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Research Article

On Hölder continuity and equivalent formulation of intrinsic Harnack estimates for an anisotropic parabolic degenerate prototype equation

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ABSTRACT. We give a proof of Hölder continuity for bounded local weak solutions to the equation

$$u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i}, \quad \text{in} \quad \Omega_T = \Omega \times (0,T], \quad \text{with} \quad \Omega \subset\subset \mathbb{R}^N, \tag{$*$}$$

under the condition $2 < p_i < \bar{p}(1+2/N)$ for each $i = 1,..,N$, being $\bar{p}$ the harmonic mean of the p_is, via recently discovered intrinsic Harnack estimates. Moreover, we establish an equivalent formulation of these Harnack estimates within the proper intrinsic geometry.

Keywords: Anisotropic p-Laplacian, Hölder continuity, Harnack estimates, intrinsic scaling.

2020 Mathematics Subject Classification: 35K65, 35K92, 35B65.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION AND MAIN RESULT

Equations of the kind of ($*$) fall into the wide class of degenerate equations, because their coordinate modulus of ellipticity $|u_{x_i}|^{p_i-2}$ vanishes as soon as the partial derivative u_{x_i} approaches zero. This behaviour is classically studied in equations evolving as the degenerate p-Laplacian

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = 0, \quad \text{in} \quad \Omega_T = \Omega \times (0,T], \quad \Omega \subset\subset \mathbb{R}^N, \tag{1.1}$$

whose modulus of ellipticity $|\nabla u|^{p-2}\nabla u$ goes to zero when the whole gradient of the solution vanishes. Within a rich variety of other different techniques, the method of intrinsic scaling is one of the keys to access the theory of regularity for degenerate parabolic equations (see for instance [7], [10]): it provides a correct interpretation of the evolution of the equation, interpreted in a particular geometry dictated by the solution itself, hence the name. This method has proven to be powerful and flexible enough to be adapted to a wide class of equations; a sketchy example for the doubly nonlinear case can be found in [4]. Nevertheless, the application of this method to the anisotropic case is not straightforward, because the degenerative behavior of the equation is purely directional, i.e., some partial derivatives can vanish while some other ones may direct the diffusion. When $p_i \equiv p$, the equation ($*$) is a different equation from (1.1) and bears the name of *orthotropic p-Laplacian*. The prototype equation ($*$) reflects the

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modeling of many materials that reveal different diffusion rates along different directions, such as liquid crystals, wood or earth's crust (see for an example the book [26]). The regularity of bounded local weak solutions to equations as ($*$) with measurable and bounded coefficients is still an open problem. The main difference with standard nonlinear regularity theory is the directional growth of the operator, usually referred to as *nonstandard growth* (see for instance [1]). This requires the definition of a new class of function spaces, called anisotropic Sobolev spaces (see Section 2), and whose study is open and challenging.

1.1. **The elliptic problem.** Even in the elliptic case, a standard statement of regularity for such equations requires a bound on the sparseness of the powers p_i. Indeed, in general, the weak solution can be unbounded, as proved in [15], [19]. Lipschitz bounds were obtained by Marcellini for the p, q nonstandard growth in his work [21], supposing the coefficients are regular enough. This work opened one way to the regularity theory of the so-called nonstandard growth conditions. See also the work by Uralt'seva and Urdaletova in [28], and [17], [18] for more general equations. In [2], the authors proved the boundedness of solutions under the assumption

