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Research Article

Spectral properties of some differential operators of Sturm-Liouville type with homogeneous delay

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Abstract

In this paper we observe the operator $D^2 = D^2(h, H, q, \alpha)$, $h, H \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, $q(x) \in L_2\,[0, \pi]$, $\alpha \in (0, 1)$ and construct and partially transform its characteristic function. Those transformations enable more complete asymptotic decomposition of the zeroes and eigenvalues of the operator.
The goal of this paper is to contribute to the development of the spectral theory of differential operators with homogeneous delay.

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1. Introduction

The spectral theory is the part of mathematical analysis that studies sets of eigenvalues – spectra, and eigenvectors of linear operators that are defined on infinitely dimensional functional spaces. It is very efficient in solving wide classes of problems in mathematics, physics, electronics, and other natural sciences. Results presented in [1] and [2] are the most fundamental ones in this area. The most intensive development of this mathematical discipline today represents inverse spectral problems that deal with the construction of the linear operator from known spectral characteristics, which can be found in many scientific papers. Many papers are devoted to solving direct and inverse spectral problems with different types of delay. Some results for the constant delay can be found in [3, 4, 6, 8–12]. Also, the operators with delay and advance are considered in [5]. This paper deals with a homogeneous delay. Some results regarding this type of delay can be found in [7].

In this paper we solve the direct problem and give precise asymptotic behavior of the operator $D^2 = D^2(h, H, q, \alpha)\quad h, H \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}, q(x) \in L_2\,[0, \pi]\,, \alpha \in (0, 1)$, in special case when $h = \infty, H = 0$ for the operator D^2_{01}, and $h = 0, H = \infty$ for the operator D^2_{10}.

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2. Definition of operators and characteristic functions

For operators D_{01}^2 and D_{10}^2 defined by the differential equation

$$-y''(x)+q(x)y(\alpha x)=\lambda y(x)=z^2y(x),\quad q(x)\in L_2\,[0,\pi]\,,\quad \alpha\in(0,1) \tag{2.1}$$

and with boundary conditions

$$y(0)=y'(\pi)=0$$
$$y'(0)=y(\pi)=0$$

we construct the characteristic functions and use some additional transformations of integrals that enable us to introduce auxiliary functions and more precisely decompose the zeros of characteristic function and eigenvalues of the operators.

2.1. The construction of characteristic functions

Applying the method of variation of parameters and method of successive approximations on given boundary problems, we obtain the characteristic functions of observed operators.

The equation (2.1) with the boundary conditions $y(0)=0$ and $y'(0)=0$ is equivalent to the integral equations of the operator D_{01}^2,

$$y(x,z)=\sin zx+\frac{1}{z}\int_0^x q(t_1)\sin z(x-t_1)y(\alpha t_1,z)dt_1, \tag{2.2}$$

and operator D_{10}^2

$$y(x,z)=\cos zx+\frac{1}{z}\int_0^x q(t_1)\sin z(x-t_1)y(\alpha t_1,z)dt_1. \tag{2.3}$$

Introducing functions

$$\begin{aligned}
a_{s^2}(x,z)&=\int_0^x q(t_1)\sin z(x-t_1)\sin z\alpha t_1dt_1,\\
a_{c^2}(x,z)&=\int_0^x q(t_1)\cos z(x-t_1)\cos z\alpha t_1dt_1,\\
a_{cs}(x,z)&=\int_0^x q(t_1)\cos z(x-t_1)\sin z\alpha t_1dt_1,\\
a_{sc}(x,z)&=\int_0^x q(t_1)\sin z(x-t_1)\cos z\alpha t_1dt_1,\\
a_{cs^k}(x,z)&=\int_0^x q(t_1)\cos z(x-t_1)\cdot a_{s^k}(\alpha t_1,z)dt_1,\ \ k=2,3,\ldots,\\
a_{s^kc}(x,z)&=\int_0^x q(t_1)\sin z(x-t_1)\cdot a_{s^{k-1}c}(\alpha t_1,z)dt_1,\ \ k=2,3,\ldots.
\end{aligned} \tag{2.4}$$

and applying the method of successive approximations we obtain the solutions of integral equations (2.2) and (2.3) in the form

$$y(x,z)=\sin zx+\frac{1}{z}a_{s^2}(x,z)+\frac{1}{z^2}a_{s^3}(x,z)+\frac{1}{z^3}a_{s^4}(x,z)+\sum_{k=4}^{\infty}\frac{1}{z^k}a_{s^{k+1}}(x,z) \tag{2.5}$$

and

$$y(x,z)=\cos zx+\frac{1}{z}a_{sc}(x,z)+\frac{1}{z^2}a_{s^2c}(x,z)+\frac{1}{z^3}a_{s^3c}(x,z)+\sum_{k=4}^{\infty}\frac{1}{z^k}a_{s^kc}(x,z). \tag{2.6}$$

