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Research Article

Coefficients estimates of a new class of analytic bi-univalent functions with bounded boundary rotation

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Abstract

In this paper, we introduce a new subclass of analytic bi-univalent functions defined by using q-derivative operator. Further, we obtain both some initial and general coefficient bounds, and also Fekete-Szegö inequalities for bi-univalent functions that belong to this class. Our results extend and improve some known results as special cases.

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1. Introduction and definitions

Let $\mathfrak{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{P}(\beta)$ be the class of analytic functions satisfying the condition $\Re h(z) > \beta$ in $\mathbb{D}$ with $h(0) = 1$.

Definition 1.1 ([15])**.** Let $\mathbb{P}_m(\beta)$ denote the class of analytic functions $h(z)$ in $\mathbb{D}$, satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re h(z) - \beta}{1 - \beta} \right| d\theta \leq m\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $m \geq 2$ and $0 \leq \beta < 1$.

For $m = 2$, $\mathbb{P}_2(\beta) = \mathbb{P}(\beta)$. When $\beta = 0$, $\mathbb{P}_m(\beta)$ reduces to the class $\mathbb{P}_m(0) = \mathbb{P}_m$ defined by Pinchuk [16]. Also we get the well known class $\mathbb{P}_2(0) = \mathbb{P}$ of Carathéodory functions.

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Definition 1.2 ([12, 13]). For $0 < q < 1$, the q-derivative operator $\mathfrak{D}_q f$ of a function $f \in \mathfrak{A}$ given by (1.1) is defined by

$$\mathfrak{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.3}$$

provided $f'(0)$ exists.

We note from Definition 1.2 that

$$\lim_{q \to 1^-} \mathfrak{D}_q f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function f, which is differentiable in $\mathbb{D}$. It is deduced from (1.1) and (1.3) that

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

We also note that $[n]_q \to n$ as $q \to 1^-$.

Let $\mathfrak{S}$ denote the subclass of $\mathfrak{A}$ consisting of univalent functions in $\mathbb{D}$. The Koebe one-quarter theorem [7] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathfrak{S}$ contains the disk with center in the origin and radius $1/4$. Thus, every univalent function $f \in \mathfrak{S}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$, satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.5}$$

A function $f \in \mathfrak{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, supposing that $\mathbb{D} \subseteq f(\mathbb{D})$, and we denote the class of bi-univalent functions by Σ.

Using the Faber polynomial expansion [8] of functions $f \in \Sigma$, the coefficients of its inverse map $g = f^{-1}$ may be expressed [3] as follows:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} \mathbb{K}_{n-1}^{-n}(a_2, a_3 ..., a_n)\, w^n, \tag{1.6}$$

where

$$\begin{aligned} \mathbb{K}_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5}[a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6}[a_6 + (-2n+5)a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. In particular, the first three terms of $\mathbb{K}_{n-1}^{-n}$ are

$$\mathbb{K}_1^{-2} = -2a_2, \quad \mathbb{K}_2^{-3} = 3\left(2a_2^2 - a_3\right), \quad \mathbb{K}_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, an expansion of $\mathbb{K}_{n-1}^p$ is given ([1], [17]) by

$$\begin{aligned} &\mathbb{K}_{n-1}^p\left(a_2, a_3..., a_n\right) \\ &= pa_n + \frac{p!}{(p-2)!2!}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + ... + \frac{p!}{(p-n+1)!\,(n-1)\,!}D_{n-1}^{n-1}, \end{aligned} \tag{1.7}$$

where $D_{n-1}^p = D_{n-1}^p\left(a_2, a_3..., a_n\right)$ (for details, see [18]). We also have ([1], [17])

$$D_{n-1}^l\left(a_2, a_3..., a_n\right) = \sum_{n=1}^{\infty} \frac{l!}{\mu_1! \cdots \mu_{n-1}!} a_2^{\mu_1} \cdots a_n^{\mu_{n-1}},$$

and the sum is taken over all non-negative integers $\mu_1, \ldots, \mu_{n-1}$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_{n-1} = l, \\ \mu_1 + 2\mu_2 + \cdots + (n-1)\,\mu_{n-1} = n-1. \end{cases}$$

It is clear that $D_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$ [3].

Lewin [14] investigated the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [6] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the classes of bi-starlike functions and bi-convex functions, and obtained estimates on the initial coefficients. Recently, Ali et al. [4], Srivastava et al. [19], Frasin and Aouf [9], Goyal and Goswami [10], Aljouiee and Goswami [5] and many others have introduced and investigated subclasses of bi-univalent functions and investigated bounds for the initial coefficients.