$$\bar{p} < N, \quad \max\{p_1, .., p_N\} < \bar{p}^*, \tag{1.2}$$

where $\bar{p}$ and $\bar{p}^*$ are defined respectively in (2.11) and (2.12). Regularity properties are proved assuming strong conditions on the regularity of the coefficients (see [20], [21]). In this context, we quote also the contribution [12]. A recent result of Lipschitz regularity has been proven by Brasco and Bousquet in [3] for the elliptic counterpart of ($*$), by assuming that solutions are just bounded. On the other hand, structure conditions are left in big generalisation while a tighter condition on the spareness of p_i is used in the paper [14] to obtain Lipschitz bounds. This list is far from being complete, we refer to the survey [22] for an exhaustive bibliography. Nevertheless even in the elliptic case, when the coefficients are rough, Hölder continuity remains still nowadays an open problem. Indeed, continuity conditioned to boundedness has been proved in [11] through the intrinsic scaling method, but with a condition of stability on the exponents p_i which is only qualitative. Removability of singularities has been considered in [27], where the idea of working with fundamental solutions in the anisotropic framework had yet been taken into consideration.

1.2. **The parabolic problem.** To the best of our knowledge, the boundedness of local weak solutions to equations behaving as ($*$) has been proved in [13], [23]. More precisely, they prove that local weak solutions are bounded if

$$p_i < \bar{p}\left(1 + \frac{2}{N}\right), \quad i = 1, .., N. \tag{1.3}$$

Again in [13], the authors find some useful L^∞ estimates, together with finite speed of propagation and lower semicontinuity of solutions. These have been the starting point for the study of fundamental solutions to ($*$) (see for instance [5]), and the behaviour of their support in [6]. Recently, an approach based on an expansion of positivity relying on the behaviour of fundamental solutions has brought the authors to prove in [6] the following Harnack inequality, properly structured in an intrinsic anisotropic geometry that we are about to describe. Fix numbers $\theta, \rho > 0$ to be defined later, and define the anisotropic cubes

$$\mathcal{K}_\rho(\theta) := \prod_{i=1}^{N} \left\{ |x_i| < \theta^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}. \tag{1.4}$$

Next, define the following centered, forward and backward anisotropic cylinders, for a generic point (x_0, t_0):

$$\text{(1.5)} \quad \begin{cases} \text{centered cylinders: } (x_0, y_0) + \mathcal{Q}_\rho(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times (t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{forward cylinders: } (x_0, y_0) + \mathcal{Q}^+_\rho(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times [t_0, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}); \\ \text{backward cylinders: } (x_0, y_0) + Q^-_\rho(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times (t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}, t_0]. \end{cases}$$

We use the following Theorem 1.1 as an essential tool to prove the Hölder continuity of solutions, in a similar fashion to the approach firstly used by J. Moser in [24] and successively developed in [8] for degenerate parabolic equations of p-Laplacian type with measurable coefficients.

Theorem 1.1. *Let u be a non-negative local weak solution to $(*)$ such that for some point $(x_0, t_0) \in \Omega_T$, we have $u(x_0, t_0) > 0$. There exist constants c, γ depending only upon p_i, N such that the following inequality holds for all intrinsic cylinders $(x_0, t_0) + \mathcal{Q}^+_{4\rho}(\theta)$ contained in Ω_T*

$$\text{(1.6)} \qquad u(x_0, t_0) \le \gamma \inf_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}), \qquad \theta = \left(\frac{c}{u(x_0, t_0)}\right).$$

It is remarkable that estimate (1.6) is prescribed on a *space* configuration dependent on the solution. This property differs substantially from the isotropic case because it reveals a typical anisotropic intrinsic geometry. In this setting, an expansion of positivity can be performed by means of the comparison principle (see [6]). In the present work, we show that Theorem 1.1 implies local Hölder continuity of local weak solutions to $(*)$.

