INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 14 NO. 1 PAGE 174–182 (2021)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.819887

# Some Myers-Type Theorems and Comparison Theorems for Manifolds with Modified Ricci Curvature

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

**ABSTRACT**

**In this paper, we establish some new compactness criteria for complete Riemannian manifolds with Bakry-Émery Ricci curvature bounded below. These results improve or generalize previous ones obtained by H. Tadano [6], J. Wan [7], I.A. Kaboye and M. Bazanfaré [3]. We also prove a volume comparison theorem for such manifolds.**

***Keywords:*** *Bakry-Émery Ricci curvature, Myers' theorem, weighted mean curvature, volume comparison theorem.*
***AMS Subject Classification (2020):*** *Primary 53C25; Secondary 53C20, 53C21.*

## 1. Introduction

Let $(M, g)$ be a complete Riemannian manifold and $V$ a smooth vector field on $M$. For $m > n$ a m-Bakry-Émery Ricci curvature corresponding to $V$ is defined by

$$Ric_{V,m} = Ric_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m-n} V^* \otimes V^*,$$

where $Ric_g$ , $\mathcal{L}_V$ and $V^*$ denote respectively the Ricci curvature of $(M, g)$, the Lie derivative and themetric-dual of $V$. The $\infty-$ Bakry-Émery Ricci curvature or simply the Bakry-Émery curvature is defined by

$$Ric_V = Ric_g + \frac{1}{2}\mathcal{L}_V g.$$

If $V$ is a gradient of a function $f$ i.e $V = \nabla f$, we note $Ric_{V,m} = Ric_{f,m}$ and $f$ is called a potential function.

When $V$ is zero then the Bakry-Émery Ricci tensor becomes the Ricci tensor so it is natural to investigate which geometric and topological results for the Ricci tensor extend to the Bakry-Émery Ricci tensor. In Riemannian Geometry, the Myers theorem and his extensions are some of the most important results in invetigating the relation between topology and geometry of Riemannian manifolds. In [7], J. Wan gave the following th eorem

**Theorem 1.1.** *Let $M$ be an $n$-dimensional complete Riemannian manifold. If there exists $p \in M, \alpha \geq 2$ and $r_0 > 0$ such that*

$$Ric_M(x) \geq \frac{C(n, \alpha, r_0)}{(r_0 + r)^\alpha} \tag{1.1}$$

*for all $r \geq 0$, where $d(p, x) = r$ and $C(n, \alpha, r_0)$ is a constant depending on $n, \alpha, r_0$, then $M$ is compact.*

In this paper we establish a compactness theorem similar to the result of J. Wan in the context of Bakry-Émery Ricci curvature.

***Received :*** *2–November–2020,* ***Accepted :*** *16–March–2021*
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**Theorem 1.2.** *Let $M$ be a n-dimensional complete Riemannian manifold which admits a vector field $V$ satisfying $\|V\| \le a$ for some $a \ge 0$. If there exists a point $p \in M$, $r_0 > 0$ and $\alpha > 1$ such that*

$$Ric_V(x) \ge \frac{C(n,\alpha,a)}{(1+r)^\alpha}$$

*for all $r \ge r_0$ where $r = d(p,x)$ and $C(n,\alpha,a,r_0) > 0$ is a constant depending on $n$, $\alpha$, $r_0$ and $a$, then $M$ is compact. If $C > (\alpha-1)(\frac{n-1}{r_0} + 4a)(1+r_0)^{\alpha-1}$ then*

$$Diam(M) \le 2(1+r_0)\left(\frac{C}{C-(\alpha-1)(\frac{n-1}{r_0}+4a)(1+r_0)^{\alpha-1}}\right)^{\frac{1}{\alpha-1}} - 2.$$

The Myers theorem was generalized in various directions ([4], [5],...) sometimes by considering the integral condition on Ricci curvature along geodesics (Ambrose ) or by perturbing the positive lower bound on the Ricci curvature. See [2] for example. In this paper, we prove the following theorem:

**Theorem 1.3.** *Let $M$ be a n-dimensional complete Riemannian manifold which admits a vector field $V$ satisfying $\|V\|(x) \le \delta d(p,x) + \alpha$ for some constants $\delta, \alpha \ge 0$. If*

$$Ric_V \ge \lambda + \frac{d}{ds}\phi \tag{1.2}$$

*and*

$$|\phi(t)| \le at + b \tag{1.3}$$

*for some $\lambda > 0$ $a \ge 0$, $b \ge 0$ such that $a + \delta < \frac{\lambda}{2}$ then $M$ is compact with diameter*

$$Diam(M) \le \frac{2(b+\alpha) + \sqrt{4(b+\alpha)^2 + (n-1)(\lambda - 2(a+\delta))\pi^2}}{\lambda - 2(a+\delta)}.$$

*Remark* 1.1. If $\delta = 0$ and $a = 0$ the theorem above is reduced to theorem 10 in [6].