In the light of this definitions, the purpose of this paper is to define a new subclass of analytic bi-univalent functions by means of the q-derivative and to obtain both initial and general coefficient bounds for functions belonging to this new class. We also investigate Fekete-Szegö problem.

Definition 1.3. For $p \in \mathbb{N} = \{1, 2, \ldots\}$, $m \geq 2$, $0 \leq \beta < 1$, $0 < q < 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{H}_\Sigma^p(q; m; \beta)$, if the following conditions are satisfied:

$$(\mathfrak{D}_q f(z))^p \in \mathbb{P}_m(\beta) \quad (z \in \mathbb{D}) \tag{1.8}$$

and

$$(\mathfrak{D}_q g(w))^p \in \mathbb{P}_m(\beta) \quad (w \in \mathbb{D}) \tag{1.9}$$

where $g = f^{-1}$ is defined by (1.5).

Remark 1.4. As a special case to Definition 1.3, by setting $p = 1$ we have a new subclass $\mathfrak{H}_\Sigma^1(q; m; \beta) =: \mathfrak{H}_\Sigma(q; m; \beta)$ of analytic bi-univalent functions which consist of functions $f \in \Sigma$ satisfying

$$\mathfrak{D}_q f(z) \in \mathbb{P}_m(\beta) \quad (z \in \mathbb{D})$$

and

$$\mathfrak{D}_q g(w) \in \mathbb{P}_m(\beta) \quad (w \in \mathbb{D})$$

where $g = f^{-1}$ is defined by (1.5).

Remark 1.5.
(i) For $q \to 1^-$, $\mathfrak{H}_\Sigma^p(q; m; \beta)$ is the class $\mathfrak{BR}_\Sigma^p(m; \beta)$ introduced by Aljouiee and Goswami [5].
(ii) For $m = 2$ and $q \to 1^-$, $\mathfrak{H}_\Sigma^p(q; m; \beta)$ is the class introduced by Jahangiri et al. [11].
(iii) For $p = 1$, $m = 2$ and $q \to 1^-$, $\mathfrak{H}_\Sigma^p(q; m; \beta)$ is the class introduced by Srivastava et al. [19].

2. Coefficient bounds

Throughout this paper, we suppose that

$$m \geq 2, \quad 0 \leq \beta < 1, \quad 0 < q < 1, \quad p \in \mathbb{N}.$$

In this section, we investigate the bounds of coefficients of Taylor-Maclaurin series expansion for functions $f \in \mathfrak{H}_{\Sigma}^{p}(q;m;\beta)$. Firstly, by means of Faber polynomial expansion, we will obtain general coefficient bounds (Theorem 2.3 below), and then we will get initial coefficient bounds (Theorem 2.6 below).

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([16])**.** *Let the function* Φ *given by* $\Phi(z) = 1 + \sum\limits_{n=1}^{\infty} h_n z^n$ *be convex in* $\mathbb{D}$*. If* $\Phi(z) \in \mathbb{P}_m$*, then*

$$|h_n| \leq m \quad (n \in \mathbb{N}).$$

Lemma 2.2 ([2])**.** *If* $\phi(z) = 1 + \sum\limits_{n=1}^{\infty} \phi_n z^n$ *is an analytic function in* $\mathbb{D}$*, then*

$$(\phi(z))^p = 1 + \sum_{n=1}^{\infty} \mathbb{K}_n^p \left(\phi_1, \phi_2, \cdots, \phi_n\right) z^n. \tag{2.1}$$

Theorem 2.3. *Let the function* $f \in \mathfrak{H}_{\Sigma}^{p}(q;m;\beta)$ *be given by* (1.1)*. If* $a_k = 0$ *for* $2 \leq k \leq n-1$*, then we have*

$$|a_n| \leq \frac{m(1-\beta)}{p\,[n]_q} \quad (n \geq 3)\,. \tag{2.2}$$

Proof. Since f is of the form (1.1), then by (1.4) and Lemma 2.2, we obtain

$$(\mathfrak{D}_q f(z))^p = 1 + \sum_{n=1}^{\infty} \mathbb{K}_n^p \left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\right) z^n, \qquad \tilde{a}_n = [n+1]_q\, a_{n+1}. \tag{2.3}$$

