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On a Diophantine Equation of Type $p^x + q^y = z^3$

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Abstract

The exponential Diophantine equations of type $p^x + q^y = z^2$ have been widely studied over the past decade. Authors studied these equations by considering primes p and q, and in general, for positive integers p and q. In this paper, we will be extending the study to Diophantine equations of type

$$p^x + q^y = z^3.$$

In particular, we will be working with Diophantine equations of type

$$p^x + (p+4)^y = z^3, \tag{0.1}$$

where p and $p+4$ are cousin primes; that is, primes that differ by four. We state some sufficient conditions for the non-existence of solutions of equation (1) on the set of positive integers. The proof uses some results in the theory of rational cubic residues as well as results in quadratic reciprocity, and some elementary techniques. It will be shown also that other Diophantine equations of similar type can also be studied with the approaches used in this paper.

Keywords: *Diophantine equation; exponential Diophantine equation; Jacobi symbol; quadratic reciprocity; rational cubic residues*
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1. Introduction

The study of Diophantine equations can be traced back to ancient times and it is still of interest in the field of Number Theory up to the present time. The simplest type of Diophantine equations is the linear type in two variables x and y, which is of the form

$$ax + by = c,$$

for given integers a, b and c [3].
Another type of Diophantine equation that is recently being studied by numerous mathematicians is the exponential type in two variables x and y and is of the form

$$a^x + b^y = c,$$

where a, b and c arethe given integers. Interestingly, this was extended to three variables. Diophantine equations of type

$$a^x + b^y = z^2$$

have been studied for positive integer constants a and b (see, for details, [1], [2], [5], [6], [7], [8] and the references cited therein). Some authors have used results about quadratic residues to solve these equations [4]. Now, it is natural to ask for the solutions of the Diophantine equations of type

$$a^x + b^y = z^3,$$

where a and b are given positive integers.
In this paper, we use some results in the theory of rational cubic residues to some open problems. In particular, by considering certain conditions, we characterize the positive integer solutions of the Diophantine equation $p^x + (p+4)^y = z^3$, where p and $p+4$ are cousin primes and $p \equiv 1 \pmod 3$. This gives us a way on solving other similar Diophantine equations.
We first state some well-known results, and prove two lemmas because they are essential in the proof of our main result.

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Theorem 1.1. *Let n be an integer possessing a primitive root, and an integer a such that $\gcd(a,n)=1$. Then, the congruence $x^k \equiv a \pmod{n}$ has a solution if and only if the following congruence is satisfied:*

$$a^{\phi(n)/d} \equiv 1 \pmod{n},$$

where $d=\gcd(k,\phi(n))$. If it has a solution, there are exactly d solutions modulo n.

The proof of this can be seen in Theorem 8-12 of [3].

Theorem 1.2. *Let p be prime and a an integer such that $\gcd(a,p)=1$. Then, the congruence $x^3 \equiv a \pmod{p}$ has a solution if and only if $a^{(p-1)/d} \equiv 1 \pmod{p}$, where $d=\gcd(3,p-1)$. If it has a solution, then there are exactly d solutions modulo p.*

It can be easily seen that this is a corollary of the preceding theorem by assuming that $k=3$ and n to be prime.

Lemma 1.3. *Let p be prime satisfying $p\equiv 1 \pmod{3}$, and a an integer with $\gcd(a,p)=1$. If the congruence $z^3\equiv a \pmod{p}$ has no solutions $z\in\mathbb{Z}$, then the congruence $z^3 \equiv a^x \pmod{p}$ has no solutions for any $x\in\mathbb{N}$ that is not a multiple of 3.*

Proof. Assume the contrary by supposing that $z^3\equiv a^x \pmod{p}$ has a solution, where x is not a multiple of 3. By using Theorem 1.2, we get $(a^x)^{(p-1)/3} = (a^{(p-1)/3})^x \equiv 1 \pmod{p}$. Also, since $z^3\equiv a \pmod{p}$ has no solutions in z then $a^{(p-1)/3}\not\equiv 1 \pmod{p}$. Now, we consider $x=3k+2$ for some $k\in\mathbb{N}$. Note that by using Fermat's little theorem, it is true also that $(a^{(p-1)/3})^{3k}\equiv 1 \pmod{p}$. Hence, we get

$$\left(a^{\frac{p-1}{3}}\right)^{3k+2} \equiv \left(a^{\frac{p-1}{3}}\right)^{3k} \pmod{p}.$$

