





On The Solution of an Infinite System of Discrete Equations

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ABSTRACT. In this work, we construct the transformation operator for the infinite system of the difference equations

$$a_{n-2}y_{n-2} + b_{n-1}y_{n-1} + c_n y_n + b_n y_{n+1} + a_n y_{n+2} = \lambda y_n \quad (n = 1, 2, \dots),$$

where $a_n \neq 0, b_n, c_n$ ($n = 1, 2, 3, \dots$) are given complex numbers, investigate some important properties of the special solutions of the difference system.

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1. INTRODUCTION

Consider the linear difference equation of the second order

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n = 1, 2, \dots \quad (1.1)$$

where $\{y_n\}_{n=0}^{\infty}$ is an unknown infinite vector called a solution, λ is a complex parameter and $a_n \neq 0, b_n$ are given complex numbers. G.Sh. Guseinov [1] investigated this infinite system of difference equations to define the three diagonal Jacobi matrix by the scattering data. He proved that if $a_n > 0, \text{Im } b_n = 0$ ($n = 1, 2, \dots$) and

$$\sum_{n=1}^{\infty} n(|1 - a_n| + |b_n| + |c_n|) < \infty$$

then the scattering data for the equation (1.1) uniquely defined the sequences (a_n) and (b_n) . In this work G.Sh. Guseinov improved the Marchenko method for the difference system (1.1) by constructing an analogue of the transformation operator for the Sturm-Liouville operator. Direct and inverse spectral problems for nonselfadjoint Jacobi matrices also were investigated by G.Sh. Guseinov [3] who gave a procedure for construction of a spectral function for the system (1.1). The full line inverse scattering problem for the second order difference system (1.1) in various statements were investigated in [2, 4, 5] by using transformation operators and with application of the Marchenko method.

Nonselfadjoint difference operators of an even order was considered by Yu.L. Kishakevich [6] for constructing the Parsevall-Marchenko identity. Using matrix moment problem he showed how to obtain the Parsevall identity for the selfadjoint case. Recently, S. M Zagorodnyuk in the series of his studies [8, 9] has considered the direct and inverse spectral problems for $(2N + 1)$ - diagonal Jacobi matrices. In these works the author used a different method for the construction of the spectral function and extended Guseinov's results to the case of $(2N + 1)$ -diagonal complex

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symmetric matrices. But in these studies the special Fourier type series representations of the solutions of an even order difference system was not considered and logically the direct and inverse scattering problems for the $(2N + 1)$ -diagonal Jacobi matrices as generalization of G.Sh. Guseinov’s result have not yet investigated.

In this work, as the first step of the investigation the direct and inverse scattering problems for the five-diagonal Jacobi matrices by the Marchenko method, we construct the transformation operator for the infinite system of the difference equations

$$a_{n-2}y_{n-2} + b_{n-1}y_{n-1} + c_n y_n + b_n y_{n+1} + a_n y_{n+2} = \lambda y_n, \quad n = 1, 2, \dots \tag{1.2}$$

where $a_n \neq 0, b_n, c_n (n = 1, 2, 3, \dots)$ are given complex numbers, and investigate some important properties of the special solutions of the system (1.2).

2. THE TRANSFORMATION OPERATOR FOR THE FIVE DIAGONAL JACOBI MATRIX

We consider the infinite system of discrete equations

$$\begin{aligned} c_1 y_1 + b_1 y_2 + a_1 y_3 &= \lambda y_1, \\ b_1 y_1 + c_2 y_2 + b_2 y_3 + a_2 y_4 &= \lambda y_2, \\ a_{n-2} y_{n-2} + b_{n-1} y_{n-1} + c_n y_n + b_n y_{n+1} + a_n y_{n+2} &= \lambda y_n, \quad n = 3, 4, \dots \end{aligned} \tag{2.1}$$

where $\{y_n\}_{n=1}^\infty$ is an unknown infinite vector called a solution, λ is a complex parameter and $a_n \neq 0, b_n, c_n (n = 1, 2, 3, \dots)$ are given complex numbers such that the series

$$\sum_{n=1}^\infty n (|1 - a_n| + |b_n| + |c_n|) \tag{2.2}$$

converges. The equation (2.1) is equivalent to defining a vector $\{y_n\}_{n=-1}^\infty$ which satisfies

