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A Numerical Technique for Direct Solution of Special Fourth Order Ordinary Differential Equation Via Hybrid Linear Multistep Method

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Keywords

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Fourth-order ordinary differential equation,
Numerical approximation,
Self-starting.

Abstract

We propose and present a self-starting numerical approximation with a higher order of accuracy for direct solution of a special fourth-order ordinary differential equation (ODE) using a Hybrid Linear Multistep Method (HLMM). The technique utilizes the collocation and interpolation approach with six-step numbers and two off-step points using power series as the basis function. Error constants and basic properties proved the convergence of the method. Numerical experiments involving both linear, non-linear, and linear systems of fourth-order initial value problems appearing in modeling of physical phenomenon from various areas of applied sciences were used to demonstrate the effectiveness and efficiency of the proposed method. The results revealed that the proposed method is an excellent choice for approximating general fourth-order ODE and shows the impact of choices of step sizes in the numerical solution of the problem considered. In addition, the proposed HLMM outperformed existing methods in the literature in terms of accuracy.

1. Introduction

Higher-order differential equations continue to gain more attention in applied sciences and engineering, especially the fourth-order ordinary differential equations (ODE) such as, in the modeling of deflection of beams, electric circuits, fluid flow, and neural networks [1–6].This class of differential equations has craved for novel researches because of its importance in understanding the behaviors and properties of the physical situations involved. Some of these equations appear in linear, non-linear, and system of differential equations where an analytical solution is rarely available. Therefore, we result in a numerical approximation of the solution to understand and interpret the physical situation.
In this paper, we will be considering a special fourth order ODE of the form;

$$D[y(t)] = f(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \frac{d^3y}{dt^3}) \qquad t \in (t_0, t_n) \tag{1}$$

coupled with the initial conditions

$$y(t_0) = \phi_0, \frac{dy(t_0)}{dt} = \phi_1, \frac{d^2y(t_0)}{dt^2} = \phi_2, \frac{d^3y(t_0)}{dt^3}) = \phi_3 \tag{2}$$

where D is equivalent to $\frac{d^4}{dt^4}, y(t) \in \mathbb{R}^n$ and f is a continuous-valued function.

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Solutions to this concerned equation (1) with its initial conditions (2) have been researched over the years. Several authors have attempted to reduce the higher-order ODE to systems of first-order equations, then applied suitable numerical methods to approximate the solution [6–10]. This attempt has some drawbacks, like wastage of computing time and storage. Some other authors have addressed these setbacks by directly solving them using the predictor-corrector approach. Still, the disadvantage is that the accuracy of the numerical Result depends on the starting method. Recently, researchers [11–17] have also worked on the direct solution of the fourth-order initial value problems using the linear multistep method. However, the limitations of these direct methods are the low degree of accuracy and low step-number.

In other to obtain a more efficient method than the available in literatures, it is necessary to increase the step numbers (both grid and off-grid points) of the method of solutions that will be a self-starting and higher order of accuracy to cater for stiff differential equations. This paper develops a zero-stable order nine hybrid block method that caters to the low-degree of accuracy and low step number in previous research. The method can handle linear, non-linear, and system of (1).

This paper is organized as follows; brief introduction was presented in section 1, formulation and development of the propose method is given in section 2, section 3 contains the analysis of the proposed method. numerical experiments are presented in section 4 to show the efficiency and effectiveness of the developed method, and the conclusion was presented in section 5.

2. Methodology

2.1. Preliminaries

2.1.1. Linear Multistep Method

Let $y(x)$ be the numerical solution of a differential equation, a $k-$ step general linear multistep method (LMM) is;

$$y(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h^n \sum_{j=0}^{k} \beta_j(t) f_{n+j} \tag{3}$$

where k is the step number, $\alpha_j(t)$ and $\beta_j(t)$ are the continuous coefficients to be determined, n is the order of the differential equation, h is the step-size [11].

2.1.2. Hybrid Linear Multistep Method

A linear multistep method with off-grid point(s) is called a hybrid linear multistep method. Following [3], a k-step hybrid LMM is represented by;

$$y(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h^n \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^n \phi_v(t) f_{n+j} + h^n \phi_u(t) f_{n+j} \tag{4}$$

where $phi_v(t)$ and $phi_u(t)$ are continuous coefficients on the off-grid points to be determined.

2.1.3. Power Series Function as a Basis function

In this paper, we implore the power series of the form

$$Y(t) = \sum_{j=0}^{\infty} c_j t^j \tag{5}$$

where c_j are the coefficients of the series.

2.2. Development of HLMM for Fourth-Order ODE

In this section, we develop a numerical approximation to (1) by using the basis function in (5) of the form;

$$y(t) = \sum_{j=0}^{a+b-1} c_j t^j \tag{6}$$

where a is the collocation points, b is the interpolating points. The fourth derivative of (6) is;

$$y^{iv}(t) = \sum_{j=0}^{a+b-1} j(j-1)(j-2)(j-3)c_j t^{j-4} \tag{7}$$

Then, (1) becomes

$$f(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \frac{d^3y}{dt^3}) = \sum_{j=0}^{a+b-1} j(j-1)(j-2)(j-3)c_j t^{j-4} \tag{8}$$

We propose an LMM of the form;

$$y(t) = \sum_{j=0}^{k-2} \alpha_j(t) y_{n+j} + h^4 \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^4 \beta_u(t) f_{n+j} + h^4 \beta_v(t) f_{n+j} \tag{9}$$