Using conditions $y'(\pi)=0$, and $y(\pi)=0$, from (2.5) and (2.6) we obtain the original form of characteristic functions $F_j(z),\ j=1,2$ of operators D_{01}^2 and D_{10}^2

$$F_1(z)=z\cos\pi z+a_{cs}(z)+\frac{1}{z}a_{cs^2}(z)+\frac{1}{z^2}a_{cs^3}(z)+\sum_{k=4}^{\infty}\frac{1}{z^{k-1}}a_{cs^k}(z), \tag{2.7}$$

$$F_2(z) = \cos \pi z + \frac{1}{z} a_{sc}(z) + \frac{1}{z^2} a_{s^2c}(z) + \frac{1}{z^3} a_{s^3c}(z) + \sum_{k=4}^{\infty} \frac{1}{z^k} a_{s^kc}(z), \tag{2.8}$$

where $a_{s^2}(\pi, z) = a_{s^2}(z)$, $a_{c^2}(\pi, z) = a_{c^2}(z)$, $a_{s^kc}(\pi, z) = a_{s^kc}(z)$, $a_{cs^k}(\pi, z) = a_{cs^k}(z)$.

The obtained form of characteristic functions given by (2.7) and (2.8) is then transformed by the introduction of so-called auxiliary functions, that have a fundamental role in solving the inverse problem. Namely, these auxiliary functions will show to carry potential q and coefficient of delay α.

Using trigonometric product formulas in (2.4) with $x = \pi$ and using the appropriate substitution we can form auxiliary functions

$$\tilde{q}_j(\theta) = \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right] \\ \frac{1}{1+\alpha} q\left(\frac{2\theta}{1+\alpha}\right), & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1+\alpha}{2}\pi\right] \ j = 1, 2 \\ \frac{1}{1+\alpha} q\left(\frac{2\theta}{1+\alpha}\right) + (-1)^j \frac{1}{1-\alpha} q\left(\frac{2\theta}{1-\alpha}\right), & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right], \end{cases}$$

and get

$$\begin{aligned} a_{s^2}(z) &= \int_0^\pi \tilde{q}_1(\theta) \cos z(\pi - 2\theta) d\theta = \tilde{a}^{(1)}(z), \\ a_{c^2}(z) &= \int_0^\pi \tilde{q}_2(\theta) \cos z(\pi - 2\theta) d\theta = \tilde{a}^{(2)}(z), \\ a_{sc}(z) &= \int_0^\pi \tilde{q}_2(\theta) \sin z(\pi - 2\theta) d\theta = \tilde{b}^{(2)}(z), \\ a_{cs}(z) &= -\int_0^\pi \tilde{q}_1(\theta) \sin z(\pi - 2\theta) d\theta = -\tilde{b}^{(1)}(z). \end{aligned}$$

Furthermore, the additional transformations of multiple integrals are made and auxiliary functions are introduced. This enables a more complete asymptotic decomposition of the zeros of the characteristic function, and eigenvalues of the operator.

Let us now introduce functions

$$\tilde{Q}_{21}(\theta, q(\theta)) = \frac{1}{2}\left(\tilde{Q}_2^{(1)}(\theta, q(\theta)) + \tilde{Q}_2^{(2)}(\theta, q(\theta)) - \tilde{Q}_2^{(3)}(\theta, q(\theta)) - \tilde{Q}_2^{(4)}(\theta, q(\theta))\right),$$

and

$$\tilde{Q}_{22}(\theta, q(\theta)) = \frac{1}{2}\left(\tilde{Q}_2^{(1)}(\theta, q(\theta)) - \tilde{Q}_2^{(2)}(\theta, q(\theta)) + \tilde{Q}_2^{(3)}(\theta, q(\theta)) - \tilde{Q}_2^{(4)}(\theta, q(\theta))\right),$$