$$a_{n-2}y_{n-2} + b_{n-1}y_{n-1} + c_n y_n + b_n y_{n+1} + a_n y_{n+2} = \lambda y_n, \quad n = 1, 2, \dots; a_{-1} = a_0 = 1, b_0 = 0 \tag{2.3}$$

under conditions

$$y_{-1} = 0, \quad y_0 = 0. \tag{2.4}$$

This system of difference equations can be written by the help of the five-diagonal Jacobi matrix in the form

$$JY = \lambda Y$$

where $Y = (y_1, y_2, \dots)^T$ and

$$J = \begin{pmatrix} c_1 & b_1 & a_1 & 0 & 0 & 0 & \dots & \dots \\ b_1 & c_2 & b_2 & a_2 & 0 & 0 & \dots & \dots \\ a_1 & b_2 & c_3 & b_3 & a_3 & 0 & \dots & \dots \\ 0 & a_2 & b_3 & c_4 & b_4 & a_4 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Difference or discrete equations of type (2.3) often arises in various problems related to physics and engineering. The second order difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n = 1, 2, \dots$$

is a simple discretization of the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda y$$

which has a lot of applications in physics and quantum mechanics [7].

If $a_n = 1, b_n = 0, c_n = 0$ then for $\lambda = 2 \cos 2\xi$ Eq. (2.3) has the simple solution $f_n^0(\xi) = e^{in\xi}$. Using this fact we require that the problem (2.3)-(2.4) has a solution $f_n(\xi)$ which is presented as

$$f_n(\xi) = \alpha_n e^{in\xi} \left(1 + \sum_{m=1}^\infty K_{n,m} e^{im\xi} \right), \quad n = 1, 2, \dots \tag{2.5}$$

Here α_n and $K_{n,m} (n = 1, 2, \dots; m = 1, 2, \dots)$ are complex numbers and satisfy the conditions

$$\lim_{n \rightarrow \infty} \alpha_n = 1, \quad \lim_{n \rightarrow \infty} K_{n,m} = 0 \quad (m = 1, 2, \dots).$$

We can extend $K_{n,m}$ as $K_{n,m} = 0$ for $m \leq 0$ if we need it elsewhere.

The solution (2.5) of the discrete system (2.3) is called transformation operator for the fourth order difference operator and it is important in the investigation of the spectral problems, instantly, inverse scattering problems for the equation (2.3) (see [1, 3, 7]). Substituting $y_n = f_n(\xi)$ in equations (2.3) after some simple transformation we have

$$\alpha_n = \alpha_{n-2}a_{n-2}, \tag{2.6}$$

$$\alpha_{n-1}b_{n-1} + \alpha_n K_{n-2,1} - \alpha_n K_{n,1} = 0, \tag{2.7}$$

$$c_n \alpha_n + \alpha_n K_{n-2,2} - \alpha_n K_{n,2} + \alpha_{n-1} b_{n-1} K_{n-1,1} = 0, \tag{2.8}$$

$$b_n \alpha_{n+1} + \alpha_n K_{n-2,3} - \alpha_n K_{n,3} + \alpha_{n-1} b_{n-1} K_{n-1,2} + \alpha_n c_n K_{n,1} = 0, \tag{2.9}$$

$$\begin{aligned} \alpha_n a_n^2 - \alpha_n + \alpha_n K_{n-2,4} - \alpha_n K_{n,4} + \alpha_{n-1} b_{n-1} K_{n-1,3} + \\ + c_n \alpha_n K_{n,2} + \alpha_{n+1} b_n K_{n+1,1} = 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \alpha_n a_n^2 K_{n+2,m} - \alpha_n K_{n,m} + \alpha_{n+1} b_n K_{n+1,m+1} + c_n \alpha_n K_{n,m+2} + \\ + \alpha_{n-1} b_{n-1} K_{n-1,m+3} + \alpha_n K_{n-2,m+4} - \alpha_n K_{n,m+4} = 0, \quad m \geq 1. \end{aligned} \tag{2.11}$$

