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Generalized R-contraction by using triangular α-orbital admissible

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Abstract *– This study presents Ciric type generalization of R-contraction and generalized R-contraction by using an α-orbital admissible function in metric spaces using the definition of R-contraction introduced by Roldan-Lopez-de-Hierro and Shahzad [New fixed-point theorem under R-contractions, Fixed Point Theory and Applications, 98(2015): 18 pages, 2015] and prove some fixed-point theorems for this type contractions. Thanks to these theorems, we generalize some known results.*

Subject Classification (2020): 54H25, 47H10.

1. Introduction

This section provides some of basic notions. The concept of fixed point appeared in 1922 with the Banach contraction principle (BCP) [1]. So far, many studies [2-5] has been conducted on this concept applied in many areas, such as differential equations theory and economics. The most striking of the results obtained by generalizing BCP is the Meir-Keeler contraction (MKC) provided in [6]:

Let T be a self-mapping on a complete metric space (X, d). Given $\varepsilon > 0$, there exist $\delta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies that } d(Tx,Ty) < \varepsilon$$

After that, many authors studied extensions of MKC. In [7], the authors presented the notion of simulation function (SF), an auxiliary function for improving BCP, and generalized MKC:

A simulation function ξ is a mapping from $[0,\infty) \times [0,\infty)$ to $\mathbb{R}$ such that

ξ_1) $\xi(0,0) = 0$

ξ_2) $\xi(t,s) < s - t$, for all $s, t \in \mathbb{N}$

ξ_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$

Afterwards, [8] modified the condition ξ_3 of SF to expand the family of SFs:

ξ_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$.

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In [7], the researchers then put forward the Z-contraction mapping as follows:

Let (X, d) be a metric space and T be a self-mapping on X. If there exists $\xi \in Z$ (Z is the family of SFs), for all $x, y \in X$ with $x \neq y$,

$$\xi\big(d(Tx, Ty), d(x, y)\big) \geq 0$$

then T is Z-contraction concerning ξ. So, they generalized the Banach fixed point theorem in metric space using the auxiliary function ξ. Furthermore, the concept of manageable function (MF) provided by [2] to work multivalued contraction mappings is as follows:

A function $\eta: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is manageable if

η_1) $\eta(t, s) < s - t$, for all $s, t > 0$

η_2) For a bounded sequence $\{t_n\} \subset (0, \infty)$, a non-increasing sequence $\{s_n\} \subset (0, \infty)$, η provides

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1.$$

Besides, [7] defined $\widehat{\mathrm{Man}(R)}$-contraction for single-valued mapping as follows:

Let (X, d) be a metric space and T be self-mapping on X. If there exists $\eta \in \widehat{\mathrm{Man}(R)}$ such that

$$\eta(d(Tx, T^2x), d(x, Tx)) \geq 0$$

for all $x \in X$, then T is $\widehat{\mathrm{Man}(R)}$-contraction.

Recently, [9] have introduced R-function for considering a true extension of MKC as follows:

Let $A \subset \mathbb{R}$, $A \neq \emptyset$, and $\varrho: A \times A \to \mathbb{R}$ be a function. Then, ϱ is called an R-function:

(ϱ_1) If a sequence $\{a_n\} \subset (0, \infty) \cap A$ and $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

(ϱ_2) If two sequence $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ converges to $L \geq 0$ such that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

(ϱ_3) If $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$ are two sequences such that $\{b_n\} \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$.

Let R_A denote the family of all R-functions, (X, d) be a metric space, and T be a mapping on X. T is R-contraction concerning ϱ if there exist $\varrho \in R_A$ such that $\mathrm{ran}(d) \subset A$ and

$$\varrho\big(d(Tx, Ty), d(x, y)\big) > 0$$

for all $x, y \in X$ with $x \neq y$

$$\mathrm{ran}(d) = \{d(x, y): x, y \in X\} \subset [0, \infty)$$

[9] also gave R-contraction concerning ϱ and showed a relationship between the class of some known functions and R-function and between some known contractions and R-contraction relating to ϱ as follows:

i. A SF is an R-function and verifies (ϱ_3),

ii. Any MF is an R-function and confirms (ϱ_3),

iii. A Geraghty function (GF) $\phi: [0, \infty) \to [0,1)$ holds if $\{t_n\} \subset [0, \infty)$ and $\{\phi(t_n)\} \to 1$, then $\{t_n\} \to 0$ [10]