Let $d\mu = e^{-\int_0^r \langle V, \nabla t\rangle dt} dv_g$ be a weighted measure on $M$. We assume that $N$ is an hypersurface of $M$. Let $\nu$ be the outward pointing unit normal vector to $N$. Let $II$ denote the second fundamental form of $N$ with respect to $\nu$. Then, the mean curvature of $N$ is $(n-1)H = traceII$. The weighted mean curvature of $N$ is defined by

$$(n-1)H_V = (n-1)H + \langle V, \nu\rangle. \tag{1.4}$$

In [3] the first and the third authors of this paper proved the following theorem:

**Theorem 1.4.** *Let $(M, g, e^{-f}dvol_g)$ be a metric espace such that $Ric_f \ge -(n-1)k^2$. Suppose that $M$ contains a ball $B(x_0, r)$ of center $x_0$ and radius $r$ such that the mean curavture of the geodesic sphere $S(x_0, r)$ with the inward pointing normal vector $m(r)$. If $|f| \le c$ and and $m(r) < -(n-1)k$ then $M$ is compact and*

$$diam(M) \le 2r + \frac{\ln(h_0 - k)/(h_0 + k)}{2k}$$

*where*

$$h_0 = \sup_{x \in S(p,r)} \frac{m(x)}{n-1}$$

In this paper we prove the following theorem:

**Theorem 1.5.** *Let $(M, g, d\mu)$ be a weighted measure espace. Assume that $Ric_V \ge (n-1)k$ $(k \le 0)$ and $N$ is a compact hypersurface of $M$ with weighted mean curvature with respect the outward pointing unit normal vector $H_V > h$.*

1. *if, for any geodesic $\gamma$ we have $\langle V, \dot\gamma\rangle \le (n-1)a$ and $h > \sqrt{-k} + a$ then $M$ is compact and*

$$Diam(M) < Diam(N) + \frac{1}{\sqrt{-k}} \ln\left(\frac{h + \sqrt{-k} - a}{h - \sqrt{-k} - a}\right). \tag{1.5}$$

2. *if there exists $b > 0$ such that , for any geodesic $\gamma$, we have $\left|\int_0^r \langle V, \nabla s\rangle ds\right| \leq (n-1)b$ and $h > \sqrt{-k}(1+2b)$ then $M$ is compact and*
$$Diam(M) < Diam(N) + \frac{1}{\sqrt{-k}} \ln\left(\frac{\sqrt{-k}(1+2b)+h}{h-\sqrt{-k}(1+2b)}\right). \tag{1.6}$$

In [8], G. Wei and W. Wylie proved a volume comparison result under the Bakry-Émery Ricci curvature condition. In this paper, we give stronger volume comparison result in the context of the modified Ricci curvature.

In the geodesic polar coordinates the volume element can be written as:
$$dvol_g = A(t,\theta)dt \wedge d\theta^{n-1}$$
where $d\theta^{n-1}$ is the standard volume element on the unit sphere $S^{n-1}$. Let
$$A_V(t,\theta) = e^{-\int_0^t g(V,\dot{\gamma})(s)ds} A(t,\theta)$$
where $\gamma$ is a geodesic such that $\dot{\gamma}(0) = \theta$. By the first variation of area we have:
$$\frac{d}{dt}\ln(A(t,\theta)) = \frac{A'(t,\theta)}{A(t,\theta)} = m(t) \tag{1.7}$$
$$\frac{d}{dt}\ln(A_V(t,\theta)) = \frac{A'_V(t,\theta)}{A_V(t,\theta)} = m_V(t) \tag{1.8}$$
Hence, for $r \geq r_0 > 0$, we have
$$\frac{A_V(r,\theta)}{A_V(r_0,\theta)} = e^{\int_{r_0}^r m_V(t)dt}$$
Let $M_k^n$ be the simply connected space with constant curvature $k$ and $m_k(r)$ be the mean curvature of the geodesic sphere of radius $r$ in the model space $M_k^n$. Set $Vol_V B(p,r) = \int_{B(p,r)} d\mu$ the weighted volume (or $V$-volume) of the ball of center $p$ and radius $r$ in $M$ and let $Vol_k^n(r)$ be the volume of the ball of radius $r$ in the model space $M_k^n$. Let $sn_k(r)$ be the solution of equation
$$y''(t) + ky(t) = 0$$
with initial conditions $y(0) = 0$ and $y'(0) = 1$. Then we have
$$m_k(r) = (n-1)\frac{sn'_k(r)}{sn_k(r)}. \tag{1.9}$$