The method is specified with $a = 9$, $b = 4$, $k = 6$, $u = \frac{1}{4}$, and $\frac{1}{2}$. Collocating (7) at $t = t_{n+j}; j = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 5, 6$ and interpolating (6) at $t = t_{n+j}; j = 0, \frac{1}{4}, \frac{1}{2}, 1$ to obtain a 13×13 systems of equation. We solve the system of equations using the matrix inversion method with the aid of Maple 2015 software to obtain the unknown coefficients $\alpha_{n+j}; j = 0, \frac{1}{4}, \frac{1}{2}, 1$ and $\beta_{n+j}; j = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 5, 6$.
The continuous hybrid LMM is of the form;

$$y(t) = \sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_{\frac{1}{4}} y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + \sum_{j=0}^{6} \beta_j f_{n+j} + \beta_{\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} \tag{10}$$

Evaluating (10) at $t = t_j; j = 2, 3, 4, 5, 6$ to obtain the following scheme

$$\begin{aligned} y_{n+2} = &-21y_n + 64y_{n+\frac{1}{4}} - 56y_{n+\frac{1}{2}} + 14y_{n+1} + h^4 \frac{5081221}{3397386240} f_n + h^4 \frac{15163}{15574680} f_{n+\frac{1}{4}} \\ &+ h^4 \frac{2417567}{21897216} f_{n+\frac{1}{2}} + h^4 \frac{174693371}{1698693120} f_{n+1} + h^4 \frac{11529503}{3397386240} f_{n+2} - h^4 \frac{4042693}{9342812160} f_{n+3}) \\ &+ \frac{297619}{3397386240} f_{n+4} - h^4 \frac{1351813}{96825507840} f_{n+5} + h^4 \frac{984133}{859538718720} f_{n+6} \end{aligned} \tag{11}$$

$$\begin{aligned} y_{n+3} = &-110y_n + 320y_{n+\frac{1}{4}} - 264y_{n+\frac{1}{2}} + 55y_{n+1} - h^4 \frac{172374109}{4756340736} f_n + h^4 \frac{4563485}{21804552} f_{n+\frac{1}{4}} \\ &+ h^4 \frac{3101429}{9953280} f_{n+\frac{1}{2}} + h^4 \frac{2460549085}{2460549085} f_{n+1} + h^4 \frac{1003712473}{4756340736} f_{n+2} - h^4 \frac{969550831}{65399685120} f_{n+3}) \\ &+ \frac{18513413}{4756340736} f_{n+4} - h^4 \frac{93833123}{135555710976} f_{n+5} + h^4 \frac{6659209}{109395836928} f_{n+6} \end{aligned} \tag{12}$$

$$\begin{aligned} y_{n+4} = &-315y_n + 896y_{n+\frac{1}{4}} - 720y_{n+\frac{1}{2}} + 140y_{n+1} - h^4 \frac{16748755}{113246208} f_n + h^4 \frac{112261}{136620} f_{n+\frac{1}{4}} \\ &+ h^4 \frac{2096243}{3649536} f_{n+\frac{1}{2}} + h^4 \frac{210331987}{56623104} f_{n+1} + h^4 \frac{165979543}{113246208} f_{n+2} + h^4 \frac{39403411}{311427072} f_{n+3}) \\ &+ \frac{5122903}{566231040} f_{n+4} - h^4 \frac{283583}{169869312} f_{n+5} + h^4 \frac{4475117}{28651290624} f_{n+6} \end{aligned} \tag{13}$$

$$
\begin{aligned}
y_{n+5} &= -684y_n + 1920y_{n+\frac{1}{4}} - 1520y_{n+\frac{1}{2}} + 285y_{n+1} - h^4\frac{1558245841}{3963617280}f_n + h^4\frac{58297}{27324}f_{n+\frac{1}{4}} \\
&+ h^4\frac{19348669}{25546752}f_{n+\frac{1}{2}} + h^4\frac{3610519517}{396361728}f_{n+1} + h^4\frac{3780601817}{792723456}f_{n+2} + h^4\frac{2734237085}{2179989504}f_{n+3}) \\
&+ \frac{148043269}{792723456}f_{n+4} - h^4\frac{30115061}{5945425920}f_{n+5} + h^4\frac{99230179}{200559034368}f_{n+6}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
y_{n+6} &= -1265y_n + 3520y_{n+\frac{1}{4}} - 2760y_{n+\frac{1}{2}} + 506y_{n+1} - h^4\frac{3523805183}{4756340736}f_n + h^4\frac{3858329}{948024}f_{n+\frac{1}{4}} \\
&+ h^4\frac{177977887}{153280512}f_{n+\frac{1}{2}} + h^4\frac{213617542267}{11890851840}f_{n+1} + h^4\frac{7622336597}{679477248}f_{n+2} + h^4\frac{56393830847}{13079937024}f_{n+3}) \\
&+ \frac{947872849}{679477248}f_{n+4} + h^4\frac{20444080127}{135555710976}f_{n+5} + h^4\frac{267530371}{261598740480}f_{n+6}
\end{aligned} \tag{15}
$$

Obtaining the first, second, and third derivatives (with respect to t) of the continuous hybrid LMM in (10), and evaluating at all the collocation points $t = t_{n+j}; j = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 5, 6$. Then, we obtain the desired discrete schemes for y_{n+j} as follows;