If $\phi: [0, \infty) \to [0,1)$ is a GF, then $\varrho'_\phi: [0, \infty) \times [0, \infty) \to \mathbb{R}$, defined with

$$\varrho'_\phi(t, s) = \phi(s)s - t$$

for all $t, s \in [0, \infty)$, is an R-function on $[0, \infty)$ satisfying condition (ϱ_3), *iv.* Any MKC is R-contraction in respect of ϱ,

v. A Geraghty contraction (GC) is a self-mapping T on X such that for every $x, y \in X$ and ϕ is a GF $d(Tx, Ty) \leq \phi\big(d(x, y)\big)d(x, y)$ [10].

Every GC is R-contraction in respect of ϱ.

In [9], it is claimed that if $\varrho(t, s) \leq s - t$ for all $t, s \in A \cap (0, \infty)$, then (ϱ_3) is held.

[11] presented the concept of weakly Picard operator as follows:

Let (X, d) be a metric space and T be a self-mapping on X. Given a point $x_0 \in X$, the Picard sequence $\{x_n\}$ of T started with x_0 is given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. T defined as a weakly Picard operator if, for all $x_0 \in X$, the Picard sequence of T converges to a fixed point of T. Also, T is a Picard operator if it is a weakly Picard operator, and T has a unique fixed point.

2. Main Result

This section proves the Ciric type generalization of R-contraction concerning ϱ, and presents a generalization of known results and illustrates them.

Definition 2.1. Let (X, d) be a metric space T be a self-mapping on X and $\varrho \in R_A$. T is generalized *R*-contraction in respect of ϱ the following case satisfying $\mathrm{ran}(d) \subset A$ and

$$\varrho\big(d(Tx, Ty), M(x, y)\big) > 0\#(2.1)$$

for all $x, y \in X$ and $x \neq y$, where

$$\mathrm{M}(\mathrm{x}, \mathrm{y}) = \max\left\{\mathrm{d}(\mathrm{x}, \mathrm{y}), \mathrm{d}(\mathrm{x}, \mathrm{Tx}), \mathrm{d}(\mathrm{y}, \mathrm{Ty}), \frac{1}{2}[\mathrm{d}(\mathrm{x}, \mathrm{Ty}) + \mathrm{d}(\mathrm{y}, \mathrm{Tx})]\right\}.$$

Theorem 2.2. Let (X, d) be a complete metric space and T be generalized *R*-contraction on X in respect of ϱ. Suppose that one of the followings hold.

i. T is continuous,
ii. ϱ satisfies the condition (ϱ_3),
iii. $\varrho(t, s) \leq s - t$ for all $t, s \in A \cap (0, \infty)$.

Then T is a Picard operator, and T has a unique fixed point.

Proof.

Let we take any $x_0 \in X$ and $\{x_n\}$ is a Picard sequence of T started with x_0. If there exists some $n_0 \in \mathbb{N}$, $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ then x_{n_0} is a fixed point of T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since T is generalized R-contraction in respect of ϱ,

$$\varrho\big(d(Tx_{n-1}, Tx_n), M(x_{n-1}, x_n)\big) > 0\#(2.2)$$

where

$$\begin{aligned}\mathrm{M}(x_{n-1}, x_n) &= \max\left\{\mathrm{d}(x_{n-1}, x_n), \mathrm{d}(x_{n-1}, \mathrm{T}x_{n-1}), \mathrm{d}(x_n, \mathrm{T}x_n), \frac{\mathrm{d}(x_{n-1}, \mathrm{T}x_n) + \mathrm{d}(x_n, \mathrm{T}x_{n-1})}{2}\right\}\\ &= \max\left\{\mathrm{d}(x_{n-1}, x_n), \mathrm{d}(x_{n-1}, x_n), \mathrm{d}(x_n, x_{n+1}), \frac{\mathrm{d}(x_{n-1}, x_{n+1}) + \mathrm{d}(x_n, x_n)}{2}\right\}\\ &= \max\{\mathrm{a}_{\mathrm{n}-1}, \mathrm{a}_{\mathrm{n}}\}.\end{aligned}$$

From (2.2), we get

$$\varrho(a_n, \max\{a_{n-1}, a_n\}) > 0 \#(2.3)$$

If $a_{n-1} \leq a_n$ for some $n \in \mathbb{N}$, then from (2.3)

$$\varrho(a_n, a_n) > 0$$

which is a contradiction. Therefore, $a_{n-1} > a_n$ for all $n \in \mathbb{N}$ and $\varrho(a_n, a_{n-1}) > 0$.