**Theorem 1.6.** *Let $M$ be a n-dimensional complete Riemannian manifold and $V$ be a vector field on $M$. Suppose*
$$Ric_V \geq (n-1)k$$

1. *If $\langle V, \nabla r\rangle \geq -a$ along all minimal segments from $p$ then for $R \geq r > 0$*
$$\frac{Vol_V B(p,R)}{Vol_V B(p,r)} \leq e^{aR}\frac{Vol_k^n(R)}{Vol_k^n(r)}$$

2. *If, for all $r \geq 0$ we have*
$$\left|\int_0^r \langle V, \nabla t\rangle dt\right| \leq c$$
*then*
$$m_V(r) \leq m_k^{n+4c}(r)$$
*and for all $R \geq r > 0$(assume $R \leq \frac{\pi}{4\sqrt{k}}$ if $k > 0$),*
$$\frac{Vol_V(B(p,R))}{Vol_V(B(p,r))} \leq \frac{Vol_k^{n+4c}(R)}{Vol_k^{n+4c}(r)}$$
*where $m_k^{n+4c}(r)$ denotes the mean curvature of the geodesic sphere of radius $r$ in the model space $M_k^{n+4c}$.*

## 2. Proofs

### 2.1. Proof of the theorem 1.2

Assume that $M$ is noncompact. Then there exists a ray $\gamma$ issuing from $p$. Let $r(x) = d(p,x)$ be the distance function from $p$ to $x$. Then $r$ is smooth on $M \backslash \{C_p, p\}$, where $C_p$ is the cut locus of the point $p$. Let $u$ be a function on $M$. For all vectors fields $X, Y$ on $M$, the gradient, Hessian and laplacian of $u$ are defined as:

$$g(\nabla u, X) = du(X), \quad (Hess(u))(X,Y) = g(\nabla_X \nabla u, Y) \quad \Delta u = tr(Hess(u)).$$

In geodesic polar coordinates at $p$, we have $\nabla r = \partial r$ and $|\nabla r| = 1$ in $M \backslash \{C_p, p\}$.

Let $A(r)$ and $m(r)$ be respectively the second fundamental form and the mean curvature of the geodesic sphere $\mathcal{S}(r)$ of radius $r$ in $M$ with respect the outer normal direction. Write $A(t) = A(\gamma(t))$. Then we have:

$$A = Hess(r); \quad m(r) = \Delta r. \tag{2.1}$$

The Bochner-Weitzenböck formula applied to the function $r(x)$ becomes:

$$|Hess(r)|^2 + \langle \nabla \Delta r, \nabla r \rangle + Ric(\nabla r, \nabla r) = 0. \tag{2.2}$$

Set $m_V(r) = m(r) - \langle V, \nabla r \rangle$. By the relation $Ric_V = Ric_g + \frac{1}{2}\mathcal{L}_V g$ the equation (2.2) gives:

$$|Hess(r)|^2 + \frac{\partial}{\partial r}(\Delta r) + Ric(\nabla r, \nabla r) = 0. \tag{2.3}$$

Noticing that

$$m'_V(r) = m'(r) - \langle \nabla_{\partial r} V, \nabla r \rangle = m'(r) - \frac{1}{2}\mathcal{L}_V g(\partial r, \partial r).$$

From (2.3) we have

$$|Hess(r)|^2 = -m'_V(r) - Ric_V(\nabla r, \nabla r). \tag{2.4}$$

Integrating (2.4) over the interval $[r_0, r]$, we get

$$0 \leq \int_{r_0}^{r} \|A(t)\|^2 dt = -\int_{r_0}^{r} m'_V(t) - \int_{r_0}^{r} Ric_V dt \leq m_V(r_0) - m_V(r) - \int_{r_0}^{r} \frac{C}{(1+t)^\alpha} dt. \tag{2.5}$$

In [9], J.Y. Wu proved the following result:

