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On construction of a quadratic Sturm-Liouville operator pencil with impulse from spectral data

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Abstract

In this study, we are studied on construction of a quadratic Sturm-Liouville operatör pencil with impulsive from spectral data. The expressions of the eigenvalues and normalizing numbers of the given problem were obtained. Eigenfunctions corresponding to eigenvalues were obtained. The expressions of the solution functions at intervals $(0,a)$ and (a,π) were obtained. Derivation of fundamental equations of the inverse spectral problem for a quadratic Sturm-Liouville operator pencil with impulse is presented and an algorithm for solving the inverse problem is offered.

Keywords: Sturm- Liouville operator, Inverse problem, Spectral Data.

1. Introduction

From the point of view application natural sciences in quantum mechanics different version of the inverse spectral problems (the inverse problem from eigenvalues and normalizing numbers and the inverse problem from two spectra) were considered for the following Sturm- Liouville eigenvalue problem with quadratic dependence on the spectral parameter and with impulsive:

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$$-y'' + [2\lambda p(x) + q(x)]y = \lambda^2 y, \quad x \in [0,a) \cup (a,\pi] \tag{1}$$

$$y'(0) = 0, \ y(\pi) = 0 \tag{2}$$

$$y(a+0) = \alpha y(a-0), \quad y'(a+0) = \alpha^{-1} y'(a-0) \tag{3}$$

where λ is the spectral parameter and $q(x) \in L_2[0,\pi]$, $p(x) \in W_2^1[0,\pi]$ are real function, α is real number and $\alpha > 0$, $\alpha \neq 1$, $a \in \left(\frac{\pi}{2}, \pi\right)$.

Inverse spectral problems for regular and singular diffusion operators have been studied by many authors (see (Aktosun 1998- Nabiev 2006, Guseinov 1985- Nabiev 2013, Gelfand 1995- Yurko 2006*)*. The papers where devoted to the investigation of the solutions of the singular diffusion operatör and it has been proved the existence of the integral representation of type of the transformation operatör fort he solution. In particular, in the papers Guseinov 1985- Nabiev 2013, Marchenko 1986- pöschel 1987, Guseinov 1985, Hüseynov 2013The uniqueness theorems for the solutions of the inverse problems by different spectral data and by the Weyl function have been proved.

In this work, u,nlike other studies where the inverse spectral problems for the diffusion operator where investigated by using the results of Aktosun 1998- Nabiev 2006, Guseinov 1985, the distinct method is presented for the recoustraction of the diffusion operator by the spectral data. Namely, in this work the main integral equations of Gelfand- Levitan-Marchenko type are obtained for the problem $(1)-(3)$ and by the help these integral equations it is proved the sufficient
condition for the inverse problem.

2. Preliminaries

In this section, we present the facts on the problem $(1)-(3)$ needed ın the subsequent sections. Denote by $W_2^n[0,\pi]$ the Sobolev space consisting of complex valued functions on $[0,\pi]$ having $n-1$ absolutely continuous derivatives and n th order derivative that is square- integrable on $[0,\pi]$. Note that $W_2^0[0,\pi]=L_2[0,\pi]$

The following theorem shows that for equation (1) there exists the so- called transformation operator which is crucial in the inverse spectral theory.

Theorem 2.1

Let $q(x)\in W_2^m[0,\pi]$, $p(x)\in W_2^{m+1}[0,\pi]$ and $\varphi(x,\lambda)$ be the solution of equation (1) satisfying the initial conditions

$$\varphi(0,\lambda)=1,\ \varphi'(0,\lambda)=h \qquad (4)$$

and jump conditions (3). Then there exists real-valued functions $A(x,t)$ and $B(x,t)$ having $m+1$ square-integrable derivatives with respect to the both variables such that

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x A(x,t)\cos\lambda t dt+\int_0^x B(x,t)\sin\lambda t dt \qquad (5)$$

$$\alpha^+\beta^+(x)=\alpha^+ xp(0)+2\int_0^x\left[A(\xi,\xi)\sin\beta^+(\xi)-B(\xi,\xi)\cos\beta^+(\xi)\right]d\xi \qquad (6)$$

$$2\frac{d}{dx}\left[A(x,x)\cos\beta^+(x)+B(x,x)\sin\beta^+(x)\right]=\alpha^+\left[q(x)+p^2(x)\right] \qquad (7)$$