From (ϱ_1), we have $\{a_n = d(x_n, x_{n+1})\} \to 0$.

Now, we show the sequence $\{x_n\}$ is Cauchy. Assume $\{x_n\}$ is not a Cauchy sequence. There exist $\varepsilon > 0$, for all $k \geq n_1$, there exist $m(k) > n(k) > k$ and $d(x_{n(k)}, x_{m(k)}) \geq \varepsilon$. Let $m(k)$ be the smallest number and satisfies the conditions above. Then $d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$. Hence,

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < \varepsilon + d(x_{m(k)-1}, x_{m(k)}).$$

As $k \to \infty$, $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$. Since

$$\left|d(x_{n(k)}, x_{m(k)-1}) - d(x_{n(k)}, x_{m(k)})\right| \leq d(x_{m(k)-1}, x_{m(k)})$$

we get $\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon$. Similarly, we obtain

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}).$$

Let $L = \varepsilon > 0$, $\{t_k = d(x_{n(k)}, x_{m(k)})\} \to L$, $\{s_k = d(x_{n(k)-1}, x_{m(k)-1})\} \to L$ and

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq M(x_{n(k)-1}, x_{m(k)-1}) = \max\left\{\begin{matrix} d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, Tx_{n(k)-1}), d(x_{m(k)-1}, Tx_{m(k)-1}), \\ \frac{1}{2}[d(x_{n(k)-1}, Tx_{m(k)-1}) + d(x_{m(k)-1}, Tx_{n(k)-1})] \end{matrix}\right\}$$

Taking a limit $k \to \infty$, we have $\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = L$. Since $L = \varepsilon < d(x_{n(k)}, x_{m(k)}) = t_k$ and

$$\varrho\left(d(x_{n(k)}, x_{m(k)}), M(x_{n(k)-1}, x_{m(k)-1})\right) > 0$$

for all $k \in \mathbb{N}$, then (ϱ_2) guarantees $L = \varepsilon = 0$. Consequently, $\{x_n\}$ is Cauchy. Since the metric space (X, d) is complete, there exist $z \in X$ such that $x_n \to z$. Let show that z fixed point.

Case 1: Suppose T is a continuous function. So $\{Tx_n = x_{n+1}\} \to Tz$, and $Tz = z$.

Case 2: In propositional logic, $p \Rightarrow q \equiv q' \Rightarrow p'$. Now we look at the proof of a fixed point of T concerning this point of view. Assume $d(z, Tz) > 0$.

$$a_n = d(Tx_n, Tz) = d(x_{n+1}, Tz) \text{ and so } \lim_{n\to\infty} a_n = d(z, Tz) > 0 \text{ and}$$

$$b_n = M(x_n, z) = \max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2}[d(z, Tx_n) + d(x_n, Tz)]\right\}$$

Let $n \to \infty$, we get $\lim_{n\to\infty} b_n = M(x_n, z) = d(z, Tz) > 0$, but

$$\varrho(d(Tx_n, Tz), M(x_n, z)) > 0.$$

It contradicts to (ϱ_3). Consequently, $d(z, Tz) = 0$.

Case 3: Assume $\varrho(t, s) < s - t$ for all $t, s \in A \cap (0, \infty)$. Proposition 1.2 means that Case 2 is applicable. z is a fixed point, so T is a weakly Picard operator.

Let $z \neq y$ and $z, y \in X$ be two fixed points. In this case, $a_n = d(z, y) > 0$ for all $n \in \mathbb{N}$.

$$\varrho(a_{n+1}, a_n) = \varrho(d(z, y), d(z, y)) = \varrho(d(Tz, Ty), M(z, y)) > 0$$

Applying (ϱ_1), $\{a_n\} \to 0$, which is a contradiction.