**Lemma 2.1.** *(Theorem 2.2) Let (M, g) be an n-dimensional complete Riemannian manifold. Assume that (M, g) admits a smooth vector field V satisfying*

$$Ric_V(\partial r, \partial r) \geq (n-1)k, \quad k \in \mathbb{R},$$

*along a minimal geodesic segment from a fixed point p and $|V| \leq a$ for some real constant $a \geq 0$ (when $k > 0$ assume $r \leq \frac{\pi}{2\sqrt{k}}$). Then, $m_V(r) - m_k(r) \leq a$ along that minimal geodesic segment from $p$. Equality holds if and only if the radial sectional curvatures are equal to $k$ and $V = -a\nabla r$.*

We have

$$m_V(r) \geq -3a$$

To see it, for any points $p, q$ on the ray $\gamma$, let

$$e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)$$

be the excess function related to $p$ and $q$. We have $e(x) \geq 0$ and $e(\gamma(t) = 0$ for $0 \leq t \leq d(p,q)$. So

$$\Delta e(\gamma(t)) = \Delta\left(d(p,x) + d(q,x) - d(p,q)\right)_{/\gamma(t)} = \Delta\left(d(p,x) + d(q,x)\right)_{/\gamma(t)} \geq 0$$

Set $q = \gamma(i)$; hence $\Delta d(p,x)_{/x=\gamma(t)} = m(t) \geq -\Delta d(\gamma(i), x)_{/x=\gamma(t)}$.

Set $\gamma_1 = \gamma^-$ be the reverse path of $\gamma$. For $r \leq i$ we get:

$$m(r) + a \geq m(r) + \langle V, \nabla r \rangle_{/\gamma_1(i-r)} = \Delta d(p,x)_{/x=\gamma(r)} + \langle V, \nabla r \rangle_{/\gamma_1(i-r)}$$

$$\geq -\Delta d(\gamma(i), x)_{/x=\gamma(r)} + \langle V, \nabla r \rangle_{/\gamma_1(i-r)} \geq -m_V^q(i-r)$$

$$\geq -m_k(r-i) - a \geq -\frac{.n-1}{i-r} - a$$

where $m_V^q(i-r) = m^q(i-r) - \langle V, \nabla r\rangle_{/\gamma_1(i-r)}$ and $m^q(i-r)$ denotes the mean curvature of the geodesic sphere of center $q$ and radius $i-r$ in $M$.

Let $i \to +\infty$, $m(r) \geq -2a$ and consequently $m_V(r) \geq -3a$.

Since $Ric_V \geq \frac{C(n,\alpha,a)}{(1+r)^\alpha} > 0$ and $|V| \leq a$, we have from lemma, 2.1 $m_V(r) < \frac{n-1}{r} + a$. So, from (2.5) we have:

$$0 < \frac{n-1}{r_0} + 4a - \int_{r_0}^{r} \frac{C}{(1+t)^\alpha} dt = \frac{n-1}{r_0} + 4a + \frac{C}{\alpha-1}\left(\frac{1}{(1+r)^{\alpha-1}} - \frac{1}{(1+r_0)^{\alpha-1}}\right). \tag{2.6}$$

Taking $r \to \infty$, we have:

$$0 < \frac{n-1}{r_0} + 4a - \int_{r_0}^{+\infty} \frac{C}{(1+t)^\alpha} dt = \frac{n-1}{r_0} + 4a - \frac{C}{(\alpha-1)(1+r_0)^{\alpha-1}}.$$

Hence, if $C \geq \left(\frac{n-1}{r_0} + 4a\right)(\alpha-1)(1+r_0)^{\alpha-1}$ then

$$0 < \frac{n-1}{r_0} + 4a - \int_{r_0}^{+\infty} \frac{C}{(1+t)^\alpha} dt \leq 0$$

which is a contradiction. Thus $M$ is compact. If $C > \left(\frac{n-1}{r_0} + 4a\right)(\alpha-1)(1+r_0)^{\alpha-1}$ and $\gamma$ is a minimal geodesic issuying from $p$ then from (2.6) we have :

$$\frac{1}{(1+r)^{\alpha-1}} > \frac{1}{(1+r_0)^{\alpha-1}} - \frac{\alpha-1}{C}\left(\frac{n-1}{r_0} + 4a\right)$$

which means that

$$r < (1+r_0)\left(\frac{C}{C - (\alpha-1)(\frac{n-1}{r_0} + 4a)(1+r_0)^{\alpha-1}}\right)^{\frac{1}{\alpha-1}} - 1$$

and the conclusion follows.