Theorem 1.2. *Let u be a local weak solution to $(*)$. Then u is locally Hölder continuous in Ω_T, i.e., there exist constants $\gamma > 1$, $\alpha \in (0, 1)$ depending only upon p_i, N, such that for each compact set $K \subset\subset \Omega_T$ we have*

$$\text{(1.7)} \qquad |u(x, t) - u(y, s)| \le \gamma ||u||_\infty \left(\frac{\sum_{i=1}^N |x_i - y_i|^{\frac{p_i}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-p_i}{p_i}} + |t - s|^{\frac{1}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-2}{\bar{p}}}}{\pi\text{-}dist(K, \partial\Omega_T)} \right)^\alpha,$$

for every pair of points (x, t), $(y, s) \in K$, with

(1.8)

$$\pi\text{-}dist(K, \partial\Omega_T) := \inf \left\{ \left(|x_i - y_i|^{\frac{p_i}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-p_i}{p_i}} \wedge |t - s|^{\frac{1}{\bar{p}}} ||u||_\infty^{\frac{\bar{p}-2}{\bar{p}}} \right) : (x, t) \in K, (y, s) \in \partial\Omega_T, \ i = 1..N \right\}.$$

Moreover, through a similar approach to the isotropic case in [9], we show that the classical Pini-Hadamard estimate can be recovered (see [25] for the complete reference)

$$\text{(1.9)} \qquad \gamma^{-1} \sup_{K_\rho(x_0)} u(\cdot, t_0 - \rho^2) \le u(x_0, t_0)n \le \gamma \inf_{K_\rho(x_0)} u(\cdot, t_0 + \rho^2), \quad \gamma > 0,$$

when $p_i \equiv 2$ for all $i = 1, .., N$ and provided the parabolic cylinders $(x_0, t_0) + \mathcal{Q}^\pm_{4\rho}$ are contained in Ω_T. Indeed, the following theorem can be shown to be sole consequence of Theorem 1.1.

Theorem 1.3. *Let u be a non-negative local weak solution to $(*)$ such that for some point $(x_0, t_0) \in \Omega_T$ we have $u(x_0, t_0) > 0$. There exist constants c, γ depending only upon p_i, N such that for all intrinsic cylinders $(x_0, t_0) + \mathcal{Q}_{4\rho}(\theta)$ contained in Ω_T as in (1.5) we have*

(1.10)

$$\gamma^{-1} \sup_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 - \theta^{2-\bar{p}}\rho^{\bar{p}}) \le u(x_0, t_0) \le \gamma \inf_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 + \theta^{2-\bar{p}}\rho^{\bar{p}}), \qquad \theta = \left(\frac{c}{u(x_0, t_0)}\right).$$

Remark 1.1. *When the elliptic counterpart is considered, we deal with stationary solutions to* (∗)*, so that the behavior at each time is always the same. In this context, easily deductible by* (1.10)*, we get the usual sup-inf estimate with no need of a waiting time. What is essential (at least for the proof of* (1.6) *in* [6]*) and deeply different from the isotropic case where the elliptic estimate holds in classic cubes, is the intrinsic space geometry* (1.4) *of* $\mathcal{K}_\rho(\theta)$*.*

2. Preliminaries

Given $\mathbf{p} := (p_1, .., p_N)$, $\mathbf{p} > 1$ with the usual meaning, we assume that the harmonic mean is smaller than the dimension of the space variables

$$\bar{p} := \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \right)^{-1} < N, \tag{2.11}$$

and we define the Sobolev exponent of the harmonic mean $\bar{p}$,

$$\bar{p}^* := \frac{N\bar{p}}{N - \bar{p}}. \tag{2.12}$$

We will suppose all along this note that the p_is are ordered increasingly, without loss of generality. Next, we introduce the natural parabolic anisotropic spaces. Given $T > 0$ and a bounded open set $\Omega \subset \mathbb{R}$, we let $\Omega_T = \Omega \times (0, T]$ and we define

$$W_o^{1,\mathbf{p}}(\Omega) := \{u \in W_o^{1,1}(\Omega) |\, D_i u \in L^{p_i}(\Omega)\},$$

$$L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega)) := \{u \in L^1_{loc}(0, T; W_o^{1,1}(\Omega)) |\, D_i u \in L^{p_i}_{loc}(0, T; L^{p_i}_{loc}(\Omega))\}.$$