Example 2.3. Let $X = [0,1]$ and $d: X \times X \to \mathbb{R}$ be a usual metric. Let $T: X \to X$ as $Tx = \frac{x}{x+1}$ for all $x \in X$. We define $\varrho: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varrho(t,s) = \frac{s}{s+1} - t$. From Theorem 2.2, $x = 0$ is a fixed point of T.

We have the following corollaries by using Theorem 2.2. In this case, we generalized Corollary 28-33 in [9] by using similar $M(x,y)$.

Corollary 2.4. Any continuous generalized R-contraction has a unique fixed point.

Corollary 2.5. Any generalized Z-contraction has a unique fixed point.

Corollary 2.6. Every generalized $\widehat{\text{Man(R)}}$-contraction has a unique fixed point.

Corollary 2.7. Let (X,d) be a complete metric space and $T: X \to X$. Assume that there exist $\varphi, \psi: [0,\infty) \to [0,\infty)$ such that

$$\psi(d(Tx,Ty)) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big)$$

for all $x, y \in X$. If φ is lower semi-continuous, ψ is nondecreasing, ψ continuous from right and $\varphi^{-1}(\{0\}) = \{0\}$, then Thas a unique fixed point.

Proof.

It is obvious Theorem 2.2 and Theorem 22 in [9].

Corollary 2.5. Every generalized GC has a unique fixed point.

Proof.

It is obvious Theorem 2.2 and Corollary 26 in [9].

Corollary 2.6. Every generalized MKC has a unique fixed point.

Proof.

It is obvious from Theorem 2.2 and Theorem 25 in [9].

3. Admissible Functions

[12] gave α-admissible concept as follows: let $T: X \to X$, $\alpha: X \times X \to \mathbb{R}$. T is said to be α-admissible if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$. Then, [3] added the condition; $\alpha(x,z) \geq 1$, $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$, nearby the α-admissible condition and so they introduced triangular α-admissible notion. We understand from these definitions, triangular α-admissible implies α-admissible, but the converse is not valid. In 2014, Popescu [4] introduced α-orbital and triangular α-orbital admissible notions as follows:

Definition 3.1. [4] Let $T: X \to X$, $\alpha: X \times X \to \mathbb{R}$. T is said to be α-orbital admissible if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$.

Definition 3.2. [4] Let $T: X \to X$, $\alpha: X \times X \to \mathbb{R}$. T is said to be triangular α-orbital admissible if $T$is α-orbital admissible, $\alpha(x,y) \geq 1$, if $\alpha(y,Ty) \geq 1$ implies $\alpha(x,Ty) \geq 1$.

Every α-admissible mapping is an α-orbital admissible and every triangular α-admissible mapping is a triangular α-orbital admissible mapping. So that a triangular α-orbital admissible mapping is a very wide function class in the literature.

Lemma 3.3. [9] Let $T: X \to X$ be a triangular α-orbital admissible mapping. Assume that there exist $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$ with $n < m$.

Definition 3.4. Let (X, d) be a metric space, $T: X \to X$. T is an α-admissible R-contraction in respect of ϱ if there exist $\varrho: A \times A \to \mathbb{R}$ such that for all $x, y \in X$, $\alpha(x, y)d(Tx, Ty) \in A$, $\text{ran}(d) \subset A$, $\alpha: X \times X \to [0, \infty)$,

$$\varrho(\alpha(x, y)d(Tx, Ty), d(x, y)) > 0$$

for all $x, y \in X$ with $x \neq y$. If $\alpha(x, y) = 1$, then T is a R-contraction.

Theorem 3.5. Let (X, d) be a complete metric space, $\alpha: X \times X \to \mathbb{R}$, $T: X \to X$. If

T is an α-admissible R-contraction type mapping in respect of ϱ,

T is a triangular α-orbital admissible mapping, there exist $x_0 \in X$ and $\alpha(x_0, Tx_0) \geq 1$,

$\varrho(t, s) < s - t$ for all $t, s \in A \cup (0,1)$,

T is a continuous function.

Then, T is a Picard operator and has a fixed point in X.

Proof.

Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and let $\{x_n\}$ be a Picard sequence of T started with x_0 such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exist $n_0 \in \mathbb{N}$, $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T. In this case, suppose that $x_{n+1} \neq x_n$ or all $n \in \mathbb{N}$. Because of (*ii*) and (*iii*), we obtain

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) \geq 1$$

similarly,

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) \geq 1$$

continuing this process, we derive $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. T is an α-admissible R-contraction, then

$$0 < \varrho\big(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), d(x_n, x_{n-1})\big) < d(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n)$$

as a result, we get for all $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) < \alpha(x_n, x_{n-1})d(x_{n+1}, x_n) < d(x_n, x_{n-1})\#(3.1)$$

Hence, the sequence $\{x_n\}$ is decreasing, bounded from below. Consequently, there exists $L \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n-1}) = L$. From equation (3.1), we get

$$\lim_{n\to\infty} \alpha(x_n, x_{n-1})d(x_{n+1}, x_n) = L.$$

Let $s_n = \alpha(x_n, x_{n-1})d(x_{n+1}, x_n)$, $t_n = d(x_n, x_{n-1})$ and we can easily see that $L < s_n$ for $n \in \mathbb{N}$. In this case, from the (ϱ_2) property, we have $L = 0$.

The sequence $\{x_n\}$ is Cauchy in X. Assume the sequence $\{x_n\}$ is not Cauchy. There exist $\varepsilon > 0$, for all $k \geq n_1$, there exist $m(k) > n(k) > k$ and $d(x_{n(k)}, x_{m(k)}) \geq \varepsilon$. Let $m(k)$ be the smallest and satisfies the above conditions. So $d\big(x_{n(k)-1}, x_{m(k)}\big) < \varepsilon$. Then

$$\varepsilon \leq d\big(x_{n(k)}, x_{m(k)}\big) \leq d\big(x_{n(k)}, x_{m(k)-1}\big) + d\big(x_{m(k)-1}, x_{m(k)}\big) < \varepsilon + d\big(x_{m(k)-1}, x_{m(k)}\big)$$

As $k \to \infty$, we get $\lim_{k\to\infty} d\big(x_{n(k)}, x_{m(k)}\big) = \varepsilon$. Since

$$\big|d\big(x_{n(k)}, x_{m(k)-1}\big) - d\big(x_{n(k)}, x_{m(k)}\big)\big| \leq d\big(x_{m(k)-1}, x_{m(k)}\big),$$

we get $\lim_{k\to\infty} d\big(x_{n(k)}, x_{m(k)-1}\big) = \varepsilon$. Similarly, we obtain

$$\lim_{k\to\infty} d\big(x_{n(k)-1}, x_{m(k)}\big) = \varepsilon = \lim_{k\to\infty} d\big(x_{n(k)-1}, x_{m(k)-1}\big).$$

By Lemma 4.3, we have $\alpha\big(x_{n(k)-1}, x_{m(k)-1}\big) \geq 1$. Thus, we deduce that

$$0 < \varrho\left(\alpha\big(x_{n(k)-1}, x_{m(k)-1}\big)d\big(Tx_{n(k)-1}, Tx_{m(k)-1}\big), d\big(x_{n(k)-1}, x_{m(k)-1}\big)\right)$$
$$< d\big(x_{n(k)-1}, x_{m(k)-1}\big) - \alpha\big(x_{n(k)-1}, x_{m(k)-1}\big)d\big(Tx_{n(k)-1}, Tx_{m(k)-1}\big)$$

for all $k \geq n_1$. Consequently,

$$0 < d\big(x_{n(k)}, x_{m(k)}\big) < \alpha\big(x_{n(k)-1}, x_{m(k)-1}\big)d\big(Tx_{n(k)-1}, Tx_{m(k)-1}\big) < d\big(x_{n(k)-1}, x_{m(k)-1}\big)$$

for all $k \geq n_1$. Let $k \to \infty$, we have

$$\lim_{k\to\infty} \alpha\big(x_{n(k)-1}, x_{m(k)-1}\big)d\big(Tx_{n(k)-1}, Tx_{m(k)-1}\big) = \varepsilon.$$

Let $a_k = \alpha\big(x_{n(k)-1}, x_{m(k)-1}\big)d\big(Tx_{n(k)-1}, Tx_{m(k)-1}\big)$ and $b_k = d\big(x_{n(k)}, x_{m(k)}\big)$. We show that $\varepsilon < a_k$ for all $k \geq n_1$. In this case, from the (ϱ_2) property, we have $\varepsilon = 0$, which is a contradiction. Hence, the sequence $\{x_n\}$ is Cauchy. From (X, d) is complete, there exist $z \in X$, $\{x_n\} \to z$.