$$2\frac{d}{dx}\left[A(x,x)\cos\beta^-(x)-B(x,x)\sin\beta^-(x)\right]_{t=2a-x-0}^{t=2a-x+0}=\alpha^-\left[q(x)+p^2(x)\right] \qquad (8)$$

$$A_t(x,t)\big|_{t=0}=B(x,0)=0 \qquad (9)$$

Where

$$\varphi_0(x,\lambda)=l^+(x)\cos\left[\lambda x-\beta^+(x)\right]+l^-(x)\cos\left[\lambda(2a-x)+\beta^-(x)\right],$$

$$l^\pm(x)=\frac{1}{2}\left(l(x)\pm\frac{1}{l(x)}\right),\ l(x)=\begin{cases}1, & 0\le x<a\\ \alpha, & a<x\le\pi\end{cases},$$

$$\beta^\pm(x)=\int_{\frac{\alpha\pm a}{2}}^x p(t)dt.$$

Next, if $m\ge 1$ then

$$\begin{aligned}&A_{xx}(x,t)-q(x)A(x,t)\\&-2p(x)B_t(x,t)=A_{tt}(x,t)\end{aligned} \qquad (10)$$

$$\begin{aligned}&B_{xx}(x,t)-q(x)B(x,t)\\&+2p(x)A_t(x,t)=B_{tt}(x,t)\end{aligned} \qquad (11)$$

Conversely, if given functions $A(x,t)$ and $B(x,t)$ have second order square-integrable partial derivatives satisfying equations (10), (11) and satisfying conditions (6)-(9), then the function $\varphi(x,\lambda)$ constructed by formula (5) is the solution equation (1) subject to initial conditions (4) and jump conditions (3).

Let us denote by D the subspace $W_2^2[0,\pi]$, consisting of functions $y(x)\in W_2^2[0,\pi]$ satisfying the boundary conditions in (2) and the jump conditions in (3).

$$D=\{y(x):y(x)\in W_2^2[0,\pi];y'(0)=0,\ y(\pi)=0,\ y(a+0)=\alpha y(a-0),\ y'(a+0)=\alpha^{-1}y'(a-0)\}$$

Further we will assume that $q(x)$, $p(x)$ are real-valued function with $q(x)\in L_2[0,\pi]$, $p(x)\in W_2^1[0,\pi]$ such that

$$\int_0^\pi\left\{|y'(x)|^2+q(x)|y(x)|^2\right\}dx>0 \qquad (12)$$

For all functions $y(x)\in D$ being not identically zero (the last condition is automatically satisfied if $q(x)>0$). Under these conditions boundary value problem $(1)-(3)$ possesses the following spectral properties.

1) The eigenvalues of boundary value problem (1)-(3) are real, different from zero and simple. This problem does not have associated functions attached to the eigenfunctions.

2) The eigenfunctions $y(x)$ and $z(x)$ of problem (1)-(3) corresponding to the different eigenvalues λ and μ, respectively, satisfy the ''ortagonality'' relation

$$(\lambda+\mu)\int_0^\pi y(x)\cdot\overline{z}(x)\,dx-2\int_0^\pi p(x)\,y(x)\cdot\overline{z}(x)\,dx=0$$

3) The problem $(1)-(3)$ has countably many eigenvalues which can be arranged in the sequence

$$\dots<\lambda_{-2}<\lambda_{-1}<\lambda_{-0}<0<\lambda_{+0}<\lambda_1<\lambda_2<\dots$$

So that for large negative and positive values of n the asymptotic formula

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{\delta_n}{\lambda_n^0} \quad (13)$$

Holds, where

$$d_n\in l_\infty,\ \delta_n\in l_2,\ \lambda_n^0=n+\frac{1}{\pi}\beta^+(\pi)+h_n,\ |h_n|\le M,$$

$$d_n=\frac{1}{2\Delta_0(\lambda_n^0)}\Big[\big\{\alpha^+\sin\big(\lambda_n^0\pi-\beta^+(\pi)\big)+\alpha^-\sin\big(\lambda_n^0(2a-\pi)+\beta^-(\pi)\big)\big\}\int_0^\pi\big[q(x)+p^2(x)\big]dx \quad (14)$$
$$-\big\{\alpha^+\cos\big(\lambda_n^0\pi-\beta^+(\pi)\big)+\alpha^-\cos\big(\lambda_n^0(2a-\pi)+\beta^-(\pi)\big)\big\}\big(p(\pi)-p(0)\big)\Big]$$

(Recall that $\lambda=0$ is not an eigenvalues of $(1)-(3)$ .Note also that the notations $n=-0$ and $n=+0$ are used fort he eigenvalues λ_n to have the asymptotic formula just in the form (13)).