$$
\begin{aligned}
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hz_n + \frac{1}{32}h^2v_n + \frac{1}{384}h^3u_n + h^4\left(\frac{27284911601}{251134790860800}f_n + \frac{955447}{10964574720}f_{n+\frac{1}{4}}\right) \\
&+ h^4\left(-\frac{29704723}{735746457600}f_{n+\frac{1}{2}} + \frac{352916869}{41855798476800}f_{n+1} - \frac{10433279}{7175279738880}f_{n+2} + \frac{337034191}{690620674867200}f_{n+3}\right) \\
&h^4\left(-\frac{2394167}{16742319390720}f_{n+4} + \frac{200451599}{7157341539532800}f_{n+5} - \frac{15152717}{5776100189798400}f_{n+6}\right)
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hz_n + \frac{1}{8}h^2v_n + \frac{1}{48}h^3u_n + h^4\left(\frac{74804309}{61312204800}f_n + \frac{4286321}{2248594425}f_{n+\frac{1}{4}}\right) \\
&+ h^4\left(-\frac{119869}{188179200}f_{n+\frac{1}{2}} + \frac{4036169}{30656102400}f_{n+1} - \frac{13133}{583925760}f_{n+2} + \frac{1268749}{168608563200}f_{n+3}\right) \\
&h^4\left(-\frac{1753}{796262400}f_{n+4} + \frac{250907}{582465945600}f_{n+5} - \frac{625519}{15511987814400}f_{n+6}\right)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
y_{n+1} &= y_n + hz_n + \frac{1}{2}h^2v_n + \frac{1}{6}h^3u_n + h^4\left(\frac{126731}{10886400}f_n + \frac{3100672}{107075925}f_{n+\frac{1}{4}}\right) \\
&+ h^4\left(-\frac{5056}{15436575}f_{n+\frac{1}{2}} + \frac{2251}{1425600}f_{n+1} - \frac{5881}{23950080}f_{n+2} + \frac{13399}{164656800}f_{n+3}\right) \\
&h^4\left(-\frac{947}{39916800}f_{n+4} + \frac{7907}{1706443200}f_{n+5} - \frac{13127}{30296851200}f_{n+6}\right)
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
y_{n+2} &= y_n + 2hz_n + 2h^2v_n + \frac{4}{3}h^3u_n + h^4\left(\frac{48229}{467775}f_n + \frac{8912896}{29202525}f_{n+\frac{1}{4}}\right) \\
&+ h^4\left(\frac{714752}{5145525}f_{n+\frac{1}{2}} + \frac{2210}{18711}f_{n+1} + \frac{1}{891}f_{n+2} + \frac{148}{467775}f_{n+3}\right) \\
&h^4\left(-\frac{61}{467775}f_{n+4} + \frac{254}{8887725}f_{n+5} - \frac{67}{23669415}f_{n+6}\right)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
y_{n+3} &= y_n + 3hz_n + \frac{9}{2}h^2v_n + \frac{9}{2}h^3u_n + h^4\left(\frac{156069}{492800}f_n + \frac{12275712}{9253475}f_{n+\frac{1}{4}}\right) \\
&+ h^4\left(+\frac{864}{1925}f_{n+\frac{1}{2}} + \frac{134217}{123200}f_{n+1} + \frac{20007}{98560}f_{n+2} - \frac{1179}{96800}f_{n+3}\right) \\
&h^4\left(+\frac{1539}{492800}f_{n+4} - \frac{1269}{2340800}f_{n+5} + \frac{531}{11334400}f_{n+6}\right)
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
y_{n+4} = {} & y_n + 4hz_n + 8h^2v_n + \frac{32}{3}h^3u_n + h^4\left(\frac{327808}{467775}f_n + \frac{536870912}{149906295}f_{n+\frac{1}{4}}\right)\\
& + h^4\left(\frac{2097152}{2205225}f_{n+\frac{1}{2}} + \frac{600064}{155925}f_{n+1} + \frac{19328}{13365}f_{n+2} + \frac{684032}{5145525}f_{n+3}\right)\\
& h^4\left(+\frac{32}{4455}f_{n+4} - \frac{34816}{26663175}f_{n+5} + \frac{14464}{118347075}f_{n+6}\right)
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
y_{n+5} = {} & y_n + 5hz_n + \frac{25}{2}h^2v_n + \frac{125}{6}h^3u_n + h^4\left(\frac{6125125}{4790016}f_n + \frac{688640000}{89943777}f_{n+\frac{1}{4}}\right)\\
& + h^4\left(+\frac{320000}{205821}f_{n+\frac{1}{2}} + \frac{1020625}{108864}f_{n+1} + \frac{7553125}{1596672}f_{n+2} + \frac{8344375}{6586272}f_{n+3}\right)\\
& h^4\left(+\frac{876875}{4790016}f_{n+4} - \frac{14125}{3250368}f_{n+5} + \frac{518125}{1211874048}f_{n+6}\right)
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
y_{n+6} = {} & y_n + 6hz_n + 18h^2v_n + 36h^3u_n + h^4\left(\frac{4167}{1925}f_n + \frac{127401984}{9253475}f_{n+\frac{1}{4}}\right)\\
& + h^4\left(+\frac{55296}{21175}f_{n+\frac{1}{2}} + \frac{35478}{1925}f_{n+1} + \frac{4293}{385}f_{n+2} + \frac{91764}{21175}f_{n+3}\right)\\
& h^4\left(+\frac{243}{175}f_{n+4} + \frac{5562}{36575}f_{n+5} - \frac{63}{69575}f_{n+6}\right)
\end{aligned} \tag{23}
$$

where z,v, and u represent the first, second, and third derivatives of equation (10).

3. Analysis

3.1. Consistency

Conventionally, the developed method in section 2 can be written as;

$$\sum_{j=0}^{k}\alpha_j y_{n+j} - h^4\sum_{j=0}^{k}\beta_j f_{n+j} = 0 \tag{24}$$

following [5] and [7], the local truncation error is defined as;

$$L[y(t);h] = \sum_{j=0}^{k}(\alpha_j y(t+jh) - h^4\beta_j f(t+jh)) \tag{25}$$

Suppose $y(t)$ and $f(t)$ are sufficiently differentiable, we express (25) in Taylor series about point x to obtain;

$$L[y(t);h] = C_0y(t) + C_1hy'(t) + C_2h^2y''(t) + \cdots + c_ph^py^p(t) + \dots \tag{26}$$

