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Research Article

On the approximation properties of bi-parametric potential-type integral operators

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Abstract

In this work we study the approximation properties of the classical Riesz potentials $I^\alpha f \equiv (-\Delta)^{-\alpha/2} f$ and the so-called bi-parametric potential-type operators $J^\alpha_\beta f \equiv (E+(-\Delta)^{\beta/2})^{-\alpha/\beta} f$ as $\alpha \to \alpha_0 > 0$ where, $\alpha > 0$, $\beta > 0$, E is the identity operator and Δ is the laplacian. These potential-type operators generalize the famous Bessel potentials when $\beta = 2$ and Flett potentials when $\beta = 1$. We show that, if A^α is one of operators J^α_β or I^α, then at every Lebesgue point of $f \in L_p(\mathbb{R}^n)$ the asymptotic equality $(A^\alpha f)(x) - (A^{\alpha_0} f)(x) = O(1)(\alpha - \alpha_0)$, $(\alpha \to \alpha_0^+)$ holds. Also the asymptotic equality $\|A^\alpha f - A^{\alpha_0} f\|_p = O(1)(\alpha - \alpha_0)$, $(\alpha \to \alpha_0^+)$ holds when $A^\alpha = J^\alpha_\beta$.

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1. Introduction

The famous Riesz potentials $I^\alpha f$, Bessel potentials $J^\alpha f$, parabolic Riesz potentials $H^\alpha f$ and parabolic Bessel potentials $\mathcal{H}^\alpha f$ play an important role in analysis and its applications (see, e.g [3], [6], [7], [12], [13], [17] and references therein). These potentials are interpreted as negative fractional powers of the differential operators $(-\Delta)$, $(E-\Delta)$, $(\frac{\partial}{\partial t} - \Delta)$ and $(\frac{\partial}{\partial t} + E - \Delta)$, respectively. Here Δ is the laplacian and E is the identity operator.

The boundedness and other properties of these operators and their explicit inverses in the framework of L_p-theory were studied by many authors (see, e.g. [3], [4], [12], [13], [17]).

The approximation properties of these operators and their various modifications as $\alpha \to 0^+$ have been studied by T. Kurokawa [11], A. D. Gadjiev, A. Aral, İ. A. Aliev [2], [8], S. Sezer [14], S. Uyhan, A. D. Gadjiev, İ. A. Aliev [18]. Also the nice paper [9] by S. G. Gal should be mentioned, where the exact order of approximation of analytic functions is

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obtained by several potential type operators generated by the gamma function and some singular integrals.

Note that in the one dimensional case, the approximation properties of the fractional integrals

$$(I^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt,\ (a < x < b),$$

as $\alpha \to \alpha_0 \geq 0$ and the strongly continuity of the semigroup $I^{\alpha}_{a+}\varphi$, $(\alpha \geq 0)$ has been studied in the book [13], (p. 48-53), by S. Samko, A. Kilbas, O. Marichev.

In this work we study the approximation properties of the so-called bi-parametric potential-type operators $J^{\alpha}_{\beta}\varphi$ and classical Riesz potentials $I^{\alpha}\varphi$ as $\alpha \to \alpha_0^+$. Note that the operators J^{α}_{β} have been introduced by İ. A. Aliev [1] and are defined as follows:

$$(J^{\alpha}_{\beta}\varphi)(x) = \frac{1}{\Gamma(\alpha/\beta)}\int_0^{\infty} e^{-t}t^{\frac{\alpha}{\beta}-1}(W^{(\beta)}_t\varphi)(x)dt,$$

where $x \in \mathbb{R}^n$, $\varphi \in L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$, and $\alpha, \beta \in (0,\infty)$. Here $\{W^{(\beta)}_t\varphi\}_{t\geq 0}$ is the β-semigroup defined by

$$(W^{(\beta)}_t\varphi)(x) = \int\limits_{\mathbb{R}^n}\varphi(x-y)w^{(\beta)}(y;t)dy,\ (t > 0)$$

and $W^{(\beta)}_0\varphi = E$ (the identity operator). The kernel function $w^{(\beta)}(y;t)$ is the inverse Fourier transform of $\exp(-t\,|x|^{\beta})$, i.e.,

$$w^{(\beta)}(y;t) = F^{-1}(e^{-t|x|^{\beta}})(y),\ (y \in \mathbb{R}^n).$$

The β-semigroup $W^{(\beta)}_t\varphi$ is the generalization of the Gauss-Weierstrass semigroup (for $\beta = 2$) and Abel-Poisson semigroup (for $\beta = 1$). Besides that, the bi-parametric potentials $J^{\alpha}_{\beta}\varphi$ generalize the Bessel potentials (for $\beta = 2$) and Flett potentials (for $\beta = 1$).