### 2.2. *Proof of the theorem 1.3*

Take two arbitrary points $p$ and $q$ in $M$. Let $\gamma$ be a minimizing geodesic segment lying $p$ to $q$. Set $l = d(p,q)$. Let $(\dot\gamma, e_2, \ldots, e_n)$ be a parallel orthonormal frame along $\gamma$. Let $I(.,.)$ be the index form of $\gamma$. If $h$ is any smooth function on the interval $[0,l]$ satisfying $h(0) = h(l) = 0$ then

$$\sum_{i=2}^{n} I(he_i, he_i) = \int_0^l [(n-1)\dot h^2 - h^2 Ric(\dot\gamma,\dot\gamma)]dt = \int_0^l [(n-1)\dot h^2 - h^2 Ric_V(\dot\gamma,\dot\gamma) + \frac{1}{2}\mathcal{L}_V g(\dot\gamma,\dot\gamma)]dt$$

$$= \int_0^l [(n-1)\dot h^2 - h^2 Ric_V(\dot\gamma,\dot\gamma) + h^2 g(\nabla_{\dot\gamma} V, \dot\gamma)]dt = \int_0^l [(n-1)\dot h^2 - h^2 Ric_V(\dot\gamma,\dot\gamma) + h^2\frac{d}{dt}(g(V,\dot\gamma)]dt$$

$$= \int_0^l [(n-1)\dot h^2 - h^2 Ric_V(\dot\gamma,\dot\gamma) + \frac{d}{dt}(h^2 g(V,\dot\gamma)) - 2\dot h h g(V,\dot\gamma)]dt$$

Since $h(l) = h(0) = 0$ and by relation (1.2) and (1.3) we have:

$$\sum_{i=2}^{n} I(hE_i, hE_i) \leq \int_0^l \left[(n-1)\dot h^2 - \lambda h^2 - h^2\frac{d\phi}{dt}\right] dt - 2\int_0^l \dot h h g(V,\dot\gamma)dt$$

$$\leq \int_0^l ((n-1)\dot h^2 - \lambda h^2)dt - \int_0^l \frac{d}{dt}(h^2\phi)dt - 2\int_0^l \dot h h(-\phi + g(V,\dot\gamma))dt$$

$$\leq \int_0^l ((n-1)\dot h^2 - \lambda h^2)dt + 2\int_0^l |\dot h h|(|\phi| + |g(V,\dot\gamma)|)$$

$$\leq \int_0^l ((n-1)\dot{h}^2 - \lambda h^2)dt + 2\int_0^l \left[|\dot{h}h|\,(a+\delta)\,t + (b+\alpha)\right] dt.$$

Set $h(t) = \sin(\frac{\pi t}{l})$; hence $\dot{h}(t) = \frac{\pi}{l}\cos(\frac{\pi t}{l})$.

It follows that

$$\sum_{i=2}^{n} I(hE_i, hE_i) \leq \int_0^l \left((n-1)\frac{\pi^2}{l^2}\cos^2(\frac{\pi t}{l}) - \lambda \sin^2(\frac{\pi t}{l})\right) dt + \frac{\pi}{l}\int_0^l [At+B]\,|\sin\frac{2\pi t}{l}|dt$$

where $A = a+\delta$; $\quad B = b+\alpha$. By integrating the last term of the right-hand side by parts, we get

$$\sum_{i=2}^{n} I(hE_i, hE_i) \leq \int_0^l \left((n-1)\frac{\pi^2}{l^2}\cos^2(\frac{\pi t}{l}) - \lambda \sin^2(\frac{\pi t}{l})\right) dt + Al + 2B \tag{2.7}$$

$$\leq (n-1)\frac{\pi^2}{2l} - \frac{\lambda l}{2} + (a+\delta)l + 2(b+\alpha) \tag{2.8}$$

$$\leq \frac{1}{2l}\left[-(\lambda - 2(a+\delta))\,l^2 + 4(b+\alpha)l + (n-1)\pi^2\right]. \tag{2.9}$$

Since $\gamma$ is a minimizing geodesic we have:

$$-(\lambda - 2(a+\delta))\,l^2 + 4(b+\alpha)l + (n-1)\pi^2 \geq 0$$

which implies that

$$l \leq \frac{2(b+\alpha) + \sqrt{4(b+\alpha)^2 + (n-1)(\lambda - 2(a+\delta))\pi^2}}{\lambda - 2(a+\delta)}$$

This proves the theorem 1.3.