for $j = 1, 2$, where

$$\tilde{Q}_2^{(1)}(\theta, q(\theta)) = \begin{cases} 0, & \theta \in \left(\frac{1+\alpha^2}{2}\pi, \pi\right] \\ \int_{\frac{2\theta}{1+\alpha^2}}^{\pi} q(t_1) q\left(\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1\right) \frac{dt_1}{1+\alpha}, & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1+\alpha^2}{2}\pi\right] \\ \int_{\frac{2\theta}{1+\alpha^2}}^{\frac{2\theta}{1-\alpha}} q(t_1) q\left(\frac{2\theta}{1+\alpha} - \frac{1-\alpha}{1+\alpha} t_1\right) \frac{dt_1}{1+\alpha}, & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right] \end{cases},$$

$$\tilde{Q}_2^{(2)}(\theta, q(\theta)) = \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right] \\ \int_{\frac{2\theta}{1+\alpha}}^{\pi} q(t_1) q\left(t_1 - \frac{2\theta}{1+\alpha}\right) \frac{dt_1}{1+\alpha}, & \theta \in \left(\frac{1-\alpha^2}{2}\pi, \frac{1+\alpha}{2}\pi\right] \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{2\theta}{1-\alpha^2}} q(t_1) q\left(t_1 - \frac{2\theta}{1+\alpha}\right) \frac{dt_1}{1+\alpha}, & \theta \in \left[0, \frac{1-\alpha^2}{2}\pi\right] \end{cases},$$

$$\tilde{Q}_2^{(3)}(\theta, q(\theta)) = \begin{cases} 0, & \theta \in \left(\frac{1-\alpha^2}{2}\pi, \pi\right] \\ \int_{\frac{2\theta}{1-\alpha^2}}^{\pi} q(t_1) q\left(\frac{2\theta}{1-\alpha} - t_1\right) \frac{dt_1}{1-\alpha}, & \theta \in \left(\frac{1-\alpha}{2}\pi, \frac{1-\alpha^2}{2}\pi\right] \\ \int_{\frac{2\theta}{1-\alpha^2}}^{\frac{2\theta}{1-\alpha}} q(t_1) q\left(\frac{2\theta}{1-\alpha} - t_1\right) \frac{dt_1}{1-\alpha}, & \theta \in \left[0, \frac{1-\alpha}{2}\pi\right] \end{cases},$$

$$\tilde{Q}_2^{(4)}(\theta, q(\theta)) = \begin{cases} 0, & \theta \in \left(\frac{1+\alpha}{2}\pi, \pi\right] \\ \int_{\frac{2\theta}{1+\alpha}}^{\pi} q(t_1) q\left(\frac{1+\alpha}{1-\alpha} t_1 - \frac{2\theta}{1-\alpha}\right) \frac{dt_1}{1-\alpha}, & \theta \in \left(\frac{1+\alpha^2}{2}\pi, \frac{1+\alpha}{2}\pi\right] \\ \int_{\frac{2\theta}{1+\alpha}}^{\frac{2\theta}{1+\alpha^2}} q(t_1) q\left(\frac{1+\alpha}{1-\alpha} t_1 - \frac{2\theta}{1-\alpha}\right) \frac{dt_1}{1-\alpha}, & \theta \in \left[0, \frac{1+\alpha^2}{2}\pi\right] \end{cases}.$$

Now, we can write

$$a_{cs^2}(z) = \int_0^{\pi} \tilde{Q}_{21}(\theta, q(\theta)) \cos z(\pi - 2\theta) d\theta = a^{(2,1)}(z),$$
$$a_{s^2c}(z) = -\int_0^{\pi} \tilde{Q}_{22}(\theta, q(\theta)) \cos z(\pi - 2\theta) d\theta = -a^{(2,2)}(z).$$