Assume the condition (v) satisfied. In this case, $\{x_{n+1} = Tx_n\} \to Tz$, and so $Tz = z$. Therefore, T is a weakly Picard operator.

Theorem 3.6. Let (X, d) be complete, $\alpha: X \times X \to \mathbb{R}$ and $T: X \to X$. Assume the followings are satisfied:

T is a α- admissible R-contraction type mapping concerning ϱ;

T is a triangular α- orbital admissible mappings,

There exist $x_0 \in X$ and $\alpha(\alpha, Tx_0) \geq 1$;

$\varrho(t, s) < s - t$ for all $t, s \in A \cup (0,1)$;

if $\{x_n\} \in X$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

So, T is a Picard operator and has a fixed point in X.

Proof.

From the proof of the above theorem, the sequence $\{x_n\}$, $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, converges to $z \in X$. By the condition (v), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. Applying (i) for all k, we get

$$0 < \varrho\big(\alpha(x_{n_k}, z)d\big(Tx_{n_k-1}, Tz\big), d(x_{n_k}, z)\big) = \varrho\big(\alpha(x_{n_k}, z), d\big(x_{n_k}, Tz\big), d(x_{n_k}, z)\big)$$
$$< d\big(x_{n_k}, z\big) - \alpha(x_{n_k}, z)d\big(x_{n_k}, Tz\big)$$

which is equivalent to

$$d\big(x_{n_k}, Tz\big) = d(Tx_{n_k-1}, Tz) \leq \alpha(x_{n_k}, z)d\big(x_{n_k}, Tz\big) < d\big(x_{n_k}, z\big).$$

Let $k \to \infty$, we have $d(z, Tz) = 0$, i.e., $z = Tz$.

From the uniqueness of fixed point of α-admissible R-contraction type mapping,

(H) For all $x \neq y$, there exists $v \in X$ and $\alpha(x, v) \geq 1$, $\alpha(y, v) \geq 1$, $\alpha(v, Tv) \geq 1$.

Replacing (*iii*) with (*H*) in the hypothesis of Theorem 3.5 and Theorem 3.6, we get the uniqueness of the fixed point of T. Assume z, t are two fixed points of T and $z \neq t$. From the condition (*H*), there exists $v \in X$ and

$$\alpha(z,v) \geq 1, \alpha(t,v) \geq 1, \alpha(v,Tv) \geq 1.$$

Because T is triangular α-orbital admissible, we obtain $\alpha(z,T^n v) \geq 1$ and $\alpha(t,T^n v) \geq 1$ for all $n \in \mathbb{N}$, we get

$$0 < \varrho\big(\alpha(z,T^n v)d(Tz,T^{n+1}v), d(z,T^n v)\big)$$
$$< d(z,T^n v) - \alpha(z,T^n v)d(Tz,T^{n+1}v)$$

and so

$$d(z,T^n v) = d(Tz,T^n v) \leq \alpha(z,T^n v)d(Tz,T^{n+1}v) < d(z,T^n v)$$

By the Theorem 3.5, we know that the sequence $\{T^n v\}$ converges to a fixed point t of T. As $n \to \infty$,

$s_n = (z,T^n v)d(Tz,T^{n+1}v) \to d(z,t)$ and $t_n = d(z,T^n v) \to d(z,t)$

From (ϱ_2), we $d(z,t) = 0$, which is a contradiction. Therefore, $z = t$.

Now, we can give some corollaries by using Theorem 3.5 and Theorem 3.6.

Corollary 3.7. Every α-admissible Z-contraction has a unique fixed point.

Corollary 3.8. Every α-admissible $\widehat{\mathrm{Man}(R)}$-contraction has a unique fixed point.

We prove the following corollary by using Theorem 3.5 and Theorem 2.2.

Corollary 3.9. Every α-admissible Z-contraction has a unique fixed point.

Corollary 3.10. Every α-MKC has a unique fixed point.

Conflicts of Interest

The author declares no conflict of interest.

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