This gives

$$\left(a^{\frac{p-1}{3}}\right)^{2} \equiv 1 \pmod{p},$$

which implies that $a^{(p-1)/3}\equiv -1 \pmod{p}$. This further tells that $a^{p-1}\equiv(-1)^3\equiv -1 \pmod{p}$, which is a contradiction. We also get a contradiction if x is assumed to be $3k+1$, for some positive integer k. □

Lemma 1.4. *Let $q\neq 7$ be an odd integer. Then the Jacobi symbol $\left(\frac{7}{q}\right)$ is equal to 1 if and only if $q\equiv 1,3,9,19,25,27 \pmod{28}$.*

Proof. By the Generalized Quadratic Reciprocity Law, we have that

$$\left(\frac{7}{q}\right)\left(\frac{q}{7}\right) = \begin{cases} 1 & \text{if } q\equiv 1 \pmod{4} \\ -1 & \text{if } q\equiv 3 \pmod{4} \end{cases} \Longrightarrow \left(\frac{7}{q}\right) = \begin{cases} \left(\frac{q}{7}\right) & \text{if } q\equiv 1 \pmod{4} \\ -\left(\frac{q}{7}\right) & \text{if } q\equiv 3 \pmod{4} \end{cases}.$$

We also know that

$$\left(\frac{q}{7}\right) = \begin{cases} 1 & \text{if } q\equiv 1,2,4 \pmod{7} \\ -1 & \text{if } q\equiv 3,5,6 \pmod{7} \end{cases}.$$

The conclusion follows. □

The following result on the theory of rational cubic residues is also needed to justify our main result.

Lemma 1.5. *The integer 2 is a cubic residue modulo $p=\frac{1}{4}(L^2+27M^2)$ if and only if $L\equiv M\equiv 0 \pmod{2}$.*

The proof of this lemma can be seen in Proposition 7.1 of [4].

2. Main Result

We now present the main theorem.

Theorem 2.1. *Let p and $p+4$ be primes such that $p\equiv 1 \pmod{3}$. If $p=\frac{1}{4}(L^2+27M^2)$, where $L\not\equiv 0 \pmod{2}$ (or $M\not\equiv 0 \pmod{2}$), then the exponential Diophantine equation $p^x+(p+4)^y=z^3$ has no solutions (x,y,z) in the set of positive integers.*

Proof. Take the equation $p^x+(p+4)^y=z^3$ modulo p to get $z^3\equiv 2^{2y} \pmod{p}$. Now, if $LM\not\equiv 0 \pmod{2}$, then by Lemma 1.5, $z^3\equiv 2 \pmod{p}$ has no solution in z. By using Lemma 1.3, we observe that $z^3\not\equiv 2^{2y} \pmod{p}$ except when y is a multiple of 3. Hence, we only consider $y=3y_1$ where $y_1\in\mathbb{N}$. In this case, we have

$$p^x = z^3-(p+4)^{3y_1} = (z-(p+4)^{y_1})(z^2+(p+4)^{y_1}z+(p+4)^{2y_1}).$$

Since p is prime, there exist integers α and β, $\alpha<\beta$ such that $\alpha+\beta=y$ and

$$\frac{p^\beta}{p^\alpha} = p^{\beta-\alpha} = \frac{z^2+(p+4)^{y_1}z+(p+4)^{2y_1}}{z-(p+4)^{y_1}} = z+2(p+4)^{y_1}+\frac{3(p+4)^{2y_1}}{z-(p+4)^{y_1}}.$$

This implies that the term $z-(p+4)^{y_1}$ must divide the expression $3(p+4)^{2y_1}$. So, we can write $z-(p+4)^{y_1}=3(p+4)^j$ or $z-(p+4)^{y_1}=(p+4)^j$, where $0\le j\le 2y_1$. Note that $j>0$ is not possible since this will give us $(p+4)\mid z$, which is clearly a contradiction. Hence, $j=0$ and we get either $z-(p+4)^{y_1}=3$ or $z-(p+4)^{y_1}=1$. For the first case, we have

$$p^x+(p+4)^y = ((p+4)^{y_1}+3)^3 = (p+4)^y+9(p+4)^{2y_1}+27(p+4)^{y_1}+27.$$

We get $p^x = 9(p+4)^{2y_1} + 27(p+4)^{y_1} + 27$, but this is contrary to the fact that $p \equiv 1 \pmod 3$. Hence, we are left with the case where $z = (p+4)^{y_1} + 1$. For this one, we get

$$p^x + (p+4)^y = ((p+4)^{y_1} + 1)^3 = (p+4)^y + 3(p+4)^{2y_1} + 3(p+4)^{y_1} + 1.$$