A function

$$u \in C^0_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega))$$

is a *local weak solution* of (∗) if for all $0 < t_1 < t_2 < T$ and any test function $\varphi \in C^\infty_{loc}(0, T; C^\infty_o(\Omega))$ it satisfies

$$\int_\Omega u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega (-u\, \varphi_t + \sum_{i=1}^{N} |u_{x_i}|^{p_i - 2} u_{x_i} \varphi_{x_i}) \, dxdt = 0. \tag{2.13}$$

By a density and approximation argument, this actually holds for any test function of the kind

$$\varphi \in W^{1,2}_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0, T; W_o^{1,\mathbf{p}}(\Omega))$$

for any *p_i-semirectangular domain* $\Omega \subset\subset \mathbb{R}^N$, where traces can be properly defined (see Theorem 3 in [16]).

Definition 2.1. *(*[16]*) If the set of N elements of the vector $(p_1, .., p_N)$ consists of L distinct values, let us denote the multiplicity of each of the values by n_i, $i = 1, ..., L$ such that $n_1 + ... + n_L = N$. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the p_i-semirectangular restriction related to the vector $(p_1, .., p_N)$, if there exist bounded Lipschitz domains $\Omega_i \subset \mathbb{R}^{n_i}$, $i = 1, .., L$, such that $\Omega = \Omega_1 \times ... \times \Omega_L$.*

We will suppose all along the work that $\Omega \subset\subset \mathbb{R}^N$ is a *p_i-semirectangular domain*, being considerations and estimates of local nature.

3. Proof of Theorem 1.2

We employ the intrinsic Harnack inequality (1.6) to establish locally quantitative Hölder estimates for local, weak solutions u of ($*$), conditioned to the boundedness condition $2 < p_i < \bar{p}(1+2/N)$ for each $i = 1,..,N$. Fix a point $(x_0,t_0) \in \Omega_T$ which, up to translations, we will consider to be the origin in $\mathbb{R}^{N+1}$, and for an initial radius $\rho_0 > 0$, consider the cylinder $Q' = K_{\rho_0} \times (-\rho_0^2, 0]$, with vertex at $(0,0)$, and set

$$M_0 = \sup_{Q'} u, \quad m_0 = \inf_{Q'} u, \quad \text{and} \quad \omega_0 = M_0 - m_0 = \operatorname*{osc}_{Q'} u.$$

With ω_0 at hand, we can construct the initial cylinder for our purpose, intrinsically scaled:

$$\mathcal{Q}_0 = \mathcal{K}_{\rho_0} \times (-\theta_0^{\bar{p}-2}\rho_0^{\bar{p}}, 0] = \prod_{i=1}^{N} \left\{ |x_i| < \theta_0^{\frac{\bar{p}-p_i}{p_i}} \rho_0^{\frac{\bar{p}}{p_i}} \right\} \times \left(-\theta_0^{\bar{p}-2}\rho_0^{\bar{p}}, 0 \right], \text{ where } \quad \theta_0 = \left(\frac{c}{\omega_0} \right),$$

and c is a constant to be determined later only in terms of the data and independent of u, ρ_0. The accommodation of degeneracy deals with the following fact: if $\omega_0 > c\rho_0$, then $\mathcal{Q}_0 \subset Q'$. Converse inequality would lead directly to continuity in $\mathcal{Q}_0$.