An LMM is said to be consistent if $p \geq 1$ for $C_0 = C_1 = C_2 = \cdots = C_{p+3} = 0$, and $C_{p+4} \neq 0$ is the error constant, where p is the order of the method [7]. Therefore, each scheme in (11) to (15) has equal order 9 with error constants C_{13} obtained as $-\frac{411605}{1826434842624}, -\frac{156416461}{9132174213120}, -\frac{79500469}{1522029035520}, -\frac{213114317}{1522029035520}, -\frac{264703481}{1304596316160}$ respectively. Since $p > 1$, then the method is consistent. Furthermore, in closed form, (24) can be rewritten as;

$$\rho(z) = h^4\sigma(z) \tag{27}$$

where $\rho(z)$ and $\sigma(z)$ are the first and second characteristic polynomials of the method, respectively. The consistency of a linear multistep method can be strengthened with the following conditions [11]

1. $\sum_{j=0}^{k}\alpha_j = 0$
2. $\rho(z) = \rho'(z) = \rho''(z) = \cdots = \rho^{(n-1)}(z) = 0$
3. $\rho^n(z) = n!\sigma(z)$

where $z = 1$ is the principal root and $n = 4$ is the root of the differential equation. Equation (11) to (15) satisfied the above conditions. Hence, the hybrid block method is consistent.

3.2. Zero Stability

We analyze the zero stability of the developed discrete schemes in (16) to (23) by normalizing the first characteristic polynomial $\rho(z)$ as

$$\rho(z) = det(zA^0 - A^1) = z^{n-1}(z-1) \tag{28}$$

where $|z| \leq 1$ and the roots $|z| = 1$ has multiplicity not exceeding the order 4 of the differential equation, with $A^0 = 8 \times 8$ identity matrix and

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, $\rho(z) = z^7(z-1)$. Hence, the method is zero stable.

3.3. Convergence

The necessary and sufficient condition for the convergence of an LMM is stability and consistency [7, 18, 19]. Since the newly developed method is consistent and zero-stable, the convergence holds.

4. Numerical Experiments

In this section, we shall investigate the effectiveness of the newly proposed method with four different problems consisting of;

1. **Linear Homogeneous Initial Value Problem of type (1)**

$$\frac{d^4y(t)}{dt^4} = 2\frac{d^3y(t)}{dt^3} - \frac{d^2y(t)}{dt^2} + y(t) \tag{29}$$

with the initial conditions;

$$y(0) = 1, y'(0) = -1, y''(0) = 0, y'''(0) = 1, \qquad t \in [0,1] \tag{30}$$

The exact solution $Y(t_i)$ and absolute error, $E_i = |Y(t_i) - y(t_i)|$ at $h = 10^{-1}$ and $h = 10^{-2}$ are presented in Table 1

2. **Linear inhomogeneous Initial Value Problem of type (1)**

$$\frac{d^4y(t)}{dt^4} = y(t) + \sin(2t+1) \tag{31}$$

with the initial conditions;

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 0, \qquad t \in [0,1] \tag{32}$$

The exact solution $Y(t_i)$ and absolute error, $E_i = |Y(t_i) - y(t_i)|$ at $h = 10^{-1}$ and $h = 10^{-2}$ are presented in Table 2

3. **Non-linear inhomogeneous Initial Value Problem of type (1)**

$$\frac{d^4y(t)}{dt^4} = y(t)^2 + sin^2(t) - cos(t) - 1 \tag{33}$$

coupled with the initial conditions;

$$y(0) = -1, y'(0) = 0, y''(0) = 1, y'''(0) = 0, \qquad t \in [0, 1] \tag{34}$$

The exact solution $Y(t_i)$ and absolute error, $E_i = |Y(t_i) - y(t_i)|$ at $h = 10^{-1}$ and $h = 10^{-2}$ are presented in Table 3

4. **Linear system of Fourth Order Initial Value Problem of type (1)**

$$\frac{d^4y(t)}{dt^4} = w(t) - 3t + 1 \qquad \frac{d^4w(t)}{dt^4} = y(t) - 17t, \qquad t \in [0, 1] \tag{35}$$

coupled with the initial conditions;

$$y(0) = y'(0) = 0, y''(0) = 1, y'''(0) = 2, \quad w(0) = 0, w'(0) = 1, w''(0) = 2, w'''(0) = 3 \tag{36}$$

The exact solution $Y(t_i)$ and $W(t_i)$, and absolute error, $E_i = |Y(t_i) - y(t_i)|$ and $F_i = |W(t_i) - w(t_i)|$ at $h = 10^{-1}$ and $h = 10^{-2}$ are presented in Table 4

4.1. Numerical Results

Tables 1-4 show the numerical approximation for fourth-order ODE problems 1, 2, 3, and 4, respectively. In table 1, 2, and 3, we present the exact solution of the linear homogenous problem 1, linear inhomogeneous problem 2, and the non-linear inhomogeneous problem 3, respectively, and further compare the absolute error at $h = 10^{-1}$ and $h = 10^{-2}$. We observe and validate the notion that smaller step sizes improve the accuracy of numerical results. In table 4, we compare the absolute error generated at the two solutions of the system of linear fourth-order initial value problem 4, at $h = 10^{-1}$ and $h = 10^{-2}$. As well, $h = 10^{-2}$ produce better results.
Furthermore, we performed a comparative study of the numerical solution of the proposed method and a method from literature. The numerical results in tables 5 – 8 were presented and we observed that our proposed method agreed more with the exact solution better than the method in the recent literature. Also, Figures 1 – 8 shows the plot of the absolute errors in table 1 – 8. Logarithm was used to rescale the absolute errors in figure 5 – 8.