4) Obviously, $\varphi_n(x)=\varphi(x,\lambda_n)$ is an eigenvalues λ_n . Let us set

$$\alpha_n:=\int_0^\pi\varphi_n^2(x)\,dx-\frac{1}{\lambda_n}\int_0^\pi p(x)\,\varphi_n^2(x)\,dx.$$

The numbers α_n we call the normalizing numbers of problem $(1)-(3)$.

5) the normalizing numbers α_n are positive and large negative and positive values of n the asymptotic formula

$$\alpha_n=\frac{\pi}{2}\Big[\left(\alpha^+\right)^2+\left(\alpha^-\right)^2\Big]+\frac{\alpha_{11}}{\lambda_n^0}+\frac{\alpha_{1n}}{n}$$

holds, where

$$\alpha_{11}=-\frac{\alpha\pi}{2}p(0),\ (\alpha_{1n})\in l_2 \quad (15)$$

6) For arbitrary function $f(x)$ in $L_2[0,\pi]$ the ''two fold'' expansion formulas

$$\sum_n\frac{1}{\lambda_n\alpha_n}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)\,dt=0 \quad (16)$$

$$\sum_n\frac{1}{2\alpha_n}\varphi(x,\lambda_n)\int_0^\pi f(t)\varphi(t,\lambda_n)\,dt=f(x) \quad (17)$$

Hold, where series converge in the metric of space $L_2[0,\pi]$. Every where in the infinite sums n runs all the values $n=\pm0,\pm1,\dots$.

7) The equality

$$\sum_n\frac{1}{\lambda_n\alpha_n}=0$$

Holds, where the infinite sum is understood in the sense of a principal value, i.e. as the limit of sums $\sum_{-N}^{N}$ as $N\to\infty$.

The inverse spectral problem consists in recovering the coefficient functions $q(x)$, $p(x)$ in equation (1) and the coefficient number α and a in jump conditions (3) from the spectral data $\{\lambda_n,\alpha_n\}$ of problem $(1)-(3)$.

3. Fundamental equations of the inverse problem

Assume that for problem $(1)-(3)$ the conditions stated above in section 2 are satisfied (see (12)). Let $\{\lambda_n\}$ be the eigenvalues and $\{\alpha_n\}$ be the corresponding normalizing numbers of problem $(1)-(3)$, where $n=\pm0,\pm1,\dots$. In this section we will derive some equations which allow formally solve the inverse spectral problem.

Let us set

$$H(x,t)=\sum_n\frac{1}{2\lambda_n\alpha_n}\varphi(x,\lambda_n)\exp(i\lambda_n t) \quad (18)$$

where n in the sum runs all the values $\pm 0, \pm 1, \pm 2, \ldots$.

Lemma 3.1. The equality

$$H(x,t) = 0 \ for \ 0 \le t < x \tag{19}$$

holds.

Theorem 3.1. The kernels $A(x,t)$ and $B(x,t)$ involved in representation (5) of $\varphi(x,\lambda)$ satisfy the following system of linear integral equations:

$$\begin{aligned} & l^+(x) F_{11}(x,t)\cos\beta^+(x) \\ & + l^+(x) F_{12}(x,t)\sin\beta^+(x) \\ & + l^-(x) F_{11}(2a-x,t)\cos\beta^-(x) \\ & + l^-(x) F_{12}(2a-x,t)\sin\beta^-(x) \\ & + A(x,t) + \int_0^x A(x,\xi) F_{11}(\xi,t)\,d\xi \\ & + \int_0^x B(x,\xi) F_{12}(\xi,t)\,d\xi = 0 \end{aligned} \tag{20}$$

$$\begin{aligned} & l^+(x) F_{21}(x,t)\cos\beta^+(x) \\ & + l^+(x) F_{22}(x,t)\sin\beta^+(x) \\ & + l^-(x) F_{21}(2a-x,t)\cos\beta^-(x) \\ & + l^-(x) F_{22}(2a-x,t)\sin\beta^-(x) \\ & + B(x,t) + \int_0^x A(x,\xi) F_{21}(\xi,t)\,d\xi \\ & + \int_0^x B(x,\xi) F_{22}(\xi,t)\,d\xi = 0 \end{aligned} \tag{21}$$