Proposition 3.1. *Either $\omega_0 \leq c\rho_0$, or there exist numbers $\gamma > 1$, $\delta, \varepsilon \in (0,1)$, that can be quantitatively determined only in terms of the data p_i, N and independent of u, ρ_0, such that if we set*

$$\omega_n = \delta\omega_{n-1}, \quad \theta_n = \left(\frac{c}{\omega_n} \right), \quad \rho_n = \varepsilon\rho_{n-1}, \tag{3.14}$$

and

$$\mathcal{Q}_n = \mathcal{Q}_{\rho_n}(\theta_n) = \prod_{i=1}^{N} \left\{ |x_i| < \theta_n^{\frac{\bar{p}-p_i}{p_i}} \rho_n^{\frac{\bar{p}}{p_i}} \right\} \times (-\theta_n^{\bar{p}-2}\rho_n^{\bar{p}}, 0], \quad \textit{for} \quad n \in \mathbb{N}, \quad \textit{holding} \quad \mathcal{Q}_{n+1} \subset \mathcal{Q}_n,$$

then the oscillation in $\mathcal{Q}_n$ can be controlled with the same constant by the oscillation in $\mathcal{Q}_{n-1}$, i.e.,

$$\operatorname*{osc}_{\mathcal{Q}_n} u \leq \omega_n. \tag{3.15}$$

Proof. The proof is by induction: we show constants δ, ε, c depending only upon the data p_i, N, such that if the statement holds for the n-th step, then it holds the the $(n+1)$-th. Coherently with the accommodation of degeneracy, the first inductive step has already been stated. Assume now that $\mathcal{Q}_n$ has been constructed and that the statement holds up to n. Set

$$M_n = \sup_{\mathcal{Q}_n} u, \quad m_n = \inf_{\mathcal{Q}_n} u, \quad \text{and} \quad P_n = \left(0, -\frac{1}{2}\theta_n\rho_n^{\bar{p}} \right).$$

The point P_n is roughly speaking the point whose coordinates are the mid-point of each coordinate of $\mathcal{Q}_n$. On a first glance, we observe that we can assume

$$\omega_n \leq M_n - m_n = \operatorname*{osc}_{\mathcal{Q}_n} u,$$

because otherwise there is nothing to prove. At least one of the two inequalities

$$M_n - u(P_n) > \frac{1}{4}\omega_n, \quad \text{or} \quad u(P_n) - m_n > \frac{1}{4}\omega_n$$

must hold. Indeed, if it is not so, we arrive at the contradiction

$$\omega_n \leq M_n - m_n \leq \frac{1}{2}\omega_n.$$

We assume that the first inequality holds true, the proof for the second case is similar. The function $(M_n - u)$ is a nonnegative weak solution of ($*$) in $\mathcal{Q}_n$, and satisfies the intrinsic Harnack inequality (1.6) with respect to P_n, if its waiting time and space levels

$$\left(\frac{c_1}{M_n - u(P_n)}\right)^{\bar{p}-2}, \left(\frac{c_1}{M_n - u(P_n)}\right)^{\frac{\bar{p}-p_i}{p_i}},$$

are inside the respective time and space ranges of $\mathcal{Q}_n$. To this aim, we define c to be greater than c_1 so that

$$\left(\frac{c_1}{M_n - u(P_n)}\right) \le \left(\frac{c}{\omega_n}\right), \quad \text{as} \quad \omega_n \le 4\Big(M_n - u(P_n)\Big).$$

Finally, we have the estimate

$$\inf_{\mathcal{Q}_{\rho_n/4}} (M_n - u) \ge \frac{1}{\gamma}(M_n - u(P_n)) > \frac{1}{4\gamma}\omega_n.$$

This means, as $\inf(-u) = -\sup u$, that

$$M_n \ge \sup_{\mathcal{Q}_{\rho_n/4}} u + \frac{1}{4\gamma}\omega_n \ge \sup_{\mathcal{Q}_{n+1}} + \frac{1}{4\gamma}\omega_n, \quad \text{if} \quad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n},$$

leading us, by subtracting $\inf_{\mathcal{Q}_{n+1}} u$ from both sides, to

$$M_n - \inf_{\mathcal{Q}_n} u \ge M_n - \inf_{\mathcal{Q}_{n+1}} u \ge \operatorname*{osc}_{\mathcal{Q}_{n+1}} u + \frac{1}{4\gamma}, \quad \text{if} \qquad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n},$$

and thus to

$$\operatorname*{osc}_{\mathcal{Q}_{n+1}} u \le \delta\omega_n = \omega_{n+1}, \quad \text{if} \qquad \mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n}.$$