Similarly,

$$a_{cs^3}(z) = -\int_0^{\pi} \tilde{Q}_{31}(\theta, q(\theta)) \sin z(\pi - 2\theta) d\theta = -b^{(3,1)}(z),$$
$$a_{s^3c}(z) = \int_0^{\pi} \tilde{Q}_{32}(\theta, q(\theta)) \sin z(\pi - 2\theta) d\theta = b^{(3,2)}(z),$$

where

$$\tilde{Q}_{31}(\theta, q(\theta)) = \frac{1}{4}\Big(- \tilde{Q}_3^{(1)}(\theta, q(\theta)) - \tilde{Q}_3^{(2)}(\theta, q(\theta)) + \tilde{Q}_3^{(3)}(\theta, q(\theta)) + \tilde{Q}_3^{(4)}(\theta, q(\theta))$$
$$- \tilde{Q}_3^{(5)}(\theta, q(\theta)) - \tilde{Q}_3^{(6)}(\theta, q(\theta)) + \tilde{Q}_3^{(7)}(\theta, q(\theta)) + \tilde{Q}_3^{(8)}(\theta, q(\theta))\Big),$$

$$\tilde{Q}_{32}(\theta, q(\theta)) = \frac{1}{4}\Big(\tilde{Q}_3^{(1)}(\theta, q(\theta)) - \tilde{Q}_3^{(2)}(\theta, q(\theta)) + \tilde{Q}_3^{(3)}(\theta, q(\theta)) - \tilde{Q}_3^{(4)}(\theta, q(\theta))$$
$$+ \tilde{Q}_3^{(5)}(\theta, q(\theta)) - \tilde{Q}_3^{(6)}(\theta, q(\theta)) + \tilde{Q}_3^{(7)}(\theta, q(\theta)) - \tilde{Q}_3^{(8)}(\theta, q(\theta))\Big).$$

Here

$$\tilde{Q}_3^{(7)}(\theta, q(\theta)) =$$

$$= \begin{cases} 0, & \theta \in \left[\frac{1+\alpha^2}{2}\pi, \pi\right] \\ \left.\begin{array}{l} \int_{\frac{2\theta}{1+\alpha^2}}^{\pi} \frac{1}{1-\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha}-\frac{1-\alpha}{1+\alpha}t_1}^{\alpha t_1} q(t_2) q\Big(t_1 + \frac{1+\alpha}{1-\alpha}t_2 - \\ \quad - \frac{2\theta}{1-\alpha}\Big) dt_2 dt_1, \end{array}\right\} & \theta \in \left[\frac{1+\alpha^3}{2}\pi, \frac{1+\alpha^2}{2}\pi\right] \\ \left.\begin{array}{l} \int_{\frac{2\theta}{1+\alpha^3}}^{\pi} \frac{1}{1-\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha}-\frac{1-\alpha}{1+\alpha}t_1}^{\frac{2\theta}{1+\alpha^2}-\frac{1-\alpha}{1+\alpha^2}t_1} q(t_2) q\Big(t_1 + \frac{1+\alpha}{1-\alpha}t_2 - \\ - \frac{2\theta}{1-\alpha}\Big) dt_2 dt_1 + \\ + \int_{\frac{2\theta}{1+\alpha^2}}^{\frac{2\theta}{1+\alpha^3}} \frac{1}{1-\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha}-\frac{1-\alpha}{1+\alpha}t_1}^{\alpha t_1} q(t_2) q\Big(t_1 + \frac{1+\alpha}{1-\alpha}t_2 - \\ - \frac{2\theta}{1-\alpha}\Big) dt_2 dt_1, \end{array}\right\} & \theta \in \left[\frac{1-\alpha}{2}\pi, \frac{1+\alpha^3}{2}\pi\right] \\ \left.\begin{array}{l} \int_{\frac{2\theta}{1+\alpha^3}}^{\frac{2\theta}{1-\alpha}} \frac{1}{1-\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha}-\frac{1-\alpha}{1+\alpha}t_1}^{\frac{2\theta}{1+\alpha^2}-\frac{1-\alpha}{1+\alpha^2}t_1} q(t_2) q\Big(t_1 + \frac{1+\alpha}{1-\alpha}t_2 - \\ - \frac{2\theta}{1-\alpha}\Big) dt_2 dt_1 + \\ + \int_{\frac{2\theta}{1+\alpha^2}}^{\frac{2\theta}{1+\alpha^3}} \frac{1}{1-\alpha} q(t_1) \int_{\frac{2\theta}{1+\alpha}-\frac{1-\alpha}{1+\alpha}t_1}^{\alpha t_1} q(t_2) q(t_1 + \frac{1+\alpha}{1-\alpha}t_2 - \\ - \frac{2\theta}{1-\alpha}\Big) dt_2 dt_1, \end{array}\right\} & \theta \in \left(0, \frac{1-\alpha}{2}\pi\right] \end{cases}$$

and other auxiliary functions can be obtained analogously.