### 2.3. *Proof of the theorem 1.5*

For $x \in M$ set $r = d(x,N)$. There exists a point $p \in N$ such that $d(x,p) = d(x,N)$. Let $\gamma$ be the shortest geodesic between $p$ and $x$. Let $(e_1, e_2, \ldots, e_{n-1}, \dot{\gamma})$ be a parallel orthonormal frame along $\gamma$. Let $\varphi$ the solution of equation:

$$y''(t) + ky(t) = 0, \quad k \leq 0 \tag{2.10}$$

satisfying $\varphi(0) = a_0 > 0$; $\varphi'(0) = a_1 < 0$. Hence $\varphi(t) = \frac{a_1}{\sqrt{-k}}\sinh\sqrt{-k}t + a_0\cosh\sqrt{-k}t$.

Set $E_i(t) = \frac{\varphi(t)}{\varphi(r)}e_i(t)$. We have

$$\Delta r(x) = \sum_{i=1}^{n-1} Hessr(e_i, e_i) \leq \sum_{i=1}^{n-1} I(E_i, E_i)$$

$$= \frac{1}{(\varphi(r))^2}\sum_{i=1}^{n-1}\int_0^r \left[|\nabla_{\dot{\gamma}}E_i|^2 - R(E_i,\dot{\gamma})\dot{\gamma}, E_i)\right] ds - \frac{\varphi(0)^2}{\varphi(r)^2}\sum_{i=1}^{n} II(e_i, e_i) \tag{2.11}$$

$$\leq \frac{1}{(\varphi(r))^2}\int_0^r \left[(n-1)\varphi'(t)^2 - \varphi(t)^2 Ric(\dot{\gamma}(t))\right] dt - \frac{a_0^2}{\varphi^2(r)}(n-1)H \tag{2.12}$$

$$\leq \frac{1}{(\varphi(r))^2}\int_0^r \left[(n-1)\varphi'(t)^2 - \varphi(t)^2 Ric_V(\dot{\gamma}(t)) + \varphi^2(t)\langle\nabla_{\dot{\gamma}}V, \dot{\gamma}\rangle(t)\right] dt - \frac{a_0^2}{\varphi^2(r)}(n-1)H. \tag{2.13}$$

Note that

$$\frac{1}{\varphi^2(r)}\int_0^r \varphi^2(t)\langle\nabla_{\dot{\gamma}}V,\dot{\gamma}\rangle dt = \frac{1}{\varphi^2(r)}\int_0^r \frac{d}{dt}(\varphi^2(t)\langle V,\dot{\gamma}\rangle dt - \frac{1}{\varphi^2(r)}\int_0^r (\varphi^2)'(t)\langle V,\dot{\gamma}\rangle dt$$

$$= \langle V,\dot{\gamma}\rangle(r) - \frac{a_0^2}{\varphi^2(r)}\langle V,\dot{\gamma}\rangle(0) - \frac{2}{\varphi^2(r)}\int_0^r \varphi'(t)\varphi(t)\langle V,\dot{\gamma}\rangle dt.$$

Hence from the relation (2.13) and the assumption $Ric_V \geq (n-1)k$ we have

$$\Delta r(x) \leq \frac{n-1}{(\varphi(r))^2}\int_0^r \left[\varphi'(t)^2 - k\varphi(t)^2\right] dt - \frac{a_0^2}{\varphi^2(r)}\left((n-1)H + \langle V,\dot{\gamma}\rangle(0)\right)$$

$$+\langle V,\dot{\gamma}\rangle(r)-\frac{2}{\varphi^2(r)}\int_0^r\varphi'(t)\varphi(t)\langle V,\dot{\gamma}\rangle dt. \tag{2.14}$$

Let prove the **part 1)**
Take $a_1<-a_0\sqrt{-k}$; then we have $\varphi'(t)\varphi(t)\le 0$ and since $\langle V,\dot{\gamma}\rangle\le(n-1)a$ we obtain:

$$\begin{aligned}\Delta_V r(x)&\le\frac{n-1}{(\varphi(r))^2}\int_0^r\left[\varphi'(t)^2-k\varphi(t)^2\right]dt-\frac{a_0^2}{\varphi^2(r)}(n-1)H_V-\frac{2(n-1)a}{\varphi^2(r)}\int_0^r\varphi'(t)\varphi(t)dt\\&\le\frac{n-1}{(\varphi(r))^2}\int_0^r\left[\varphi'(t)^2-k\varphi(t)^2\right]dt-\frac{a_0^2}{\varphi^2(r)}(n-1)H_V-(n-1)a+\frac{(n-1)aa_0^2}{\varphi^2(r)}.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^r\left(\varphi'(t)^2-k\varphi(t)^2)\right)dt&=[\varphi'(t)\varphi(t)]_0^r-\int_0^r(\varphi''(t)\varphi(t)+k\varphi(t)^2)dt \\&=\varphi'(r)\varphi(r)-a_1a_0.\end{aligned}\tag{2.15}$$