The article is organized as follows: Section 2 contains some necessary notations, definitions and auxilary lemmas. Section 3 and 4 include the main results of the article and are devoted to the approximation properties of the families $J^{\alpha}_{\beta}\varphi$ and $I^{\alpha}\varphi$ as $\alpha \to \alpha_0^+$. Roughly speaking, our main results assert that, if $A^{\alpha}f$ is one of $I^{\alpha}f$ or $J^{\alpha}_{\beta}f$, then under some conditions on $f \in L_p(\mathbb{R}^n)$, the asymptotic equality $((A^{\alpha}f)(x) - (A^{\alpha_0}f)(x)) = O(1)(\alpha - \alpha_0)$ as $\alpha \to \alpha_0^+$ is valid at the Lebesgue points of f. Also, we obtain asymptotic equality $\|A^{\alpha}f - A^{\alpha_0}f\|_p = O(1)(\alpha - \alpha_0)$ as $\alpha \to \alpha_0^+$, when $A^{\alpha} = J^{\alpha}_{\beta}$.

2. Notations, definitions and auxiliary lemmas

Let $L_p = L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, be the standard space of measurable functions with the norm

$$\|f\|_p = \left(\int\limits_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p} < \infty,\ \ \|f\|_{\infty} = ess\sup_{x\in\mathbb{R}^n}|f(x)|\,,$$

where $x = (x_1, x_2, ..., x_n)$ and $dx = dx_1dx_2...dx_n$.

The Fourier and inverse Fourier transforms are defined by

$$f^{\wedge}(z) \equiv Ff(z) = \int\limits_{\mathbb{R}^n} f(x)e^{-ixz}dx \text{ and } f^{\vee}(z) \equiv F^{-1}f(z) = (2\pi)^{-n}Ff(-z),$$

where $xz = x_1z_1 + x_2z_2 + ... + x_nz_n$.

Let $f \in L_p$, $1 \leq p < \infty$. The Poisson (or Abel-Poisson) semigroup associated with the function f is defined as

$$P_tf(x) = \int\limits_{\mathbb{R}^n} p(y,t)f(x-y)dy,\ 0 < t < \infty. \tag{2.1}$$

Here, the Poisson kernel $p(y,t)$ has the form

$$p(y,t)=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\frac{t}{(t^2+|y|^2)^{(n+1)/2}}=t^{-n}p(\frac{y}{t},1) \tag{2.2}$$

and is the inverse Fourier transform of $e^{-t|\xi|}$, $(\xi\in\mathbb{R}^n)$, i.e.,

$$F[p(.,t)](\xi)=e^{-t|\xi|};\ t>0,\ |\xi|=(\xi_1^2+\xi_2^2+...+\xi_n^2)^{1/2}.$$

Another important semigroup is the famous Gauss-Weierstrass semigroup:

$$W_tf(x)=\int\limits_{\mathbb{R}^n}w(y,t)f(x-y)dy,\ 0<t<\infty, \tag{2.3}$$

where the Gauss-Weierstrass kernel $w(y,t)$ is defined in Fourier terms by

$$F[w(.,t)](\xi)=e^{-t|\xi|^2},\ (t>0,\ \xi\in\mathbb{R}^n)$$

and is explicitly computed as

$$w(y,t)=(4\pi t)^{-n/2}\exp(-|y|^2/4t). \tag{2.4}$$

The Bessel potentials are defined in Fourier terms by

$$J^\alpha f=F^{-1}(1+|\xi|^2)^{-\alpha/2}Ff\equiv(E-\Delta)^{-\alpha/2}f,\ (0<\alpha<\infty).$$

E is the identity operator and $\Delta=\sum\limits_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}$ is the laplacian. If $f\in L_p$, then $J^\alpha f$ has the following integral representation:

$$(J^\alpha f)(x)=\frac{1}{\lambda_n(\alpha)}\int\limits_{\mathbb{R}^n}f(y)G_\alpha(x-y)dy\ \text{([13], p.540; [17], p. 132)}, \tag{2.5}$$

where

$$\lambda_n(\alpha)=2^n\pi^{n/2}\Gamma(\alpha/2),\ G_\alpha(x)=\int_0^\infty t^{(\alpha-n)/2}e^{-t-|x|^2/4t}\frac{dt}{t}.$$

These potentials play important role in various branches of mathematics and its applications; see, e.g. [7], [12], [13], [17].

There are other fractional integral operators whose behaviours are "almost midway" between the Bessel and Riesz potentials. These potentials are introduced by T. M. Flett in his fundamental paper [7] (see, also [13], 541-542).

Flett potentials $\mathcal{F}^\alpha f$ are defined in terms of Fourier transform as

$$F[\mathcal{F}^\alpha f](x)=(1+|x|)^{-\alpha}F[f](x),\ (x\in\mathbb{R}^n,\ 0<\alpha<\infty)$$

and are interpreted as negative fractional powers of the operator $(E+\sqrt{-\Delta})$, i.e., $\mathcal{F}^\alpha f=(E+\sqrt{-\Delta})^{-\alpha}f$. These potentials can be represented as convolution

$$(\mathcal{F}^\alpha f)(x)=\int\limits_{\mathbb{R}^n}f(y)\Phi_\alpha(x-y)dy,$$

where

$$\Phi_\alpha(y)=\frac{1}{\gamma_n(\alpha)}|y|^{\alpha-n}\int_0^\infty\frac{s^\alpha e^{-s|y|}}{(1+s^2)^{(n+1)/2}}ds \tag{2.6}$$

with $\gamma_n(\alpha)=\pi^{(n+1)/2}\Gamma(\alpha)/\Gamma((n+1)/2)$; see, [5], [7], p. 447 and [13], p. 542.