Similarly, for $g = f^{-1}$ given by (1.5) and (1.6), we have

$$\mathfrak{D}_q g(w) = 1 + \sum_{n=2}^{\infty} \frac{[n]_q}{n} \mathbb{K}_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^{n-1} =: 1 + \sum_{n=1}^{\infty} \tilde{A}_n w^n. \tag{2.4}$$

Consequently, by Lemma 2.2, we get

$$(\mathfrak{D}_q g(w))^p = 1 + \sum_{n=1}^{\infty} \mathbb{K}_n^p(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n) w^n, \quad \tilde{A}_n = \frac{[n+1]_q}{n+1} \mathbb{K}_n^{-(n+1)}(a_2, a_3, \cdots, a_{n+1}). \tag{2.5}$$

Now from the Definition 1.3, there exists two functions

$$\psi(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathbb{P}_m$$

and

$$\varphi(w) = 1 + d_1 w + d_2 w^2 + \cdots \in \mathbb{P}_m$$

such that

$$(\mathfrak{D}_q f(z))^p = \beta + (1-\beta)\,\psi(z) = 1 + (1-\beta)\,c_1 z + (1-\beta)\,c_2 z^2 + \cdots \tag{2.6}$$

and

$$(\mathfrak{D}_q g(w))^p = \beta + (1-\beta)\,\varphi(w) = 1 + (1-\beta)\,d_1 w + (1-\beta)\,d_2 w^2 + \cdots, \tag{2.7}$$

respectively. Now comparing the coefficients of equations (2.3) and (2.6), it gives

$$\mathbb{K}_n^p \left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\right) = (1-\beta)\,c_n, \qquad \tilde{a}_n = [n+1]_q\, a_{n+1} \tag{2.8}$$

for all n. Similarly, from (2.5) with (2.7), we get

$$\mathbb{K}_n^p(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n) = (1-\beta)\, d_n, \qquad \tilde{A}_n = \frac{[n+1]_q}{n+1}\mathbb{K}_n^{-(n+1)}(a_2, a_3, ..., a_{n+1}) \tag{2.9}$$

for all n. If $a_k = 0$ for $2 \le k \le n-1$, then combining the equations (2.8) and (2.9) and using relation (1.7), it yields

$$\begin{aligned} p[n]_q\, a_n &= (1-\beta)\, c_{n-1}, \\ -p[n]_q\, a_n &= (1-\beta)\, d_{n-1}. \end{aligned}$$

By taking absolute value both sides on above equations and using Lemma 2.1, we get

$$|a_n| \le \frac{(1-\beta)\,|c_{n-1}|}{p\,[n]_q} = \frac{(1-\beta)\,|d_{n-1}|}{p\,[n]_q} \le \frac{m(1-\beta)}{p\,[n]_q} \qquad (n \ge 3)\,.$$

□

Remark 2.4. (i) If we take $q \to 1^-$ in Theorem 2.3, we get the result obtained by Aljouiee and Goswami [5, Theorem 4].
(ii) If we take $q \to 1^-$ and $m = 2$ in Theorem 2.3, we get the result obtained by Jahagiri et al. [11, Theorem 2.1].

Letting $p = 1$ in Theorem 2.3, we get the following consequence.

Corollary 2.5. *Let the function $f \in \mathfrak{H}_\Sigma(q; m; \beta)$ be given by (1.1). If $a_k = 0$ for $2 \le k \le n-1$, then we have*

$$|a_n| \le \frac{m(1-\beta)}{[n]_q} \quad (n \ge 3)\,.$$

If we relax the condition $a_k = 0$ in Theorem 2.3, then we have following theorem.

Theorem 2.6. *Let the function $f \in \mathfrak{H}_\Sigma^p(q; m; \beta)$ be given by (1.1). Then*

$$|a_2| \le \begin{cases} \sqrt{\frac{2m(1-\beta)}{p(p-1)[2]_q^2 + 2p[3]_q}}, & 0 \le \beta \le 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2[3]_q\right)}, \\ \frac{m(1-\beta)}{p[2]_q}, & 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2[3]_q\right)} \le \beta < 1, \end{cases} \tag{2.10}$$

$$|a_3| \le \begin{cases} \frac{2m(1-\beta)}{p(p-1)[2]_q^2 + 2p[3]_q} + \frac{m(1-\beta)}{p[3]_q}, & 0 \le \beta \le 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2[3]_q\right)}, \\ \frac{m^2(1-\beta)^2}{p^2[2]_q^2} + \frac{m(1-\beta)}{p[3]_q}, & 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2[3]_q\right)} \le \beta < 1, \end{cases} \tag{2.11}$$