Using obtained transformations we get the asymptotic decomposition of characteristic functions

$$F_1(z) = z\cos\pi z - \tilde{b}^{(1)}(z) + \frac{1}{z}a^{(2,1)}(z) - \frac{1}{z^2}b^{(3,1)}(z) + O\Big(\frac{a^{(3,1)}(z)}{z^3}\Big), \tag{2.9}$$

$$F_2(z) = \cos\pi z + \frac{1}{z}\tilde{b}^{(2)}(z) - \frac{1}{z^2}a^{(2,2)}(z) + \frac{1}{z^3}b^{(3,2)}(z) + O\Big(\frac{a^{(3,2)}(z)}{z^4}\Big). \tag{2.10}$$

3. Asymptotics of zeros of the characteristic functions and eigenvalues of the operators

Now we will find the asymptotic of zeros of the characteristic function, as well as eigenvalues of the operator.

We search the asymptotic of zeros of $F_j(z),\ \ j = 1, 2$ in the form

$$\begin{aligned} z_{n_1 j} = n_1 + \frac{C_{1j}(n_1)}{n_1} + \frac{C_{2j}(n_1)}{n_1^2} + \frac{C_{3j}(n_1)}{n_1^3} + \\ + o\Big(\frac{C_{3j}(n_1)}{n_1^3}\Big),\ \ n_1 \longrightarrow \infty, n_1 = n + \frac{1}{2}, j = 1, 2 \end{aligned} \tag{3.1}$$

From the asymptotic expansions

$$z_{n_1} \cdot \cos z_{n_1} = (-1)^{n+1}\Big(\pi C_1(n_1) + \frac{\pi C_2(n_1)}{n_1} + \frac{\pi C_3(n_1) - \frac{1}{6}\pi^3 C_1^3(n_1)}{n_1^2} + o\Big(\frac{C_1(n_1)}{n_1^2}\Big)\Big),$$

$$\cos z_{n_1} = (-1)^{n+1}\Big(\frac{\pi C_1(n_1)}{n_1} + \frac{\pi C_2(n_1)}{n_1^2} + \frac{\pi C_3(n_1) - \frac{1}{6}\pi^3 C_1^3(n_1)}{n_1^3} + o\Big(\frac{C_1(n_1)}{n_1^3}\Big)\Big)$$