By choosing

$$\delta = 1 - \frac{1}{4\gamma} \quad \text{and} \quad \varepsilon = \frac{1}{4}\delta^{\frac{\bar{p}-2}{\bar{p}}}, \tag{3.16}$$

we manage to have both the inclusion $\mathcal{Q}_{n+1} \subset Q_{\rho_n/4} \subset \mathcal{Q}_{\rho_n}$ and the $(n+1)$-th conclusion of the iterative step (3.15). Indeed, by direct computation,

$$\begin{aligned}
\theta_{n+1}^{\bar{p}-2}\rho_{n+1}^{\bar{p}} &= \left(\frac{c}{\omega_{n+1}}\right)^{\bar{p}-2}\left(\frac{\rho_n^{\bar{p}}}{4^{\bar{p}}(\frac{4\gamma-1}{4\gamma})^{\bar{p}-2}}\right)\\
&= \left(\frac{4\gamma c}{(4\gamma-1)\omega_{n+1}}\right)^{\bar{p}-2}\left(\frac{\rho_n}{4}\right)^{\bar{p}}\\
&= \left(\frac{c}{\omega_n}\right)^{\bar{p}-2}\left(\frac{\rho_n}{4}\right)^{\bar{p}}\\
&= \theta_n^{\bar{p}-2}(\rho_n/4)^{\bar{p}},
\end{aligned}$$

precisely, and for each $i \in \{1, .., N\}$ as $p_i > 2$, it holds

$$\begin{aligned}
\theta_{n+1}^{\frac{\bar{p}-p_i}{p_i}} \rho_{n+1}^{\frac{\bar{p}}{p_i}} &= \left(\frac{c}{\omega_{n+1}}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\left(1-\frac{1}{4\gamma}\right)^{\frac{\bar{p}-2}{\bar{p}}}\right)^{\frac{\bar{p}}{p_i}} \\
&= \left(\frac{c}{(1-\frac{1}{4\gamma})\omega_n}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\bar{p}-2} p_i \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \\
&= \left(\frac{c}{\omega_n}\right)^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\frac{p_i-\bar{p}}{p_i}+\frac{\bar{p}-2}{p_i}} \\
&= \theta_n \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}} \left(1-\frac{1}{4\gamma}\right)^{\frac{p_i-2}{p_i}} \leq \theta_n^{\frac{\bar{p}-p_i}{p_i}} \left(\frac{\rho_n}{4}\right)^{\frac{\bar{p}}{p_i}}.
\end{aligned}$$

□

3.1. **Conclusion of the proof of Theorem 1.2.** Owing to the previous Proposition, we conclude that on each such intrinsically scaled cylinder $\mathcal{Q}_n$ it holds $\operatorname{osc}_{\mathcal{Q}_n} u \leq \omega_n$, so that by induction and the definition of ω_n, we have

$$\operatorname*{osc}_{\mathcal{Q}_n} u \leq \delta^n \omega_0.$$

Let now $0 < \rho < r$ be fixed, and observe that there exists a $n \in \mathbb{Z}$ such that, by use of (3.16), we have

$$\epsilon^{n+1} r \leq \rho \leq \epsilon^n r.$$

This implies

$$(n+1) \geq \ln\left(\frac{\rho}{r}\right)^{\frac{1}{\ln(\epsilon)}} \Rightarrow \delta^n \leq \frac{1}{\delta}\left(\frac{\rho}{r}\right)^{\alpha}, \quad \text{with} \quad \alpha = \frac{|\ln(\delta)|}{|\ln(\epsilon)|}, \tag{3.17}$$

by an easy change of basis on the logarithm. Thus, by (3.16) and (3.17), we get

$$\operatorname*{osc}_{\mathcal{Q}_0} u \leq \operatorname*{osc}_{\mathcal{Q}_n} u \leq \frac{\omega_0}{\delta}\left(\frac{\rho}{r}\right)^{\alpha}. \tag{3.18}$$