and

$$\left|a_3 - a_2^2\right| \le \frac{m(1-\beta)}{p\,[3]_q}. \tag{2.12}$$

Proof. Since $f \in \mathfrak{H}_\Sigma^p(q; m; \beta)$, from (2.3) with (2.8) , and (2.5) with (2.9), we have

$$p[2]_q a_2 = (1-\beta)c_1 \tag{2.13}$$

and

$$-p[2]_q a_2 = (1-\beta)d_1, \tag{2.14}$$

respectively. Taking absolute value in both equations and using Lemma 2.1, it gives

$$|a_2| \le \frac{m(1-\beta)}{p[2]_q}.$$

On the other hand, from equations (2.8), (2.9) and (1.7), it follows that

$$\frac{p(p-1)}{2}[2]_q^2 a_2^2 + p[3]_q a_3 = (1-\beta)c_2 \tag{2.15}$$

and

$$\left(\frac{p(p-1)}{2}[2]_q^2 + 2p\,[3]_q\right) a_2^2 - p[3]_q a_3 = (1-\beta)d_2. \tag{2.16}$$

Adding (2.15) to (2.16), we obtain

$$a_2^2 = \frac{(1-\beta)(c_2+d_2)}{p(p-1)[2]_q^2 + 2p\,[3]_q}. \tag{2.17}$$

Taking absolute value in (2.13) and (2.17) and using Lemma 2.1, we have the coefficient bounds for a_2 given by (2.10).

Subtracting (2.16) from (2.15), we get

$$2p[3]_q a_3 - 2p[3]_q a_2^2 = (1-\beta)(c_2-d_2).$$

Again by Lemma 2.1, we get desired result in (2.12).

Further, we obtain from the above equality that

$$a_3 = a_2^2 + \frac{(1-\beta)(c_2-d_2)}{2p[3]_q}. \tag{2.18}$$

Taking absolute value in (2.18), we obtain

$$|a_3| \le |a_2|^2 + \frac{(1-\beta)(|c_2|+|d_2|)}{2p[3]_q}. \tag{2.19}$$

Applying Lemma 2.1, we get from (2.10) and (2.19) that

$$|a_3| \le \frac{2m(1-\beta)}{p(p-1)[2]_q^2 + 2p\,[3]_q} + \frac{m(1-\beta)}{p[3]_q}$$

for

$$0 \le \beta \le 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2\,[3]_q\right)}$$

and that

$$|a_3| \le \frac{m^2(1-\beta)^2}{p^2[2]_q^2} + \frac{m(1-\beta)}{p[3]_q}$$

for

$$\beta \ge 1 - \frac{2p[2]_q^2}{m\left((p-1)[2]_q^2 + 2\,[3]_q\right)}.$$

This completes the proof of Theorem 2.6. □

Remark 2.7. (i) If we take $q \to 1^-$ in Theorem 2.6, we get the result obtained by Aljouiee and Goswami [5, Theorem 5].

(ii) If we take $q \to 1^-$ and $m = 2$ in Theorem 2.6, we get the result obtained by Jahagiri et al. [11, Theorem 2.2].

Letting $p = 1$ in Theorem 2.6, we get the following consequence.

Corollary 2.8. *Let the function $f \in \mathfrak{H}_\Sigma(q;m;\beta)$ be given by (1.1). Then*

$$|a_2| \le \begin{cases} \sqrt{\frac{m(1-\beta)}{[3]_q}}, & 0 \le \beta \le 1 - \frac{[2]_q^2}{m[3]_q}, \\ \frac{m(1-\beta)}{[2]_q}, & 1 - \frac{[2]_q^2}{m[3]_q} \le \beta < 1, \end{cases}$$

$$|a_3| \leq \begin{cases} \frac{2m(1-\beta)}{[3]_q}, & 0 \leq \beta \leq 1 - \frac{[2]_q^2}{m[3]_q}, \\ \\ \frac{m^2(1-\beta)^2}{[2]_q^2} + \frac{m(1-\beta)}{[3]_q}, & 1 - \frac{[2]_q^2}{m[3]_q} \leq \beta < 1, \end{cases}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{m(1-\beta)}{[3]_q}.$$

3. Fekete-Szegö problem

In this section, we obtain Fekete-Szegö inequalities for functions $f \in \mathfrak{H}_\Sigma^p(q;m;\beta)$. For this purpose, we need the following lemma.