$$
\begin{aligned}
-\tilde{b}^{(1)}(z_{n_1}) &= (-1)^{n+1}\int_0^\pi \tilde{q}_1(\theta)\cos 2n_1\theta d\theta \\
&+(-1)^n\frac{\pi^2C_1^2(n_1)}{2n_1^2}\int_0^\pi \tilde{q}_1(\theta)\cos 2n_1\theta d\theta-(-1)^n\frac{2\pi C_1^2(n_1)}{n_1^2}\int_0^\pi \theta\tilde{q}_1(\theta)\cos 2n_1\theta d\theta \\
&+(-1)^n\frac{2C_1^2(n_1)}{n_1^2}\int_0^\pi \theta^2\tilde{q}_1(\theta)\cos 2n_1\theta d\theta-(-1)^n\frac{\pi C_1(n_1)}{n_1}\int_0^\pi \tilde{q}_1(\theta)\sin 2n_1\theta d\theta \\
&+(-1)^n\frac{2C_1(n_1)}{n_1}\int_0^\pi \theta\tilde{q}_1(\theta)\sin 2n_1\theta d\theta-(-1)^n\frac{\pi C_2(n_1)}{n_1^2}\int_0^\pi \tilde{q}_1(\theta)\sin 2n_1\theta d\theta \\
&+(-1)^n\frac{2C_2(n_1)}{n_1^2}\int_0^\pi \theta\tilde{q}_1(\theta)\sin 2n_1\theta d\theta+O\Big(\frac{C_3(n_1)}{n_1^3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z_{n_1}}\tilde{b}^{(2)}(z_{n_1}) &= (-1)^n\frac{1}{n_1}\int_0^\pi \tilde{q}_2(\theta)\cos 2n_1\theta d\theta \\
&-(-1)^n\frac{\pi^2C_1^2(n_1)}{2n_1^3}\int_0^\pi \tilde{q}_2(\theta)\cos 2n_1\theta d\theta+(-1)^n\frac{2\pi C_1^2(n_1)}{n_1^3}\int_0^\pi \theta\tilde{q}_2(\theta)\cos 2n_1\theta d\theta \\
&-(-1)^n\frac{2C_1^2(n_1)}{n_1^3}\int_0^\pi \theta^2\tilde{q}_2(\theta)\cos 2n_1\theta d\theta+(-1)^n\frac{\pi C_1(n_1)}{n_1^2}\int_0^\pi \tilde{q}_2(\theta)\sin 2n_1\theta d\theta \\
&-(-1)^n\frac{2C_1(n_1)}{n_1^2}\int_0^\pi \theta\tilde{q}_2(\theta)\sin 2n_1\theta d\theta+(-1)^n\frac{\pi C_2(n_1)}{n_1^3}\int_0^\pi \tilde{q}_2(\theta)\sin 2n_1\theta d\theta \\
&-(-1)^n\frac{2C_2(n_1)}{n_1^3}\int_0^\pi \theta\tilde{q}_2(\theta)\sin 2n_1\theta d\theta+o\Big(\frac{C_3(n_1)}{n_1^3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z_{n_1}}a^{(2,1)}(z_{n_1}) &= (-1)^n\frac{1}{n_1}\int_0^\pi \tilde{Q}_{21}(\theta,q(\theta))\sin 2n_1\theta d\theta \\
&-(-1)^n\frac{\pi C_1(n_1)}{n_1^2}\int_0^\pi \tilde{Q}_{21}(\theta,q(\theta))\cos 2n_1\theta d\theta \\
&+(-1)^n\frac{2C_1(n_1)}{n_1^2}\int_0^\pi \theta\tilde{Q}_{21}(\theta,q(\theta))\cos 2n_1\theta d\theta+O\Big(\frac{C_2(n_1)}{n_1^3}\Big),
\end{aligned}
$$

$$
\begin{aligned}
-\frac{1}{z_{n_1}^2}a^{(2,2)}(z_{n_1}) &= (-1)^{n+1}\frac{1}{n_1^2}\int_0^\pi \tilde{Q}_{22}(\theta,q(\theta))\sin 2n_1\theta d\theta \\
&+(-1)^n\frac{\pi C_1(n_1)}{n_1^3}\int_0^\pi \tilde{Q}_{22}(\theta,q(\theta))\cos 2n_1\theta d\theta \\
&-(-1)^n\frac{2C_1(n_1)}{n_1^3}\int_0^\pi \theta\tilde{Q}_{22}(\theta,q(\theta))\cos 2n_1\theta d\theta+o\Big(\frac{C_2(n_1)}{n_1^3}\Big), \\
-\frac{1}{z_{n_1}^2}b^{(3,1)}(z_{n_1}) &= (-1)^{n+1}\frac{1}{n_1^2}\int_0^\pi \tilde{Q}_{31}(\theta,q(\theta))\cos 2n_1\theta d\theta+o\Big(\frac{C_1(n_1)}{n_1^2}\Big), \\
\frac{1}{z_{n_1}^3}b^{(3,2)}(z_{n_1}) &= (-1)^{n}\frac{1}{n_1^3}\int_0^\pi \tilde{Q}_{32}(\theta,q(\theta))\cos 2n_1\theta d\theta+o\Big(\frac{C_1(n_1)}{n_1^3}\Big),
\end{aligned}
$$

and $F_j(z)=0$, $j=1,2$ we obtain constants

$$C_{1j}(n_1)=\frac{(-1)^j}{\pi}\tilde{a}_{2n_1}^{(j)}, \tag{3.2}$$

$$C_{2j}(n_1)=\frac{(-1)^{j-1}}{\pi}b_{2n_1}^{(2,j)}+\frac{1}{\pi}\tilde{a}_{2n_1}^{(j)}\cdot\tilde{b}_{2n_1}^{(j)}-\frac{2}{\pi^2}\tilde{a}_{2n_1}^{(j)}\cdot\hat{\tilde{b}}_{2n_1}^{(j)}, \tag{3.3}$$

$$C_{3j}(n_1)=O\Big(a_{2n_1}^{(3,j)}\Big). \tag{3.4}$$

Using (3.2), (3.3), and (3.4) we obtain from (3.1) the asymptotic (3.5). By squaring it, having in mind that $z^2 = \lambda$, we obtain the asymptotic of eigenvalues of the operator (3.6).