Simplifying, we obtain

$$p^x = 3(p+4)^{2y_1} + 3(p+4)^{y_1} + 1. \tag{2.1}$$

We now look the equation in modulo 4 and 8. Note that z is even. So, $z^3 \equiv 0 \pmod 8$ (and $z^3 \equiv 0 \pmod 4$). On the other hand, we have the following congruences:

$$p^x + (p+4)^y \equiv \begin{cases} 2 \pmod 4 \text{ if } p \equiv 1 \pmod 4 \\ 2 \pmod 4 \text{ if } p \equiv 3 \pmod 4 \text{ and } x \equiv y \pmod 2 \\ 0 \pmod 4 \text{ if } p \equiv 3 \pmod 4 \text{ and } x \not\equiv y \pmod 2 \end{cases}.$$

Thus, $p \equiv 3 \pmod 4$, and x and y must be of different parity. Working on this equation modulo 8 will give us the following scenarios:

$$p^x + (p+4)^y \equiv \begin{cases} 4 \pmod 8 \text{ if } p \equiv 3 \pmod 8 \text{ and } x \text{ is odd}, y \text{ is even} \\ 0 \pmod 8 \text{ if } p \equiv 3 \pmod 8 \text{ and } x \text{ is even}, y \text{ is odd} \\ 4 \pmod 8 \text{ if } p \equiv 7 \pmod 8 \text{ and } x \text{ is even}, y \text{ is odd} \\ 0 \pmod 8 \text{ if } p \equiv 7 \pmod 8 \text{ and } x \text{ is odd}, y \text{ is even} \end{cases}.$$

Here, we get two possible cases; namely, $p \equiv 3 \pmod 8$ with x even and y odd, and $p \equiv 7 \pmod 8$ with x odd and y even.

Now, we go back to equation (2) and consider it in modulo $(p+4)$. This simplifies to $p^x \equiv 1 \pmod{(p+4)}$. Assume first that x is odd (and consequently y is even). This implies that $p^{x+1} = p^{2k} \equiv p \pmod{(p+4)}$. Hence the value of the Legendre symbol $\left(\frac{p}{p+4}\right)$ is 1; that is,

$$\left(\frac{p}{p+4}\right) = 1.$$

Since $p \equiv 3 \pmod 4$ and $p+4 \equiv 3 \pmod 4$, by using the Quadratic Reciprocity Law, we have

$$\left(\frac{p}{p+4}\right) = -\left(\frac{p+4}{p}\right) = -\left(\frac{2}{p}\right)^2 = -1,$$

which leads to a contradiction.
We are only left with the case where x is even (and y is odd). For this case, taking equation (2) modulo $\frac{p+3}{2}$ will give us $p^x \equiv 7 \pmod{\frac{p+3}{2}}$. If $\gcd(\frac{p+3}{2}, 7) = 1$, then the value of the Jacobi symbol $\left(\frac{7}{\frac{p+3}{2}}\right)$ is also 1; that is,

$$\left(\frac{7}{\frac{p+3}{2}}\right) = 1.$$

Now, we look at all possible values for $\frac{p+3}{2}$ modulo 28 for prime p modulo 7. Refer to the following table. Note that $p \equiv 3 \pmod 8$ also holds.

$p \pmod 7$	$p \pmod{56}$	$\frac{p+3}{2} \pmod{28}$
$p \equiv 1 \pmod 7$	$p \equiv 43 \pmod{56}$	$\frac{p+3}{2} \equiv 23 \pmod{28}$
$p \equiv 2 \pmod 7$	$p \equiv 51 \pmod{56}$	$\frac{p+3}{2} \equiv 27 \pmod{28}$
$p \equiv 3 \pmod 7$	$p \equiv 3 \pmod{56}$	$\frac{p+3}{2} \equiv 3 \pmod{28}$
$p \equiv 4 \pmod 7$	$p \equiv 11 \pmod{56}$	$\frac{p+3}{2} \equiv 7 \pmod{28}$
$p \equiv 5 \pmod 7$	$p \equiv 19 \pmod{56}$	$\frac{p+3}{2} \equiv 11 \pmod{28}$
$p \equiv 6 \pmod 7$	$p \equiv 27 \pmod{56}$	$\frac{p+3}{2} \equiv 15 \pmod{28}$