Table 1: Comparison of Exact Solution and Absolute Errors at Different h for Problem 1

i	t_i	**Exact Solution**	E_i **at** $h = 10^{-1}$	E_i **at** $h = 10^{-2}$
0	0.0	1.00000000000000000000000000000	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$
1	0.1	0.900179507086368372903374199083	$3.6033340898 \times 10^{-18}$	$7.5186000000 \times 10^{-26}$
2	0.2	0.801544463014773780147434876987	$3.0383799877 \times 10^{-17}$	$3.5324420000 \times 10^{-24}$
3	0.3	0.705598886975389102786943445988	$2.6490566309 \times 10^{-16}$	$2.4454285000 \times 10^{-23}$
4	0.4	0.614238931291468928689510164747	$9.7651913746 \times 10^{-16}$	$9.2802195000 \times 10^{-23}$
5	0.5	0.529808143393722026230366958821	$2.9378900946 \times 10^{-15}$	$2.5992831300 \times 10^{-22}$
6	0.6	0.455160525903564167533022684300	$4.6224649408 \times 10^{-15}$	$6.0541431800 \times 10^{-22}$
7	0.7	0.393732683343854693761826491153	$3.5548475308 \times 10^{-14}$	$1.2455985530 \times 10^{-21}$
8	0.8	0.349626585274874939064008235833	$1.8393427213 \times 10^{-13}$	$2.3442494940 \times 10^{-21}$
9	0.9	0.327704761421510586558072410078	$5.6244382854 \times 10^{-13}$	$4.1256599390 \times 10^{-21}$
10	1.0	0.333700082480454521066470136026	$1.3053065440 \times 10^{-12}$	$6.8907945350 \times 10^{-21}$

5. Conclusions

In this paper, we have developed a self-starting six-step continuous LMM (10) with a higher order of accuracy via collocation and interpolation technique using power series as the basis function for solving the general fourth-order initial value problems (1). The basis of our scheme development is to produce a method with high accuracy and efficiency with a higher step-number. We choose two off-grid points and six-step points in the development of the method. The hybrid block method does not require any starting method as it can solve directly (1). We analyzed the necessary and sufficient conditions for the zero stability of the proposed method and established that

Table 2: Comparison of Exact Solution and Absolute Errors at Different h for Problem 2

i	t_i	**Exact Solution**	E_i **at** $h = 10^{-1}$	E_i **at** $h = 10^{-2}$
0	0.0	1.0000000000000000000000000000	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$
1	0.1	1.1050078428116231879275357493 3	$1.1768058627 \times 10^{-18}$	$1.3385000000 \times 10^{-25}$
2	0.2	1.2201280927303744114138938981 1	$1.0004440235 \times 10^{-17}$	$5.6943600000 \times 10^{-24}$
3	0.3	1.3456610715421682293773471933 2	$6.1097212716 \times 10^{-17}$	$3.4172400000 \times 10^{-23}$
4	0.4	1.4821271773308825486571004658 9	$2.4456364514 \times 10^{-16}$	$1.1086648000 \times 10^{-22}$
5	0.5	1.6302808494471884083047283037 7	$6.6790232440 \times 10^{-16}$	$2.6218577000 \times 10^{-22}$
6	0.6	1.7911216721936540173932516846 7	$1.1856235912 \times 10^{-15}$	$5.0889493000 \times 10^{-22}$
7	0.7	1.9659027947347328174994215924 5	$3.4063187692 \times 10^{-15}$	$8.5965198000 \times 10^{-22}$
8	0.8	2.1561370229611704585269839296 4	$1.1966112787 \times 10^{-14}$	$1.3049569100 \times 10^{-21}$
9	0.9	2.3636011079160176267726218954 1	$3.0999150442 \times 10^{-14}$	$1.8119147900 \times 10^{-21}$
10	1.0	2.5903389089705417690033464044 2	$6.5371716798 \times 10^{-14}$	$2.3199642600 \times 10^{-21}$

Table 3: Comparison of Exact Solution and Absolute Errors at Different h for Problem 3

i	t_i	**Exact Solution**	E_i **at** $h = 10^{-1}$	E_i **at** $h = 10^{-2}$
0	0.0	-1.0000000000000000000000000000	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$
1	0.1	- 0.9950041652780257660955619878 04	$3.4653522370 \times 10^{-21}$	$1.4000000000 \times 10^{-29}$
2	0.2	-0.9800665778412416311241965167 48	$2.6041766249 \times 10^{-20}$	$8.2400000000 \times 10^{-28}$
3	0.3	- 0.9553364891256060196423102275 68	$2.8318136289 \times 10^{-19}$	$7.8250000000 \times 10^{-27}$
4	0.4	- 0.9210609940028850827985267320 52	$9.0277390864 \times 10^{-19}$	$3.6969000000 \times 10^{-26}$
5	0.5	- 0.8775825618903727161162815826 04	$2.6568462816 \times 10^{-18}$	$1.2088600000 \times 10^{-25}$
6	0.6	- 0.8253356149096782972409524989 55	$3.4494683710 \times 10^{-18}$	$3.1478400000 \times 10^{-25}$
7	0.7	- 0.7648421872844884262558599901 92	$3.9132946168 \times 10^{-17}$	$7.0209100000 \times 10^{-25}$
8	0.8	- 0.6967067093471654209207499816 42	$2.0539350498 \times 10^{-16}$	$1.3999120000 \times 10^{-24}$
9	0.9	- 0.6216099682706644564847161514 07	$6.0481068347 \times 10^{-16}$	$2.5639890000 \times 10^{-24}$
10	1.0	- 0.5403023058681397174009366074 43	$1.3388092848 \times 10^{-15}$	$4.3932680000 \times 10^{-24}$