where defining $c_0 = \beta^+(x)$

$$\begin{aligned} F_{11}(x,t) &= \frac{1}{\pi}\cos c_0 x \cos c_0 t \\ & + \sum_n \left\{ \frac{1}{2a_n}\cos\lambda_n x \cos\lambda_n t \right. \\ & \left. - \frac{1}{\pi}\cos(n+c_0)x\cos(n+c_0)t \right\} \end{aligned} \tag{22}$$

$$\begin{aligned} F_{12}(x,t) &= \frac{1}{\pi}\sin c_0 x \cos c_0 t \\ & + \sum_n \left\{ \frac{1}{2a_n}\sin\lambda_n x \cos\lambda_n t \right. \\ & \left. - \frac{1}{\pi}\sin(n+c_0)x\cos(n+c_0)t \right\} \end{aligned} \tag{23}$$

$$\begin{aligned} F_{21}(x,t) &= \frac{1}{\pi}\cos c_0 x \sin c_0 t \\ & + \sum_n \left\{ \frac{1}{2a_n}\cos\lambda_n x \sin\lambda_n t \right. \\ & \left. - \frac{1}{\pi}\cos(n+c_0)x\sin(n+c_0)t \right\} \end{aligned} \tag{24}$$

$$\begin{aligned} F_{22}(x,t) &= \frac{1}{\pi}\sin c_0 x \sin c_0 t \\ & + \sum_n \left\{ \frac{1}{2a_n}\sin\lambda_n x \sin\lambda_n t \right. \\ & \left. - \frac{1}{\pi}\sin(n+c_0)x\sin(n+c_0)t \right\} \end{aligned} \tag{25}$$

Proof. Substituting here for $\varphi(x,\lambda_n)$ the expression

$$\varphi(x,\lambda_n) = \varphi_0(x,\lambda_n) +$$

$$\begin{aligned} \varphi_0(x,\lambda) &= l^+(x)\cos\left[\lambda_n x - \beta^+(x)\right] \\ & + l^-(x)\cos\left[\lambda_n(2a-x) + \beta^-(x)\right] \end{aligned}$$

and Lemma 3.1 then to zero the real and imaginary parts of obtained equation, we get

$$\begin{aligned} & \frac{1}{2a_n}\sum_n \left\{ \left[l^+(x)\cos\left[\lambda_n x - \beta^+(x)\right] \right. \right. \\ & \left. + l^-(x)\cos\left[\lambda_n(2a-x) + \beta^-(x)\right] \right]\cos\lambda_n t \\ & + \int_0^x A(x,\xi)\cos\lambda_n\xi\cos\lambda_n t d\xi \\ & \left. + B(x,\xi)\sin\lambda_n\xi\cos\lambda_n t dt \right\} = 0 \end{aligned}$$

$$\frac{1}{2a_n}\sum_n\Big\{\Big[l^+(x)\cos\big[\lambda_n x-\beta^+(x)\big]$$
$$+l^-(x)\cos\big[\lambda_n(2a-x)+\beta^-(x)\big]\Big]\sin\lambda_n t$$
$$+\int_0^x A(x,\xi)\cos\lambda_n\xi\sin\lambda_n t d\xi$$
$$+B(x,\xi)\sin\lambda_n\xi\sin\lambda_n tdt\Big\}=0$$

for $0\le t<\alpha<x.$ next, substituting equality

$$\sum_n \cos nx\sin nt=0$$
$$-\frac{1}{\pi}+\frac{1}{\pi}\sum_n\cos nx\cos nt=\delta(x-t),$$
$$\frac{1}{\pi}\sum_n\sin nx\sin nt=\delta(x-t)$$

We obtained fundamental equations (20) and (21) of the inverse problem.

4. Algorithm for solving the inverse problem

The equations (20) and (21) are called the fundamental equations of the inverse problem. They allow to solve the inverse problem and prove Theorem 2.2 as follows.

Let a collection of numbers $\{\lambda_n,a_n\}$ be given that satisfies the conditions of Theorem 2.2. Using this collection we construct the functions $F_{jx}(x,t)(j,x=1,2)$ by (22)-(25) and consider for each x the system of Fredholm linear integral equations (20)-(21) with respect to unknow functions $A(x,t)$ and $B(x,t)$ assuming $\beta^{\pm}(x)$ in this equations an arbitrary given function. It turns out that these equations are uniquely solvable and dependence of its solution on $\beta^{\pm}(x)$ can be expressed explicitly.