Herewith, we have obtained the main result of this paper.

Theorem 3.1. *If $q(x) \in L_2\,[0,\pi]$ then the zeroes of the functions* (2.9) *and* (2.10) *of the operators D^2_{01} and D^2_{10} have the following asymptotic decomposition:*

$$\begin{aligned} z_{n_1 j} &= n_1 + (-1)^j \frac{1}{n_1} \cdot \frac{1}{\pi} \tilde{a}^{(j)}_{2n_1} \\ &+ \frac{1}{n_1^2} \Big(\frac{(-1)^{j-1}}{\pi} b^{(2,j)}_{2n_1} + \frac{1}{\pi} \tilde{a}^{(j)}_{2n_1} \cdot \tilde{b}^{(j)}_{2n_1} - \frac{2}{\pi^2} \tilde{a}^{(j)}_{2n_1} \cdot \hat{\tilde{b}}^{(j)}_{2n_1} \Big) \\ &+ O\Big(\frac{a^{(3,j)}_{2n_1}}{n_1^3} \Big), \quad n_1 = n + \frac{1}{2}, n \in Z, \quad j = 1,2, \end{aligned} \tag{3.5}$$

and the eigenvalues of the operators have the asymptotic behavior

$$\begin{aligned} \lambda_{n_1 j} = z^2_{n_1 j} &= n_1^2 + (-1)^j \frac{2}{\pi} \tilde{a}^{(j)}_{2n_1} \\ &+ \frac{1}{n_1} \Big((-1)^{j-1} \frac{2}{\pi} b^{(2,j)}_{2n_1} + \frac{2}{\pi} \tilde{a}^{(j)}_{2n_1} \cdot \tilde{b}^{(j)}_{2n_1} - \frac{4}{\pi^2} \tilde{a}^{(j)}_{2n_1} \cdot \hat{\tilde{b}}^{(j)}_{2n_1} \Big) \\ &+ O\Big(\frac{a^{(3,j)}_{2n_1}}{n_1^2} \Big), \quad n_1 \longrightarrow \infty, n_1 = n + \frac{1}{2}, j = 1,2, \end{aligned} \tag{3.6}$$

where the given number sequences are

$$\begin{aligned} \tilde{a}^{(j)}_{2n_1} &= \int_0^\pi \tilde{q}_j(\theta) \cos 2n_1\theta d\theta, \\ a^{(3,j)}_{2n_1} &= \int_0^\pi \tilde{Q}_{3j}(\theta, q(\theta)) \cos 2n_1\theta d\theta, \\ \tilde{b}^{(j)}_{2n_1} &= \int_0^\pi \tilde{q}_j(\theta) \sin 2n_1\theta d\theta, \\ \hat{\tilde{b}}^{(j)}_{2n_1} &= \int_0^\pi \theta \tilde{q}_j(\theta) \sin 2n_1\theta d\theta, \\ b^{(2,j)}_{2n_1} &= \int_0^\pi \tilde{Q}_{2j}(\theta, q(\theta)) \sin 2n_1\theta d\theta, \quad j = 1,2. \end{aligned}$$

4. Conclusion

In this paper, we have constructed the characteristic functions of the operators D^2_{01} and D^2_{10} by applying additional transformations on multiple integrals and by introducing auxiliary functions. Herewith, more complete asymptotic behavior of zeros of the characteristic functions was enabled. The importance of these transformations is reflected in the fact that it was observed that a complete transformation could yield the general terms of the asymptotic decomposition of the zeros of the characteristic function. This represents a significant contribution to the theory of inverse spectral problems with homogeneous delay. This is also important in solving inverse problems and particularly in calculating regularized traces of their operators since they are obtained by using that decomposition. This leads to further exploration in that direction.

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