Now from the equations (2.6)-(2.11) we can find that all these equations are equivalent to the following equations:

$$\begin{aligned} \alpha_n^2 K_{n+2,m} - K_{n,m} + \frac{\alpha_{n+1}}{\alpha_n} b_n K_{n+1,m+1} + c_n K_{n,m+2} + \\ + \frac{\alpha_{n-1}}{\alpha_n} b_{n-1} K_{n-1,m+3} + K_{n-2,m+4} - K_{n,m+4} = 0, \quad m \geq 1, \end{aligned} \tag{2.12}$$

$$b_n = \frac{\alpha_{n+1}}{\alpha_n} (K_{n+1,1} - K_{n-1,1}), \quad n \geq 1, \tag{2.13}$$

$$c_n = K_{n-2,2} - K_{n,2} - K_{n-1,1} (K_{n,1} - K_{n-2,1}), \quad n \geq 1, \tag{2.14}$$

$$\begin{aligned} a_n^2 = 1 - K_{n-2,4} + K_{n,4} + K_{n-1,3} (K_{n,1} - K_{n-2,1}) - K_{n,2} [K_{n,2} - K_{n-2,2} - \\ - K_{n-1,1} (K_{n,1} - K_{n-2,1})] - K_{n+1,1} [K_{n,3} - K_{n-2,3} - K_{n-1,2} (K_{n,1} - K_{n-2,1})] - \\ - K_{n,1} [K_{n,2} - K_{n-2,2} - K_{n-1,1} (K_{n,1} - K_{n-2,1})], \quad n \geq 1, \end{aligned} \tag{2.15}$$

$$\frac{\alpha_n}{\alpha_{n-2}} = a_{n-2}, \quad n = 1, 2, 3. \tag{2.16}$$

Consequently we have that if α_n and $K_{n,m}$ are defined by the equations (2.12)-(2.16) then the sequence $f_n(\xi)$ which is defined by Eq. (2.5) is the solution of equations (2.3). From Eq. (2.16) we immediately obtain that

$$\alpha_n = \left(\prod_{p=n}^{\infty} a_{2p-n} \right)^{-1}. \tag{2.17}$$

Using the method of the successive approximation from the equations (2.7)-(2.11) it is not difficult to find that there exists a the unique solution $K_{n,m}$ for each $m = 1, 2, \dots$ such that

$$|K_{n,m}| \leq C \sigma \left(n + \left[\frac{m}{2} \right] \right), \quad n, m \geq 1 \tag{2.18}$$

where,

$$\sigma(n) = \sum_{p=n}^{\infty} (|1 - a_p| + |b_p| + |c_p|), \tag{2.19}$$

$C > 0$ is a constant not depending on n and m . Here $[\cdot]$ denotes the integer part of a reel number.

Further, from the estimation (2.18) and the continuity of the function $f_n(\xi)$ ($-\infty < \xi < +\infty$) we obtain that the function $f_n(\xi)$ can be analytically extended to the upper semiplane $\text{Im } z > 0$ by the same formula

$$f_n(z) = \alpha_n e^{inz} \left(1 + \sum_{m=1}^{\infty} K_{n,m} e^{imz} \right), \quad n = 1, 2, \dots; \text{Im } z > 0. \tag{2.20}$$

Now it is easy to obtain the asymptotic formula

$$f_n(z) = \alpha_n e^{inz} (1 + O(e^{-\tau})), \quad z = \xi + i\tau, \tau > 0, \tag{2.21}$$

which is uniformly on z . We also have

$$f_n(z) = e^{inz} (1 + o(1)), \quad n \rightarrow \infty, \text{Im } z \geq 0. \tag{2.22}$$

Finally from (2.20) we also have

$$f_n(z + 2\pi) = f_n(z). \quad (2.23)$$

Hence we have proved the following Theorem:

Theorem 2.1. *If*

$$\sum_{n=1}^{\infty} n (|1 - a_n| + |b_n| + |c_n|) < \infty$$

then for each value $\lambda = 2 \cos 2\xi$ the problem (2.3)-(2.4) has the unique solution $(f_n(\xi))_{n=1}^{\infty}$ which is represented as

$$f_n(\xi) = \alpha_n e^{in\xi} \left(1 + \sum_{m=1}^{\infty} K_{n,m} e^{im\xi} \right), n = 1, 2, \dots$$

Moreover the following conditions are satisfied:

(a) the number sequences α_n and $K_{n,m}$ satisfy equations (2.12)-(2.16) and these numbers can be defined uniquely by the sequences $(a_n), (b_n), (c_n)$;

(b) the estimation (2.18) is valid;

(c) $f_n(\xi)$ has properties (2.20)-(2.23).

Note that together with $f_n(\xi)$ the sequences $f_n(-\xi)$, $f_n(\pi - \xi)$, $f_n(\pi + \xi)$ are also solutions of the system of equations (2.3).

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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