For $f\in L_p$, $1\le p<\infty$, the Bessel and Flett potentials have significant "one dimensional" integral representations via the Abel-Poisson and Gauss-Weierstrass semigroup [7] (see, also [12], [13]), namely

$$(J^\alpha f)(x)=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\frac{\alpha}{2}-1}e^{-t}W_tf(x)dt,\ 0<\alpha<\infty \tag{2.7}$$

and

$$(\mathcal{F}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} P_t f(x) dt, \; 0 < \alpha < \infty, \tag{2.8}$$

where the integral operators $P_t f(x)$ and $W_t f(x)$ are defined as (2.1) and (2.3).

In the paper [1] a notion of the bi-parametric potentials, which are natural generalizations of the Bessel and Flett potentials, has been introduced. Bi-parametric potentials are defined as

$$(J_{\beta}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\infty} e^{-t} t^{\frac{\alpha}{\beta}-1} (W_t^{(\beta)}\varphi)(x) dt, \tag{2.9}$$

where $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$, and $\alpha, \beta \in (0, \infty)$.

Here $\{W_t^{(\beta)}\varphi\}_{t\ge 0}$ is the β-semigroup defined by

$$(W_t^{(\beta)}\varphi)(x) = \int\limits_{\mathbb{R}^n} \varphi(x-y) w^{(\beta)}(y;t) dy, \; (t > 0). \tag{2.10}$$

The kernel function $w^{(\beta)}(y;t)$ is the inverse Fourier transform of $\exp(-t\,|x|^{\beta})$, i.e.,

$$w^{(\beta)}(y;t) = F^{-1}(e^{-t|x|^{\beta}})(y), \; (y \in \mathbb{R}^n). \tag{2.11}$$

It is easy to see that the β-semigroup (2.10) is a natural generalization of the Gauss-Weierstrass semigroup (for $\beta = 2$) and Abel-Poisson semigroup (for $\beta = 1$). Furthermore, the bi-parametric potentials $J_{\beta}^{\alpha}\varphi$ generalize the Bessel potentials (2.7) (by setting $\beta = 2$) and Flett potentials (2.8) (by setting $\beta = 1$) and the operators $J_{\beta}^{\alpha}\varphi$ are interpreted as negative fractional powers of the operator $(E + (-\Delta)^{\beta/2})$, i.e., for Schwarz test functions φ we have

$$J_{\beta}^{\alpha}\varphi = F^{-1}(1 + |\xi|^{\beta})^{-\alpha/\beta} F\varphi \equiv (E + (-\Delta)^{\beta/2})^{-\alpha/\beta}\varphi.$$

The behaviour of these integral operators in the framework of L_p-spaces and explicit inversion formulas for them have been obtained in [1].

We give here some properties of operators $J_{\beta}^{\alpha}\varphi$, $(0 < \alpha < \infty)$.

Lemma 2.1. ([1]) *Let* $1 \le p < \infty$ *and* $\varphi \in L_p(\mathbb{R}^n)$. *Then,*

a) $J_{\beta}^{\alpha}\varphi$ *is well-defined for all* $\alpha > 0$, $\beta > 0$ *and is bounded on* L_p, *i.e.,*

$$\left\| J_{\beta}^{\alpha}\varphi \right\|_p \le c(\beta) \left\| \varphi \right\|_p .$$

Moreover, if $0 < \beta \le 2$, *then we can write* $c(\beta) = 1$.

b) *The operator* J_{β}^{α} *is a convolution type operator with the Fourier multiplier* $m(\xi) = (1 + |\xi|^{\beta})^{-\alpha/\beta}$, $(\xi \in \mathbb{R}^n)$, *i.e., for any Schwarz test function* φ *we have*

$$F[J_{\beta}^{\alpha}\varphi](\xi) = (1 + |\xi|^{\beta})^{-\alpha/\beta} F[\varphi](\xi).$$

c) *For any fixed parameter* $\beta > 0$, *the family* $\{J_{\beta}^{\alpha}\}_{\alpha\ge 0}$ *has the following semigroup property:*

$$J_{\beta}^{\alpha_1+\alpha_2}\varphi = J_{\beta}^{\alpha_1}(J_{\beta}^{\alpha_2}\varphi), \; (\alpha_1, \alpha_2 \ge 0, J_{\beta}^{0} \equiv E).$$

The following Lemma gives some properties of the semigroup $W_t^{(\beta)}\varphi$.