Table 1: Values of $\frac{p+3}{2}$ modulo 28 for each prime p modulo 7

Thus, by using Lemma 1.4, we are only going to consider the cases where $p \equiv 2 \pmod 7$ and $p \equiv 3 \pmod 7$. The case where $p \equiv 0 \pmod 7$ is omitted because it gives $p = 7$, which is not possible since $p \equiv 3 \pmod 8$. For case $p \equiv 2 \pmod 7$, by taking the equation $p^x + (p+4)^y = ((p+4)^{y_1} + 1)^3$ modulo 7 will lead us to

$$2^x + 6 \equiv (6+1)^3 \equiv 0 \pmod 7.$$

Since x is even, we have $\{4,2,1\} + 6 \equiv 0 \pmod 7$ which gives us $x \equiv 0 \pmod 6$. For the case where $p \equiv 3 \pmod 7$, we get

$$3^x + 0 \equiv 1 \pmod 7.$$

Again, since x is even, we have $\{2,4,1\} \equiv 1 \pmod 7$, which implies that $x \equiv 0 \pmod 6$. Hence, for either of the two cases, we notice that x is a multiple of 6; that is, $x = 6x_1$, for $x_1 \in \mathbb{N}$. This tells us that

$$(p^{2x_1})^3 + ((p+4)^{y_1})^3 = z^3,$$

which can be justified to have no solutions by using the Fermats Last Theorem.
The last case that we need to look at is the case where $\frac{p+3}{2} \equiv 0 \pmod 7$. This is equivalent to $p+4 \equiv 1 \pmod 7$. Using equation (2.1), we get $p^x \equiv 3+3+1 \equiv 0 \pmod 7$. Since p is prime, this means that $p \equiv 0 \pmod 7$. This is a contradiction to the fact that $p \equiv 4 \pmod 8$. □

To illustrate the paper's result, we list down all Diophantine equations (1), where p ranges from 7 to 97, and see which Diophantine equations have no solutions.

p	$p^x + (p+4)^y = z^3$	L	Conclusion
7	$7^x + 11^y = z^3$	1	no solutions
13	$13^x + 17^y = z^3$	5	no solutions
19	$19^x + 23^y = z^3$	7	no solutions
37	$37^x + 41^y = z^3$	11	no solutions
43	$43^x + 47^y = z^3$	8	no conclusion
67	$67^x + 71^y = z^3$	5	no solutions
79	$79^x + 83^y = z^3$	17	no solutions
97	$97^x + 101^y = z^3$	19	no solutions

Table 1: List of some Diophantine equations of the form (0.1) that satisfy Theorem 2.1

We can see that there is a high probability that the value of L is not even for values of p less than 100. We can see that the theorem caters *majority* of the Diophantine equations of the form (0.1).

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References

[1] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, Gen. Math. Vol:15, No.4 (2007), 145-148.
[2] J. B. Bacani and J. F. T. Rabago, The complete set of solutions of the Diophantine equation $p^x + q^y = z^2$ for twin primes p and q, Int. J. Pure Appl. Math. Vol:104, No.4 (2015), 517-521.
[3] Burton, D. M., *Elementary Number Theory*, Allyn and Bacon Inc. Boston, 1980.
[4] Lemmermeyer, F., *Reciprocity Laws from Euler to Eisenstein*, Springer-Verlag Berlin, 2000.
[5] J. F. T. Rabago, More on Diophantine equations of type $p^x + q^y = z^2$, Int. J. Math. Sci. Comp. Vol:3, No.1 (2013), 15-16.
[6] J. F. T. Rabago, On an Open Problem by B. Sroysang, Konuralp J. Math. Vol:1, No.2 (2013), 30-32.
[7] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, Int. J. Pure Appl. Math. Vol:81, No.4 (2013), 601-604.
[8] A. Suvarnamani, A. Singta, S. Chotchaisthit, On two Diophantine Equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, Sci. Technol. RMUTT J. Vol:1 (2011), 25-28.