Lemma 3.1 ([20]). *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l)\, z_1 + (k-l)\, z_2| \leq \begin{cases} 2R\,|k|\,, & |k| \geq |l|\,, \\ \\ 2R\,|l|\,, & |k| \leq |l|\,. \end{cases}.$$

Theorem 3.2. *Let the function $f \in \mathfrak{H}_\Sigma^p(q;m;\beta)$ be given by (1.1) and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2m(1-\beta)}{p(p-1)[2]_q^2+2p[3]_q}\,|1-\mu|\,, & \mu \in \left(-\infty,\, -\frac{(p-1)[2]_q^2}{2[3]_q}\right] \cup \left[2 + \frac{(p-1)[2]_q^2}{2[3]_q}, \infty\right), \\ \\ \frac{m(1-\beta)}{p[3]_q}, & \mu \in \left[-\frac{(p-1)[2]_q^2}{2[3]_q},\, 2 + \frac{(p-1)[2]_q^2}{2[3]_q}\right], \end{cases}.$$

Proof. From (2.17) and (2.18), we can write

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\,\frac{(1-\beta)\,(c_2+d_2)}{p(p-1)[2]_q^2 + 2p\,[3]_q} + \frac{(1-\beta)\,(c_2-d_2)}{2p[3]_q} \\ &= (1-\beta)\left(\frac{1-\mu}{p(p-1)[2]_q^2 + 2p\,[3]_q} + \frac{1}{2p[3]_q}\right) c_2 \\ &\quad +(1-\beta)\left(\frac{1-\mu}{p(p-1)[2]_q^2 + 2p\,[3]_q} - \frac{1}{2p[3]_q}\right) d_2. \end{aligned}$$

By Lemma 3.1, we obtain from the above equality that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 2m\,(1-\beta)\,|H(\mu)|\,, & |H(\mu)| \geq \frac{1}{2p[3]_q}, \\ \\ \frac{m(1-\beta)}{p[3]_q}, & |H(\mu)| \leq \frac{1}{2p[3]_q}, \end{cases}$$

where

$$H(\mu) = \frac{1-\mu}{p(p-1)[2]_q^2 + 2p\,[3]_q}.$$

This completes the proof. □

Remark 3.3. If we take $q \to 1^-$, $m = 2$ and $p = 1$ in Theorem 3.2, we get the result obtain in [21].

If we take $q \to 1^-$ in Theorem 3.2, we get following result.

Corollary 3.4. *Let the function $f \in \mathfrak{B}\mathfrak{R}_\Sigma^p(m;\beta)$ be given by (1.1) and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{m(1-\beta)}{p(2p+1)}\,|1-\mu|\,, & \mu \in \left(-\infty,\, -\frac{2(p-1)}{3}\right] \cup \left[\frac{2(p+2)}{3}, \infty\right), \\ \\ \frac{m(1-\beta)}{p[3]_q}, & \mu \in \left[-\frac{2(p-1)}{3},\, \frac{2(p+2)}{3}\right]. \end{cases}$$

If we take $q \to 1^-$ and $m = 2$ in Theorem 3.2, we get following result.

Corollary 3.5. *Let the function* $f \in \mathfrak{R}(p;\beta)$ *be given by* (1.1) *be in the class and* $\mu \in \mathbb{R}$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{2(1-\beta)}{p(2p+1)}\left|1-\mu\right|, & \mu \in \left(-\infty,\ -\frac{2(p-1)}{3}\right] \cup \left[\frac{2(p+2)}{3}, \infty\right), \\ \frac{2(1-\beta)}{3p}, & \mu \in \left[-\frac{2(p-1)}{3},\ \frac{2(p+2)}{3}\right]. \end{cases}.$$

Letting $p = 1$ in Theorem 3.2, we get the following consequence.

Corollary 3.6. *Let the function* $f \in \mathfrak{H}_\Sigma(q;m;\beta)$ *be given by* (1.1) *and* $\mu \in \mathbb{R}$. *Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{m(1-\beta)}{[3]_q}\left|1-\mu\right|, & \mu \in (-\infty, 0] \cup [2, \infty), \\ \frac{m(1-\beta)}{[3]_q}, & \mu \in [0, 2]. \end{cases}$$

4. Conclusion

In this paper, we have obtained the coefficient bounds for the class of $\mathfrak{H}_\Sigma^p(q;m;\beta)$ with the help of Faber polynomial. Further, we have derived the Fekete-Szegö problem for the same class.

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