Lemma 2.2. ([1]) *Let the kernel function* $w^{\beta}(y;t)$, $(y \in \mathbb{R}^n, t > 0)$ *and the* β*-semigroup* $W_t^{(\beta)}\varphi$ *be defined as (2.11) and (2.10). Then,*

a)

$$\int\limits_{\mathbb{R}^n} w^{(\beta)}(y;t) dy = 1, \; \forall t > 0, \forall \beta > 0.$$

b) *If* $1 \le p \le \infty$, *then*

$$\left\| W_t^{(\beta)} \varphi \right\|_p \le c(\beta) \left\| \varphi \right\|_p, \; \forall t > 0, \forall \beta > 0,$$

where $c(\beta) = \int\limits_{\mathbb{R}^n} \left| w^{(\beta)}(y,1) \right| dy < \infty$. *If* $0 < \beta \le 2$, *then* $c(\beta) = 1$.

c) *For almost all* $x \in \mathbb{R}^n$ *and all* $\beta > 0$

$$\sup_{t>0} \left| (W_t^{(\beta)} \varphi)(x) \right| \le c_\beta (M\varphi)(x), \; \varphi \in L_p, \; 1 \le p \le \infty,$$

where $M\varphi$ *is the well-known Hardy-Littlewood maximal function:*

$$(M\varphi)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |\varphi(y)| \, dy$$

with $B(x,r)$ *is the ball of radius* r, *centered at* $x \in \mathbb{R}^n$.

d)

$$\sup_{x \in \mathbb{R}^n} \left| (W_t^{(\beta)} \varphi)(x) \right| \le c_\beta t^{-n/\beta p} \left\| \varphi \right\|_p, \; 1 \le p < \infty.$$

e)

$$W_t^{(\beta)} (W_\tau^{(\beta)} \varphi) = W_{t+\tau}^{(\beta)} \varphi, \; \forall t, \tau > 0 \;, \text{ (the semigroup property).}$$

f) *Let* $\varphi \in L_p$, $1 \le p < \infty$. *Then*

$$\lim_{t \to 0^+} (W_t^{(\beta)} \varphi)(x) = \varphi(x)$$

with the limit being understood in the L_p*-norm or pointwise for almost all* $x \in \mathbb{R}^n$, *e.g. for any Lebesgue point of function* φ.

3. Approximation properties of the family of bi-parametric potentials $J_\beta^\alpha \varphi$ as $\alpha \to \alpha_0^+$.

In this section we will study the approximation properties of the family of bi-parametric potentials $J_\beta^\alpha \varphi$ as $\alpha \to \alpha_0^+$, where $\alpha_0 > 0$ being a fixed number and $\varphi \in L_p(\mathbb{R}^n)$. The main result of this section is the following.

Theorem 3.1. *Let* $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$ *and the family of integral operators* $J_\beta^\alpha \varphi$ *be defined as (2.9). Then*

a) *for any* $\alpha_0 > 0$ *and for almost all* $x \in \mathbb{R}^n$, *(e.g. for any Lebesgue point* x *of function* φ*)*

$$\left| (J_\beta^\alpha \varphi)(x) - (J_\beta^{\alpha_0} \varphi)(x) \right| = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+;$$

b) $\left\| J_\beta^\alpha \varphi - J_\beta^{\alpha_0} \varphi \right\|_{L_p(\mathbb{R}^n)} = O(1)(\alpha - \alpha_0)$ *as* $\alpha \to \alpha_0^+$.

Corollary 3.2. *Let* $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$ *and* $\mathcal{A}^\alpha \varphi$ *be either one of the Bessel or Flett Potentials of the function* φ. *Then,*

a) *for any* $\alpha_0 > 0$ *and for almost all* $x \in \mathbb{R}^n$ *(e.g. for any Lebesgue point* x *of function* φ*)*

$$|(\mathcal{A}^\alpha \varphi)(x) - (\mathcal{A}^{\alpha_0} \varphi)(x)| = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+;$$

b) $\| \mathcal{A}^\alpha \varphi - \mathcal{A}^{\alpha_0} \varphi \|_{L_p(\mathbb{R}^n)} = O(1)(\alpha - \alpha_0)$ *as* $\alpha \to \alpha_0^+$.

Proof. a) Given $\alpha_0 > 0$, let $\alpha \in (\alpha_0, 2\alpha_0)$. Then

$$\begin{aligned}(J^{\alpha}_{\beta}\varphi)(x) - (J^{\alpha_0}_{\beta}\varphi)(x) &= (\frac{1}{\Gamma(\frac{\alpha}{\beta})} - \frac{1}{\Gamma(\frac{\alpha_0}{\beta})}) \int_0^{\infty} e^{-t} t^{\frac{\alpha_0}{\beta}-1} (W^{(\beta)}_t \varphi)(x) dt \\ &\quad + \frac{1}{\Gamma(\frac{\alpha}{\beta})} \int_0^{\infty} e^{-t} (t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1}) (W^{(\beta)}_t \varphi)(x) dt \\ &= I_1(\alpha) + I_2(\alpha).\end{aligned}$$

(In fact, the expressions $I_1(\alpha)$ and $I_2(\alpha)$ also depend on the x-variable. However, since x is fixed, we only wrote the variable α.)