Table 4: Comparison of Absolute Errors at Different Step Size for Problem 4

i	t_i	E_i **at** $h = 10^{-1}$	E_i **at** $h = 10^{-2}$	F_i **at** $h = 10^{-1}$	F_i **at** $h = 10^{-2}$
0	0.0	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$	$0.0000000000 \times 10^{0}$
1	0.1	$4.9976250273 \times 10^{-16}$	$2.7735971761 \times 10^{-16}$	$8.8388987407 \times 10^{-10}$	$7.1727383447 \times 10^{-8}$
2	0.2	$1.0651556063 \times 10^{-14}$	$1.3944996408 \times 10^{-12}$	$5.7745406868 \times 10^{-9}$	$3.8402319854 \times 10^{-6}$
3	0.3	$2.9508564016 \times 10^{-13}$	$7.2172775853 \times 10^{-11}$	$9.5571005062 \times 10^{-8}$	$3.1298871517 \times 10^{-5}$
4	0.4	$1.7327345662 \times 10^{-13}$	$1.0548597284 \times 10^{-9}$	$2.4932225821 \times 10^{-7}$	$1.3461903866 \times 10^{-4}$
5	0.5	$1.2795102600 \times 10^{-11}$	$8.1791146832 \times 10^{-9}$	$8.7218222179 \times 10^{-7}$	$4.1422761232 \times 10^{-4}$
6	0.6	$8.1370588714 \times 10^{-11}$	$4.3065935903 \times 10^{-9}$	$1.8472152352 \times 10^{-6}$	$1.0346936017 \times 10^{-3}$
7	0.7	$3.9006503943 \times 10^{-10}$	$1.7442933422 \times 10^{-8}$	$1.7605316780 \times 10^{-4}$	$2.2405658324 \times 10^{-3}$
8	0.8	$4.1314082215 \times 10^{-9}$	$5.8419528171 \times 10^{-7}$	$1.0463737494 \times 10^{-3}$	$4.3725724503 \times 10^{-3}$
9	0.9	$4.3714003541 \times 10^{-8}$	$1.6937666861 \times 10^{-6}$	$3.1872358832 \times 10^{-3}$	$7.8834097542 \times 10^{-3}$
10	1.0	$2.7735971099 \times 10^{-7}$	$4.3847671247 \times 10^{-6}$	$3.1872358832 \times 10^{-3}$	$1.3353555708 \times 10^{-2}$

Table 5: Comparison of Computed Solution and [11] Solution for Problem 1

i	t_i	**Computed Result**	**[11] Result**
0	0.0	1.0000000000000000000000000000	1.0000000000000000000000000000
1	0.1	0.900179507086368372903374199083	0.900179351973242707537353912032
2	0.2	0.801544463014773780147434876987	0.801544463626791433337934571564
3	0.3	0.705598886975389102786943445988	0.705495644876698820940300917832
4	0.4	0.614238931291468928689510164747	0.619572272109120819023258475508
5	0.5	0.529808143393722026230366958821	0.535141294546234736647628567378
6	0.6	0.455160525903564167533022684300	0.460494563690888050849998550111
7	0.7	0.393732683343854693761826491153	0.398893819465743659183444067928
8	0.8	0.349626585274874939064008235833	0.368740268144559552520963343431
9	0.9	0.327704761421510586558072410078	0.346830880946468534592993737455
10	1.0	0.333700082480454521066470136026	0.352851809982481156753891807979

Table 6: Comparison of Computed Solution and [11] Solution for Problem 2

i	t_i	**Computed Result**	**[11] Result**
0	0.0	1.0000000000000000000000000000	1.0000000000000000000000000000
1	0.1	1.10500784281162318792753574933	1.10500775529198817591876319479
2	0.2	1.22012809273037441141389389811	1.22012809276619040882200095323
3	0.3	1.34566107154216822937734719332	1.34559968750905240186220160530
4	0.4	1.48212717733088254865710046589	1.48212717683624594791240421046
5	0.5	1.63028084944718840830472830377	1.63028073867910745725033831106
6	0.6	1.79112167219365401739325168467	1.79112164273427455960058656260
7	0.7	1.96590279473473281749942159245	1.96582619352277796774560330116
8	0.8	2.15613702296117045852698392964	2.16080128998166607911381747178
9	0.9	2.36360110791601762677262189541	2.36826520625839046078135567845
10	1.0	2.59033890897054176900334640442	2.59500343846858341914822682036

Table 7: Comparison of Computed Solution and [11] Solution for Problem 3

i	t_i	**Computed Result**	**[11] Result**
0	0.0	-1.0000000000000000000000000000	-1.0000000000000000000000000000
1	0.1	- 0.995004165278025766095561987804	- 0.995004126383223588209469314832
2	0.2	-0.980066577841241631124196516748	- 0.980066577805663811848889927646
3	0.3	- 0.955336489125606019642310227568	- 0.955306505549980353067443629653
4	0.4	- 0.921060994002885082798526732052	- 0.921060994435867925893461260988
5	0.5	- 0.877582561890372716116281582604	- 0.877582536175753210403144811227
6	0.6	- 0.825335614909678297240952498955	- 0.825335642332886842470487844905
7	0.7	- 0.764842187284488426255859990192	- 0.764815811345606363419702684906
8	0.8	- 0.696706709347165420920749981642	- 0.698783791241969344233596090398
9	0.9	- 0.621609968270664456484716151407	- 0.623687102360712351883433833488
10	1.0	- 0.540302305868139717400936607443	- 0.542379318525173566383072403243

Table 8: Comparison of Computed Solution and [11] Solution for $y(t)$ in Problem 4

i	t_i	**Computed Result** for $y(t_i)$	[11] **Result** for $y(t_i)$
0	0.0	0.000000000000000000000000000000	0.000000000000000000000000000000
1	0.1	0.005337336170588601015159770360	0.005337310731645549739432238449
2	0.2	0.022728185372858107737863502673	0.022728186631248256677142874881
3	0.3	0.054299154257895178305291350819	0.054274737471418395833836657074
4	0.4	0.102241674099634009300405386803	0.112908359229701162323943353662
5	0.5	0.168797961883331786933163948151	0.179464878857838599358704086715
6	0.6	0.256249792676434048811437843676	0.266918523282677175407813095941
7	0.7	0.366910275255487213686827518583	0.377575508790029426890743617968
8	0.8	0.503118758648318406723783714589	0.525888691857064864114881932377
9	0.9	0.667238931913925063784088510934	0.690039282924070022329539079267
10	1.0	0.861660106592471275898433381648	0.884514408445752438286251507812