Theorem 4.1: For any continuous function $\beta^{\pm}(x)$ the system of integral equations (20) and (21) has a unique solution $A(x,t),B(x,t)$ and dependence of this solution on the function $\beta^{+}(x)$ is expressed by the formulas

$$A(x,t)=A_0(x,t)\cos\beta^+(x)$$
$$+A_1(x,t)\sin\beta^+(x) \quad (26)$$

$$B(x,t)=B_0(x,t)\cos\beta^+(x)$$
$$+B_1(x,t)\sin\beta^+(x) \quad (27)$$

where $A_0(x,t)$, $B_0(x,t)$ form the solution of system (20) and (21) with $\beta^+(x)\equiv 0$,

$$l^+(x)F_{11}(x,t)+A_0(x,t)$$
$$+l^-(x)F_{11}(2a-x,t)$$
$$+\int_0^x A_0(x,\xi)F_{11}(\xi,t)d\xi \quad (28)$$
$$+\int_0^x B_0(x,\xi)F_{12}(\xi,t)d\xi=0,$$
$$0\le t<x$$

$$l^+(x)F_{21}(x,t)+B_0(x,t)$$
$$+l^-(x)F_{21}(x,t)$$
$$+\int_0^x A_0(x,\xi)F_{21}(\xi,t)d\xi \quad (29)$$
$$+\int_0^x B_0(x,\xi)F_{22}(\xi,t)d\xi=0,$$
$$0\le t<x$$

where $A_1(x,t)$, $B_1(x,t)$ form the solution of system (20) and (21) with $\beta^+(x)\equiv\frac{\pi}{2}$,

$$l^+(x)F_{12}(x,t)+A_1(x,t)$$
$$+l^-(x)F_{12}(2a-x,t)$$
$$+\int_0^x A_1(x,\xi)F_{11}(\xi,t)d\xi \quad (30)$$
$$+\int_0^x B_1(x,\xi)F_{12}(\xi,t)d\xi=0,$$
$$0\le t<x$$

$$
\begin{aligned}
&l^{+}(x)F_{22}(x,t)+B_1(x,t)\\
&+l^{-}(x)F_{22}(2a-x,t)\\
&+\int_0^x A_1(x,\xi)F_{21}(\xi,t)\,d\xi\\
&+\int_0^x B_1(x,\xi)F_{22}(\xi,t)\,d\xi=0,\\
&0\le t<x
\end{aligned}
\tag{31}
$$

We need to get an equation for $\beta^{+}(x)$. In the part of direct spectral problem we have relation (6). Substituting (26) and (27) in this relation we get for $\beta^{+}(x)$ the nonlinear Volterra integral equation

$$\alpha^{+}\beta^{+}(x)=\alpha^{+}xp(0)+\int_0^x \Phi\left(\xi,\beta^{+}(\xi)\right)d\xi \tag{32}$$

Where

$$
\begin{aligned}
&\Phi(\xi,z)=\\
&2A_1(\xi,\xi)\sin^2 z-2B_0(\xi,\xi)\cos^2 z\\
&+\left[A_0(\xi,\xi)-B_1(\xi,\xi)\right]\sin 2z
\end{aligned}
\tag{33}
$$

Thus we get the following algoritm for solution of the inverse problem.
Given a collection of numbers $\{\lambda_n,a_n\}$ satisfying the conditions of Theorem 2.2, we construct the functions $F_{jx}(x,t)(j,x=1,2)$ by (22)-(25) and consider the two sistems of equations (28)-(29) and (30)-(31) with respect to $A_0(x,t)$, $B_0(x,t)$ and $A_1(x,t)$, $B_1(x,t)$ respectively. Solving these systems we find $A_0(x,t)$, $B_0(x,t)$ and $A_1(x,t)$, $B_1(x,t)$. Then we form the function $\Phi(\xi,z)$ by (33) and consider equation (32) for $\beta^{+}(x)$ where the number $p(0)$ is taken from $p(0)=-\dfrac{2\alpha_{11}}{\alpha\pi}$ according to (15) with α_{11} given in (14). Solving this equation we find $\alpha(x)$ and then $p(x)$ *by* $p(x)=\left(\beta^{+}(x)\right)'$ according to (9). Next, define $A(x,t)$, $B(x,t)$ by (26)-(27) and then $q(x)$ by (7). These reasonings prove, in particular, the uniqueness of solution of the inverse problem: The coefficient functions $p(x)$, $q(x)$ of equation (1) and the numbers α, a in jump conditions (3) are determined uniquely from the spectral data $\{\lambda_n,a_n\}$ of the boundary value problem $(1)-(3)$.

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