By the mean value theorem we have

$$\left|\frac{1}{\Gamma(\alpha/\beta)} - \frac{1}{\Gamma(\alpha_0/\beta)}\right| = \frac{1}{\Gamma(\alpha/\beta)\Gamma(\alpha_0/\beta)} \left|\Gamma'(\theta)\right| \frac{1}{\beta}(\alpha - \alpha_0), \tag{3.1}$$

where $\theta \in (\frac{\alpha_0}{\beta}, \frac{\alpha}{\beta}) \subset (\frac{\alpha_0}{\beta}, \frac{2\alpha_0}{\beta})$.

$$\begin{aligned}|\Gamma'(\theta)| &= \left|\int_0^{\infty} e^{-t} t^{\theta-1} \ln t dt\right| \le \int_0^1 e^{-t} t^{\theta-1} \ln \frac{1}{t} dt + \int_1^{\infty} e^{-t} t^{\theta-1} \ln t dt \\ &< \int_0^1 t^{\frac{\alpha_0}{\beta}-1} \ln \frac{1}{t} dt + \int_1^{\infty} e^{-t} t^{\theta} dt \\ &\le \int_0^1 t^{\frac{\alpha_0}{\beta}-1} \ln \frac{1}{t} dt + \int_1^{\infty} e^{-t} t^{\frac{2\alpha_0}{\beta}} dt \equiv c(\alpha_0, \beta) < \infty.\end{aligned}$$

Using this and the estimate $\min\limits_{t>0} \Gamma(t) = 0,88... > \frac{1}{2}$ we have from (3.1)

$$\left|\frac{1}{\Gamma(\alpha/\beta)} - \frac{1}{\Gamma(\alpha_0/\beta)}\right| \le \frac{4}{\beta} c(\alpha_0, \beta)(\alpha - \alpha_0)$$

and therefore,

$$\begin{aligned}|I_1(\alpha)| &\le \frac{4}{\beta} c(\alpha_0, \beta) \left|(J^{\alpha_0}_{\beta})\varphi(x)\right| (\alpha - \alpha_0) \\ &= O(1)(\alpha - \alpha_0), \ (\alpha \to \alpha_0^+).\end{aligned} \tag{3.2}$$

Further,

$$\begin{aligned}|I_2(\alpha)| &= \frac{1}{\Gamma(\alpha/\beta)} \left|\int_0^{\infty} e^{-t} (t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1}) (W^{(\beta)}_t \varphi)(x) dt\right| \\ &\le \frac{1}{\Gamma(\alpha/\beta)} \int_0^1 \left|t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1}\right| \left|(W^{(\beta)}_t \varphi)(x)\right| dt \\ &\quad + \frac{1}{\Gamma(\alpha/\beta)} \left|\int_1^{\infty} e^{-t} (t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1}) (W^{(\beta)}_t \varphi)(x) dt\right| \\ &\equiv I_3(\alpha) + I_4(\alpha).\end{aligned}$$

According to Lemma 2.2,

$$\sup_{t>0} \left|(W^{(\beta)}_t \varphi)(x)\right| \le c(M\varphi)(x), \ (\varphi \in L_p, \ 1 \le p < \infty), \tag{3.3}$$

for almost all $x \in \mathbb{R}^n$, where $M\varphi$ is the Hardy-Littlewood maximal operator. By making use of these, we have for $\alpha > \alpha_0$

$$\begin{aligned}I_3(\alpha) &\le \frac{c}{\Gamma(\alpha/\beta)} (M\varphi)(x) \int_0^1 (t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1}) dt \\ &= \frac{c}{\Gamma(\alpha/\beta)} (M\varphi)(x) \beta (\frac{1}{\alpha_0} - \frac{1}{\alpha}) \\ &= \frac{c\beta}{\Gamma(\alpha/\beta)} \frac{1}{\alpha\alpha_0} (\alpha - \alpha_0).\end{aligned}$$

Since $\Gamma(\alpha/\beta) > \frac{1}{2}$, we get

$$I_3(\alpha) \le A(\alpha-\alpha_0), \text{ where } A = \frac{4c\beta}{\alpha_0^2}.$$

Thus,

$$I_3(\alpha) = O(1)(\alpha-\alpha_0) \text{ as } \alpha \to \alpha_0^+. \tag{3.4}$$

Let us estimate I_4. Using (3.3) and the mean value formula,

$$t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1} = \frac{1}{\beta}(\alpha-\alpha_0)t^{\frac{\theta}{\beta}-1}\ln t, \tag{3.5}$$

we have

$$I_4(\alpha) \le \frac{c(M\varphi)(x)}{\Gamma(\alpha/\beta)}\frac{1}{\beta}(\alpha-\alpha_0)\int_1^\infty e^{-t}t^{\frac{\theta}{\beta}-1}\ln t dt,$$

where $\alpha_0 < \theta < \alpha < 2\alpha_0$.