Table 9: Comparison of Computed Solution and [11] Solution for $w(t)$ in Problem 4

i	t_i	**Computed Result** for $w(t_i)$	[11] **Result** for $w(t_i)$
0	0.0	0.000000000000000000000000000000	0.000000000000000000000000000000
1	0.1	0.110498583878257448109870931032	0.110498718860789578690300169048
2	0.2	0.243954754921108116463530519418	0.243954761570889539077098824100
3	0.3	0.403156946378856514336340022922	0.403207922325459632312979586432
4	0.4	0.590555936573307549372616536208	0.606555699109555560542847437601
5	0.5	0.808098677127406427715346949050	0.824098366046852480671784550245
6	0.6	1.057062120416915643346918800779	1.073061621528848784192737694 04
7	0.7	1.338063442550747067048642818 54	1.354143606835153980861783996 92
8	0.8	1.651075836387403991368157984 92	1.674862124648983171395032270 55
9	0.9	1.994971154823106912881912585 82	2.016637494300246222750717806 91
10	1.0	2.367311699674422497556218220 38	2.385028482287925379193212469 45

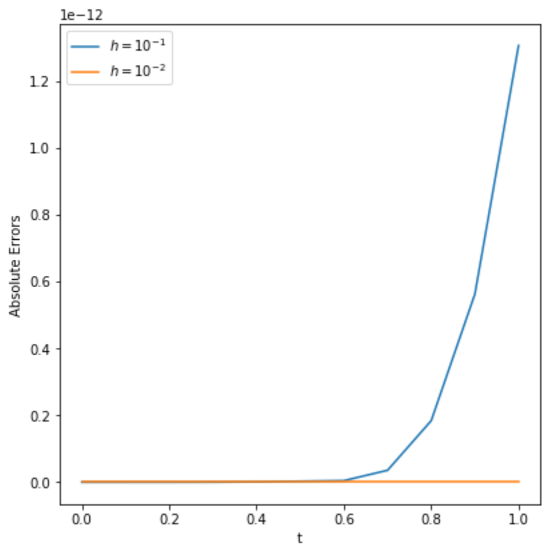

Figure 1: Absolute errors at $h = 10^{-1}$ and $h = 10^{-2}$ for problem 1.

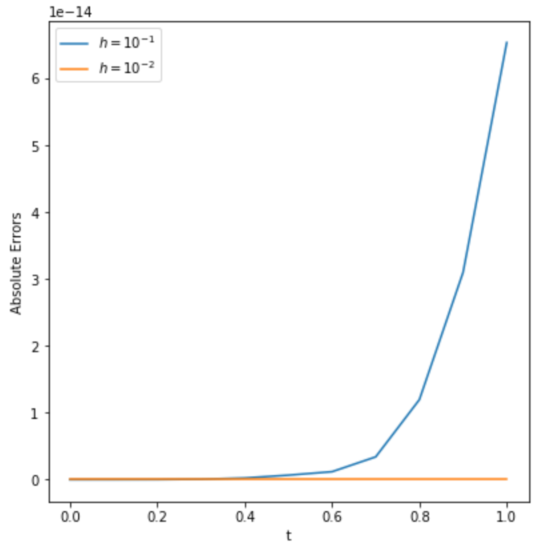

Figure 2: Absolute errors at $h = 10^{-1}$ and $h = 10^{-2}$ for problem 2.

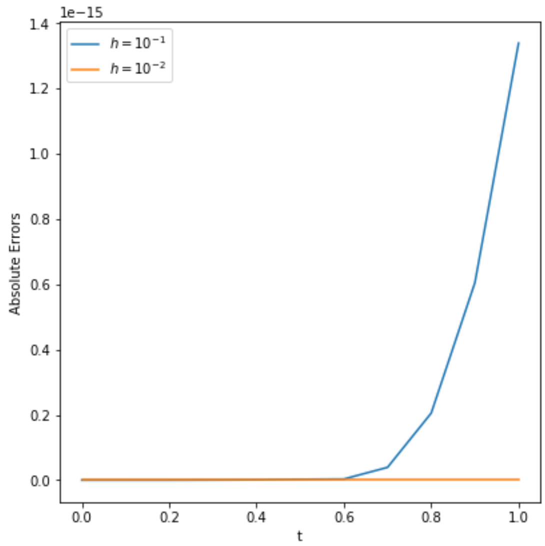

Figure 3: Absolute errors at $h = 10^{-1}$ and $h = 10^{-2}$ for problem 3.

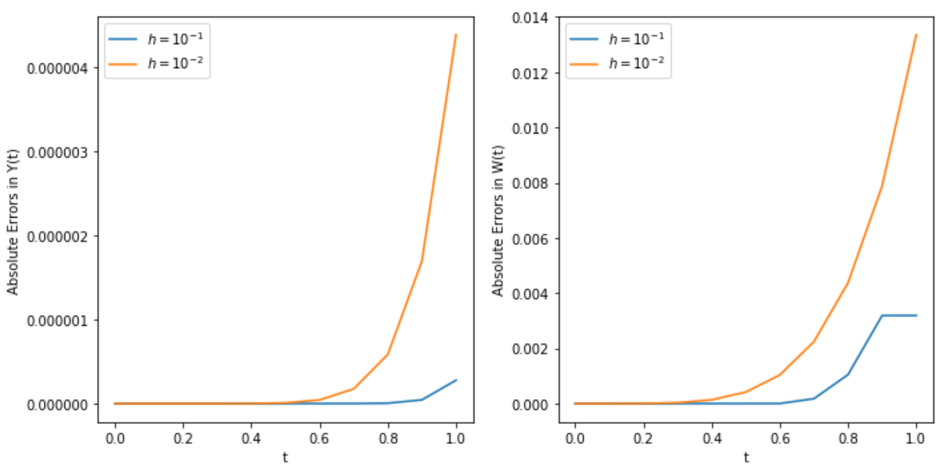

Figure 4: Absolute errors at $h = 10^{-1}$ and $h = 10^{-2}$ for problem 4.