We deduce

$$\Delta_V r(x)\le(n-1)\frac{\varphi'(r)}{\varphi(r)}+\frac{(n-1)a_0^2}{\varphi^2(r)}\left(a-H_V-\frac{a_1}{a_0}\right)-(n-1)a \tag{2.16}$$

$$\le(n-1)\frac{\varphi'(r)}{\varphi(r)}+\frac{(n-1)a_0^2}{\varphi^2(r)}\left(a-h-\frac{a_1}{a_0}\right)-(n-1)a. \tag{2.17}$$

Take $a_1=(a-h)a_0$ then we have

$$\Delta_V r(x)\le(n-1)\frac{\varphi'(r)}{\varphi(r)}-(n-1)a. \tag{2.18}$$

Since

$$\Delta r-(n-1)a\le\Delta_V r\le(n-1)\frac{\varphi'(r)}{\varphi(r)}-(n-1)a$$

we conclude that

$$\Delta r\le(n-1)\frac{\varphi'(r)}{\varphi(r)}=(n-1)\frac{(a-h)\cosh\sqrt{-k}t+\sqrt{-k}\sinh\sqrt{-k}t}{\frac{1}{\sqrt{-k}}(a-h)\sinh\sqrt{-k}t+\cosh\sqrt{-k}t}$$

which goes to $-\infty$ when

$$t\to r_0=\frac{1}{2\sqrt{-k}}\ln\left(\frac{h+\sqrt{-k}-a}{h-\sqrt{-k}-a}\right).$$

Hence we conclude that

$$Diam(M)<Diam(N)+\frac{1}{\sqrt{-k}}\ln\left(\frac{h+\sqrt{-k}-a}{h-\sqrt{-k}-a}\right).$$

To prove the **part 2)** take $f(\gamma(t))=\int_0^t\langle V,\dot{\gamma}(s)\rangle ds$. From (2.14) and (2.15), we get

$$\Delta_V r(x)\le(n-1)\frac{\varphi'(r)}{(\varphi(r))}-\frac{a_0^2}{\varphi^2(r)}\left((n-1)H_V+\frac{a_1}{a_0}\right)-\frac{1}{\varphi^2(r)}\int_0^r(\varphi^2)'(t)\langle V,\dot{\gamma}\rangle dt. \tag{2.19}$$

Since $(\varphi^2)''\ge 0$ and $\left|\int_0^r\langle V,\dot{\gamma}\rangle dt\right|\le(n-1)b$, we have:

$$\begin{aligned}-\int_0^r(\varphi^2)'(t)\langle V,\dot{\gamma}\rangle dt&=-[(\varphi^2)'(t)f(\gamma(t))]_0^r+\int_0^r((\varphi^2)''(t)f(\gamma(t))dt\\&\le-2\varphi'(r)\varphi(r)(n-1)b+(n-1)b[(\varphi^2)'(t)]_0^r\\&=-2\varphi'(r)\varphi(r)(n-1)b+2\varphi'(r)\varphi(r)(n-1)b-2\varphi'(0)\varphi(0)(n-1)b\\&=-2(n-1)ba_1a_0.\end{aligned}$$

Therefore, the inequality (2.19) gives:

$$\Delta_V r(x)\le(n-1)\frac{\varphi'(r)}{(\varphi(r))}-\frac{a_0^2}{\varphi^2(r)}\left((n-1)H_V+\frac{(n-1)a_1}{a_0}+2(n-1)\frac{a_1b}{a_0}\right) \tag{2.20}$$

$$\leq (n-1)\frac{\varphi'(r)}{(\varphi(r))} - \frac{(n-1)a_0^2}{\varphi^2(r)}\left(h + \frac{a_1}{a_0} + 2\frac{a_1 b}{a_0}\right). \tag{2.21}$$