Since

$$\int_1^\infty e^{-t}t^{\frac{\theta}{\beta}-1}\ln t dt < \int_1^\infty e^{-t}t^{2\alpha_0-1}\ln t dt < \infty,$$

we have

$$I_4(\alpha) = O(1)(\alpha-\alpha_0) \text{ as } \alpha \to \alpha_0^+. \tag{3.6}$$

Finally, taking into account (3.2), (3.4) and (3.6) we get

$$\left|(J_\beta^\alpha\varphi)(x) - (J_\beta^{\alpha_0}\varphi)(x)\right| = O(1)(\alpha-\alpha_0) \text{ as } \alpha \to \alpha_0^+,$$

for almost all $x \in \mathbb{R}^n$.

b) As in proof of part a), we have for $\alpha > \alpha_0$ and $\varphi \in L_p$, $1 \le p \le \infty$

$$(J_\beta^\alpha\varphi)(x) - (J_\beta^{\alpha_0}\varphi)(x) = I_1(x) + I_2(x),$$

where

$$I_1(x) = (\frac{1}{\Gamma(\alpha/\beta)} - \frac{1}{\Gamma(\alpha_0/\beta)})\int_0^\infty e^{-t}t^{\frac{\alpha_0}{\beta}-1}(W_t^{(\beta)}\varphi)(x)dt$$

and

$$I_2(x) = \frac{1}{\Gamma(\alpha/\beta)}\int_0^\infty e^{-t}(t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1})(W_t^{(\beta)}\varphi)(x)dt.$$

Hence,

$$\left\|J_\beta^\alpha\varphi - J_\beta^{\alpha_0}\varphi\right\|_p \le \|I_1\|_p + \|I_2\|_p\,. \tag{3.7}$$

By making use of (3.1) and Lemma 2.1-a) we have

$$\|I_1\|_p \le \frac{4}{\beta}c(\alpha_0,\beta)\left\|J_\beta^{\alpha_0}\varphi\right\|_p|\alpha-\alpha_0| \le B\|\varphi\|_p|\alpha-\alpha_0|\,, \tag{3.8}$$

where the coefficient B depends only on the parameters α_0 and β.

Further,

$$I_2(x) = I_3(x) + I_4(x), \tag{3.9}$$

where

$$I_3(x) = \frac{1}{\Gamma(\alpha/\beta)}\int_0^1 e^{-t}(t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1})(W_t^{(\beta)}\varphi)(x)dt$$

and

$$I_4(x) = \frac{1}{\Gamma(\alpha/\beta)}\int_1^\infty e^{-t}(t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1})(W_t^{(\beta)}\varphi)(x)dt.$$

By using Minkovski inequality and Lemma 2.2-b) we get

$$\begin{aligned}\|I_3\|_p &\le \frac{c(\beta)}{\Gamma(\alpha/\beta)}\|\varphi\|_p\int_0^1(t^{\frac{\alpha_0}{\beta}-1} - t^{\frac{\alpha}{\beta}-1})dt \\ &= \frac{c(\beta)}{\Gamma(\alpha/\beta)}\|\varphi\|_p\frac{\beta}{\alpha\alpha_0}(\alpha-\alpha_0) \le \frac{2\beta c(\beta)}{\alpha_0^2}\|\varphi\|_p(\alpha-\alpha_0).\end{aligned}$$

Therefore,

$$\|I_3\|_p = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+. \tag{3.10}$$

Further,

$$\|I_4\|_p \le \frac{c(\beta)}{\Gamma(\alpha/\beta)} \|\varphi\|_p \int_1^\infty e^{-t}(t^{\frac{\alpha}{\beta}-1} - t^{\frac{\alpha_0}{\beta}-1})dt.$$

By making use of the formula (3.5), we have

$$\|I_4\|_p \le \frac{c(\beta)}{\beta\Gamma(\alpha/\beta)} \|\varphi\|_p (\alpha - \alpha_0) \int_1^\infty e^{-t}t^{\frac{\theta}{\beta}-1} lntdt, (\alpha_0 < \theta < \alpha \le 2\alpha_0),$$

and therefore

$$\|I_4\|_p = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+. \tag{3.11}$$

By (3.9),

$$\|I_2\|_p \le \|I_3\|_p + \|I_4\|_p,$$

and as a result, we have from (3.10) and (3.11) that

$$\|I_2\|_p = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+. \tag{3.12}$$

By taking into account (3.12) and (3.8) in (3.7) we conclude that

$$\left\| J_\beta^\alpha \varphi - J_\beta^{\alpha_0} \varphi \right\|_p = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0^+,$$

and the proof is complete. □

4. Approximation properties of the Riesz potentials $I^\alpha f$ as $\alpha \to \alpha_0^+$.

In this section we study the approximation properties of the famous Riesz potentials (see, e.g. [17], p.117)

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \tag{4.1}$$

where $0 < \alpha < n$ and $\gamma_n(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \Gamma(\alpha/2)/\Gamma((n-\alpha)/2)$.

It is known that the operators $I^\alpha f$ are well defined for all $f \in L_p(\mathbb{R}^n)$, $1 \le p < \infty$, provided that $0 < \alpha < n/p$.

We need the following:

Lemma 4.1. ([15], p. 552) *Let $0 < \alpha < n$, $1 \le p < n/\alpha$ and $f \in L_p(\mathbb{R}^n)$. Then the Riesz potentials $I^\alpha f$ admit the following "one dimensional" integral representation via the β-semigroup $W_t^{(\beta)} f$:*

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha/\beta)} \int_0^\infty t^{\frac{\alpha}{\beta}-1} (W_t^{(\beta)} f)(x) dt, \tag{4.2}$$

where β is an arbitrary positive number.