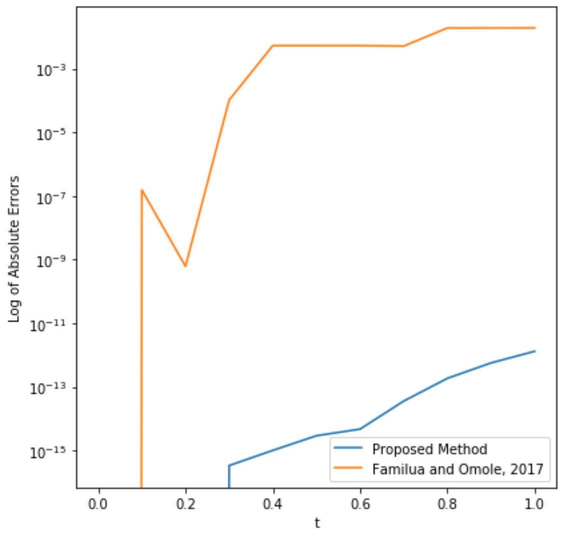

Figure 5: Log of absolute errors of the proposed method and [11] for problem 1.

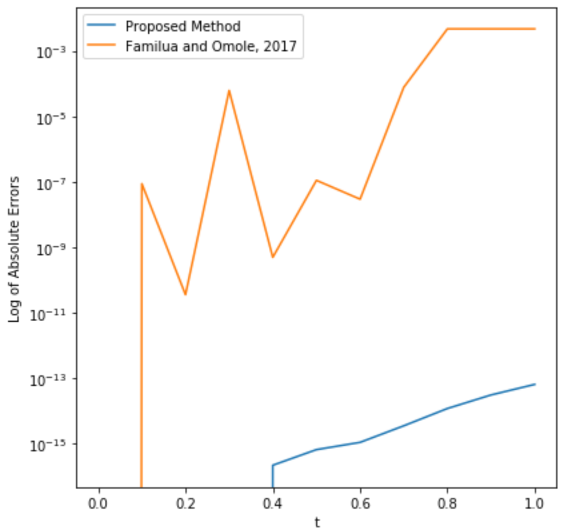

Figure 6: Log of absolute errors of the proposed method and [11] for problem 2.

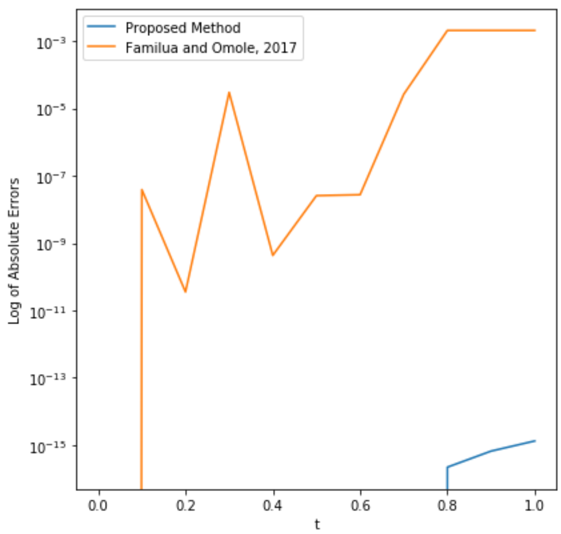

Figure 7: Log of absolute errors of the proposed method and [11] for problem 3.

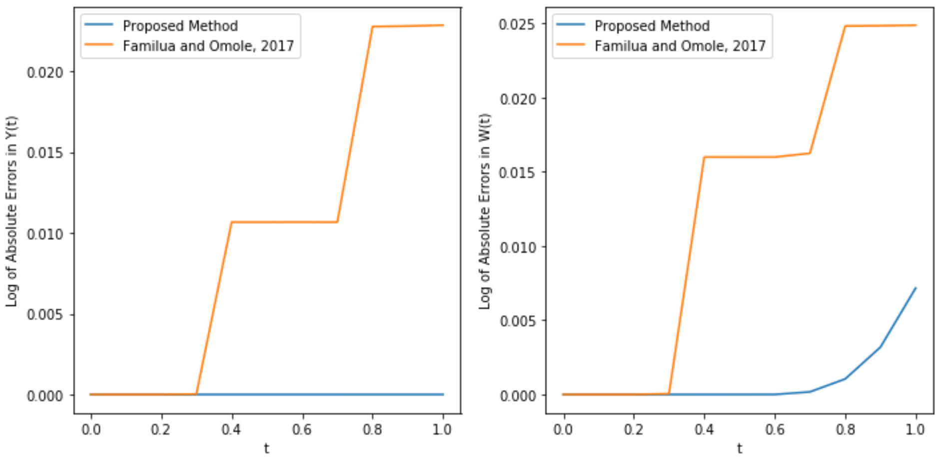

Figure 8: Log of absolute errors of the proposed method and [11] for problem 4.

the proposed method is zero-stable. Furthermore, we illustrated the efficiency of the method with four initial value problems of a linear homogeneous, non-linear homogeneous, non-linear inhomogeneous, and linear system of (1) coupled with distinct initial conditions. The numerical results obtained showed the efficiency of the proposed method as numerical approximations are closer to the exact solutions. The results proved that the proposed method is an excellent choice for approximating the numerical solution of general fourth-order initial value problems in applied sciences and engineering. Our numerical computations are performed with the aid of MAPLE 2015 software package.

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Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

Abd'gafar Tunde Tiamiyu: Conceptualization, Software, Writing- Original draft preparation, Visualization, Data Curation.
Kazeem Iyanda Falade: Supervision, Validation.
Quadri Opeyemi Rauf: Methodology, Formal analysis.
Sikirulai Abolaji Akande: Writing-Reviewing and Editing, Resources.

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