If we take

$$a_1 = -a_0\frac{h}{1+2b};$$

then we have:

$$\Delta_V r(x) \leq (n-1)\frac{\varphi'(r)}{(\varphi(r))}. \tag{2.22}$$

Since $h > \sqrt{-k}(1+2b)$ then $\frac{\varphi'(r)}{(\varphi(r))} \to -\infty$ when $r \to \frac{1}{2\sqrt{-k}}\ln\left(\frac{\sqrt{-k}(1+2b)+h}{h-\sqrt{-k}(1+2b)}\right)$. Hence

$$r < \frac{1}{2\sqrt{-k}}\ln\left(\frac{\sqrt{-k}(1+2b)+h}{h-\sqrt{-k}(1+2b)}\right)$$

and consequently

$$Diam(M) < Diam(N) + \frac{1}{\sqrt{-k}}\ln\left(\frac{\sqrt{-k}(1+2b)+h}{h-\sqrt{-k}(1+2b)}\right).$$

*2.4. Proof of the theorem 1.6*

Prove the **part** (1) of the theorem 1.6, we need to set a mean comparison theorem.
Using the same arguments as in the prove of theorem 2.2 in [9], we get

$$sn_k^2(r)m_V(r) \leq sn_k^2(r)m_k(r) - \int_0^r (sn_k^2)'\langle V, \nabla t\rangle dt. \tag{2.23}$$

Hence, since $(sn_k^2)' \geq 0$, in the hypothesis $\langle V, \nabla r\rangle \geq -a$, we deduce that

$$m_V(r) \leq m_k(r) + a. \tag{2.24}$$

This means that

$$\frac{d}{dt}\ln(A_V(t,\theta)) \leq \frac{d}{dt}\ln(A_k(t,\theta) + a. \tag{2.25}$$

So if $0 < r \leq R$ then

$$\frac{A_V(R,\theta)}{A_V(r,\theta)} \leq \frac{e^{aR}A_k(R,\theta)}{e^{ar}A_k(r,\theta)} \tag{2.26}$$

where $A_k(r,\theta)$ is the Riemannian volume element in $M_k^n$. The relation (2.26) means that the function $r \to \frac{A_V(r,\theta)}{e^{ar}A_k(r,\theta)}$ is nonincreasing in $r$. Using the lemma 3 in [1] and integrating along the unit sphere, we get, for $0 < r \leq R$

$$\frac{\int_{S^n}\int_0^R A_V(t,\theta)dtd\theta}{\int_{S^n}\int_0^r A_V(t,\theta)dtd\theta} \leq \frac{\int_{S^n}\int_0^R e^{at}A_k(t,\theta)dtd\theta}{\int_{S^n}\int_0^r e^{at}A_k(t,\theta)dtd\theta} \leq e^{aR}\frac{\int_{S^n}\int_0^R A_k(t,\theta)dtd\theta}{\int_{S^n}\int_0^r A_k(t,\theta)dtd\theta} \tag{2.27}$$

and we deduce

$$\frac{Vol_V(B(p,R))}{Vol_V(B(p,r))} \leq e^{aR}\frac{Vol_k^n(R)}{Vol_k^n(r)}. \tag{2.28}$$

To prove the **part (2)** we first show that, under those conditions, we have

$$m_V(r) \leq m_k^{n+4c}(r).$$

Let integrate by parts the last term of the right member of the relation (2.23)

$$\int_0^r (sn_k^2)'\langle V, \nabla t\rangle dt = (sn_k^2)'(r)\int_0^r \langle V, \nabla t\rangle dt - \int_0^r \left((sn_k^2)''(t)\int_0^t \langle V, \nabla s\rangle ds\right) dt.$$

In the hypothesis conditions, we have $(sn_k^2)' \geq 0$ and $(sn_k^2)'' \geq 0$, hence we deduce

$$-\int_0^r (sn_k^2)'\langle V, \nabla t\rangle dt \leq (sn_k^2)'(r)\left|\int_0^r \langle V, \nabla t\rangle dt\right| + \int_0^r \left((sn_k^2)''(t)\left|\int_0^t \langle V, \nabla s\rangle ds\right|\right) dt$$

$$\leq 2c(sn_k^2)'(r) = 4csn_k(r)sn_k'(r).$$

Hence the relation (2.23) becomes:

$$sn_k^2(r)m_V(r) \leq sn_k^2(r)m_k^n(r) + 4csn_k(r)sn_k'(r) = sn_k^2(r)(n-1+4c)\frac{sn_k'(r)}{sn_k(r)}$$
$$\leq sn_k^2(r)m_k^{n+4c}(r).$$

We conclude that

$$m_V(r) \leq m_k^{n+4k}(r).$$

Hence using the same arguments as in part 1) we show that

$$\frac{Vol_V(B(p,R))}{Vol_V(B(p,r))} \leq \frac{Vol_k^{n+4c}(R)}{Vol_k^{n+4c}(r)}. \tag{2.29}$$

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