Remark 4.2. The interested reader can find the proof of this Lemma in [15]. Recall that, the well known formulas

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (P_t f)(x) dt \text{ (E. Stein [16])}$$

and

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1} (G_t f)(x) dt \text{ (R. Johnson [10])}$$

are the special cases of the general formula (4.2). Namely, $P_t f = W_t^{(1)} f$ is the Abel-Poisson integral and $G_t f = W_t^{(2)} f$ is the Gauss-Weierstrass integral.

The following theorem shows that, the rate of pointwise a.e. convergence of family $(I^\alpha f)(x)$ to $(I^{\alpha_0} f)(x)$ as $\alpha \to \alpha_0^+$ is not worse than $O(1)(\alpha - \alpha_0)$ for a.e. $x \in \mathbb{R}^n$ (e.g. Lebesgue points of f).

Theorem 4.3. *Let $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$ and $0 < \alpha_0 < \alpha < \frac{n}{p}$. Let further $I^\alpha f$ be the Riesz potential of f, defined as in (4.1). Then, for almost all $x \in \mathbb{R}^n$ (e.g. for any Lebesgue point x of function f) the asymptotic equality*

$$(I^\alpha f)(x) - (I^{\alpha_0} f)(x) = O(1)(\alpha - \alpha_0) \text{ as } \alpha \to \alpha_0$$

holds.

Remark 4.4. As mentioned above, the operators $I^\alpha f$ and $I^{\alpha_0} f$ are well defined provided that $f \in L_p(\mathbb{R}^n)$ and $0 < \alpha_0 < \alpha < \frac{n}{p}$.

Proof. Let $f \in L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$ and $0 < \alpha_0 < \alpha < \frac{n}{p}$. According to formula (4.2), the choice of the parameter $\beta > 0$ is at our disposal and we choose it as $\beta = \alpha_0$. In this case the potential $I^{\alpha_0} f$ has the simpler form

$$(I^{\alpha_0} f)(x) = \int_0^\infty (W_t^{(\alpha_0)} f)(x) dt$$

and then we have

$$\begin{aligned}
(I^\alpha f)(x) - (I^{\alpha_0} f)(x) &= \frac{1}{\Gamma(\alpha/\alpha_0)} \int_0^\infty t^{\frac{\alpha}{\alpha_0}-1} (W_t^{(\alpha_0)} f)(x) dt \\
&\quad - \int_0^\infty (W_t^{(\alpha_0)} f)(x) dt \\
&= \frac{1}{\Gamma(\alpha/\alpha_0)} \int_0^\infty (t^{\frac{\alpha}{\alpha_0}-1} - 1)(W_t^{(\alpha_0)} f)(x) dt \\
&\quad + (\frac{1}{\Gamma(\alpha/\alpha_0)} - 1) \int_0^\infty (W_t^{(\alpha_0)} f)(x) dt \\
&\equiv A(\alpha) + B(\alpha).
\end{aligned} \tag{4.3}$$

Let us estimate $A(\alpha)$ as $\alpha \to \alpha_0$. We have

$$\begin{aligned}
A(\alpha) &= \frac{1}{\Gamma(\alpha/\alpha_0)} \left[\int_0^1 (t^{\frac{\alpha}{\alpha_0}-1} - 1)(W_t^{(\alpha_0)} f)(x) dt \right. \\
&\quad \left. + \int_1^\infty (t^{\frac{\alpha}{\alpha_0}-1} - 1)(W_t^{(\alpha_0)} f)(x) dt \right] \\
&\equiv a_1(\alpha) + a_2(\alpha)
\end{aligned} \tag{4.4}$$

By making use of Lemma 2.2- c) and the condition $\frac{\alpha}{\alpha_0} > 1$, we have

$$\begin{aligned}
|a_1(\alpha)| &\leq \frac{1}{\Gamma(\alpha/\alpha_0)} \int_0^1 \left| t^{\frac{\alpha}{\alpha_0}-1} - 1 \right| \left| (W_t^{(\alpha_0)} f)(x) \right| dt \\
&\leq \frac{1}{\Gamma(\alpha/\alpha_0)} c M_f(x) \int_0^1 (1 - t^{\frac{\alpha}{\alpha_0}-1}) dt \\
&= \frac{c}{\Gamma(\alpha/\alpha_0)} M_f(x)(1 - \frac{\alpha_0}{\alpha}) \\
&= \frac{c}{\alpha\Gamma(\alpha/\alpha_0)} M_f(x)(\alpha - \alpha_0), \text{ for almost all } x \in \mathbb{R}^n.
\end{aligned}$$

Since $\min\limits_{t>0} \Gamma(t) = 0,88... > \frac{1}{2}$ and $\alpha_0 < \alpha$, we have

$$|a_1(\alpha)| \leq c_1(\alpha - \alpha_0), \tag{4.5}$$

where $c_1 = c_1(x) = \frac{2c}{\alpha_0} M_f(x)$.

