (1), 11-22, (2016)

[26] Sarıgöl M.A. : *Norms and compactness of operators on absolute weighted mean summable series*. Kuwait J. Sci. **43**, 68-74, (2016)

[27] Sarıgöl, M.A.: *Extension of Mazhar's theorem on summability factors*. Kuwait J. Sci. **42** (3), 28-35, (2015)

[28] Sarıgöl, M.A.: *Matrix transformations on fields of absolute weighted mean summability*. Studia Sci. Math. Hungar. **48** (3), 331-341, (2011)

[29] Sarıgöl, M.A.: *On the local properties of factored Fourier series*. Appl. Math. Comput. **216** (11), 3386-3390, (2010)

[30] Stieglitz, M. and Tietz, H.: *Matrix transformationen von Folgenraumen. Eine Ergebnisübersicht*. Math. Z. **154** (1), 1-16, (1977)

[31] Sulaiman, W.T.: *On summability factors of infinite series*. Proc. Amer. Math. Soc. **115**, 313- 317, (1992)

[32] Wilansky, A.: *Summability Through Functional Analysis*, Mathematics Studies. 85. North Holland, Amsterdam, (1984)

Affiliations

FADIME GÖKÇE
ADDRESS: Pamukkale University, Dept. of Mathematics , Denizli, Turkey.
E-MAIL: fgokce@pau.edu.tr
ORCID ID:0000-0003-1819-3317