To estimate $a_2(\alpha)$, we will use Lemma 2.2-d):

$$\begin{aligned}|a_2(\alpha)| &= \frac{1}{\Gamma(\alpha/\alpha_0)}\left|\int_1^\infty (t^{\frac{\alpha}{\alpha_0}-1}-1)(W_t^{(\alpha_0)}f)(x)dt\right| \\ &\le c\|f\|_p \frac{1}{\Gamma(\alpha/\alpha_0)}\int_1^\infty (t^{\frac{\alpha}{\alpha_0}-1}-1)t^{-\frac{n}{\alpha_0 p}}dt \\ &= c\|f\|_p \frac{1}{\Gamma(\alpha/\alpha_0)}\left[\int_1^\infty t^{\frac{\alpha}{\alpha_0}-\frac{n}{\alpha_0 p}-1}dt-\int_1^\infty t^{-\frac{n}{\alpha_0 p}}dt\right] \\ &= c\alpha_0\|f\|_p \frac{1}{\Gamma(\alpha/\alpha_0)}\frac{1}{(\frac{n}{p}-\alpha)(\frac{n}{p}-\alpha_0)}(\alpha-\alpha_0).\end{aligned}$$

Since $0<\alpha_0<\alpha<\frac{n}{p}$ and $\alpha\to\alpha_0$, we can assume that $\alpha<\frac{1}{2}(\alpha_0+\frac{n}{p})$. Then, $(\frac{n}{p}-\alpha)>\frac{1}{2}(\frac{n}{p}-\alpha_0)$ and therefore, $1/(\frac{n}{p}-\alpha)(\frac{n}{p}-\alpha_0)<2/(\frac{n}{p}-\alpha_0)^2$.

By taking into account this and the estimate $\Gamma(\alpha/\alpha_0)>\frac{1}{2}$ we have

$$|a_2(\alpha)|\le c_2(\alpha-\alpha_0), \tag{4.6}$$

where $c_2=2c\alpha_0(\frac{n}{p}-\alpha_0)^{-2}\|f\|_p<\infty$.

Now, denoting $c_1+c_2=c_3$, we obtain from (4.4), (4.5) and (4.6) that

$$|A(\alpha)|\le c_3(\alpha-\alpha_0) \text{ as } \alpha\to\alpha_0^+. \tag{4.7}$$

Let us now estimate $B(\alpha)$ in (4.3). We have

$$|B(\alpha)|\le \frac{1}{\Gamma(\alpha/\alpha_0)}|1-\Gamma(\alpha/\alpha_0)|\int_0^\infty \left|(W_t^{(\alpha_0)}f)(x)\right|dt. \tag{4.8}$$

By Lemma 2.2-c) and d) we have

$$\begin{aligned}\int_0^\infty \left|(W_t^{(\alpha_0)}f)(x)\right|dt &= \int_0^1 \left|(W_t^{(\alpha_0)}f)(x)\right|dt+\int_1^\infty \left|(W_t^{(\alpha_0)}f)(x)\right|dt \\ &\le cM_f(x)+c\|f\|_p\int_1^\infty t^{-\frac{np}{\alpha_0}}dt=cM_f(x)+c\frac{\alpha_0}{(n/p)-\alpha_0}.\end{aligned}$$

Therefore,

$$\int_0^\infty \left|(W_t^{(\alpha_0)}f)(x)\right|dt\le c_4, \tag{4.9}$$

where $c_4=c_4(x)=c(M_f(x)+\frac{\alpha_0}{(n/p)-\alpha_0})<\infty$ for almost all $x\in\mathbb{R}^n$.

An application of the mean value theorem gives

$$\begin{aligned}\left|1-\Gamma(\frac{\alpha}{\alpha_0})\right| &= \left|\Gamma(1)-\Gamma(\frac{\alpha}{\alpha_0})\right|=\left|\Gamma'(\lambda)(1-\frac{\alpha}{\alpha_0})\right| \\ &= \frac{1}{\alpha_0}|\Gamma'(\lambda)|(\alpha-\alpha_0),\end{aligned} \tag{4.10}$$

where $1<\lambda<\frac{\alpha}{\alpha_0}$ and $\Gamma'(\lambda)=\int_0^\infty e^{-t}t^{\lambda-1}\ln t dt$.

Further, since $1<\lambda<\frac{\alpha}{\alpha_0}<\frac{n}{p\alpha_0}$, we have

$$\begin{aligned}|\Gamma'(\lambda)| &\le \int_0^\infty e^{-t}t^{\lambda-1}|\ln t|dt \\ &\le \int_0^1 \ln(\frac{1}{t})dt+\int_1^\infty e^{-t}t^{\frac{n}{p\alpha_0}-1}\ln t dt\equiv c_5<\infty.\end{aligned} \tag{4.11}$$

By taking into account (4.9), (4.10), (4.11) in (4.7) and using the estimate $\Gamma(\alpha/\alpha_0)>\frac{1}{2}$, we have

$$|B(\alpha)|\le c_6(\alpha-\alpha_o),(\alpha\to\alpha_0^+), \tag{4.12}$$

where $c_6=2\frac{1}{\alpha_0}c_4c_5<\infty$.

Finally, we obtain from (4.3), (4.7) and (4.12) that

$$|(I^{\alpha}f)(x) - (I^{\alpha_0}f)(x)| = O(1)(\alpha - \alpha_0),\ (\alpha \to \alpha_0^+).$$

This completes the proof. □

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