Now finally, we give Hölder conditions to each variable, irrespective to the others. Fix $(x, t), (y, s) \in K$, $s > t$, let $R > 0$ to be determined later, and construct the intrinsic cylinder $(y, s) + \mathcal{Q}_R(M)$, where $M = ||u||_{L^\infty(\Omega_T)}$. This cylinder is contained in Ω_T if the variables satisfy for each $i = 1, .., N$,

$$M^{\frac{p_i-\bar{p}}{p_i}} R^{\frac{\bar{p}}{p_i}} \leq \inf\left\{|x_i - y_i|, \quad \text{for} \quad x \in K,\, y \in \partial\Omega\right\} \quad \text{and} \quad M^{\frac{2-\bar{p}}{\bar{p}}} R \leq \inf_{t \in K} t^{\frac{1}{\bar{p}}}.$$

This is easily achieved if we set, for instance,

$$2R = \pi\text{-dist}(K; \partial\Omega).$$

To prove the Hölder continuity in the variable t, we first assume that $(s-t) \leq M^{2-\bar{p}} R^{\bar{p}}$. Then $\exists \rho_0 \in (0, R)$ such that $(s-t)^{\frac{1}{\bar{p}}} M^{\frac{\bar{p}-2}{\bar{p}}} = \rho_0$, and the oscillation (3.18) gives

$$\operatorname*{osc}_{\mathcal{Q}_{\rho_0}} u \leq \gamma\omega_0\left(\frac{\rho_0}{R}\right)^{\alpha},$$

implying

$$|u(x,s) - u(x,t)| \leq \gamma M\left(\frac{M^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}}{\pi\text{-dist}(K; \partial\Omega_T)}\right)^{\alpha},$$

as claimed. If otherwise $s-t \geq M^{2-\bar{p}}R^{\bar{p}}$ then, exploiting the fact that $\rho_0^\alpha \leq 4R$, we have

$$|u(x,s)-u(x,t)| \leq |u(x,s)|+|u(x,t)| \leq 2M \leq 4M\left(\frac{M^{\frac{\bar{p}-2}{\bar{p}}}|s-t|^{\frac{1}{\bar{p}}}}{\pi\text{-dist}(K;\partial\Omega)}\right)^\alpha.$$

About the space variables, we have for each i-th one the following alternative:

- If $|y_i-x_i| < M^{\frac{p_i-\bar{p}}{p_i}}R^{\frac{\bar{p}}{p_i}}$, and then analogously $\exists \rho_0 \in (0,R)$ such that $\rho_0 = |y_i - x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}$ and the oscillation reduction (3.18) gives

$$\operatorname*{osc}_{\mathcal{Q}_0} u \leq \operatorname*{osc}_{\mathcal{Q}_{\rho_0}(\theta_0)} u \leq \gamma\omega_0\left(\frac{\rho_0}{R}\right)^\alpha \quad \Rightarrow \quad |u(y_i,t)-u(x_i,t)| \leq \gamma M\left(\frac{|y_i-x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}}{\pi\text{-dist}(k;\partial\Omega_T)}\right)^\alpha.$$

- If otherwise $|y_i-x_i| \geq M^{\frac{p_i-\bar{p}}{p_i}}R^{\frac{\bar{p}}{p_i}}$, then similarly

$$|u(y_i,t)-u(x_i,t)| \leq 2M \leq 4M\left(\frac{|y_i-x_i|^{\frac{p_i}{\bar{p}}}M^{\frac{\bar{p}-p_i}{p_i}}}{\pi\text{-dist}(k;\partial\Omega_T)}\right)^\alpha.$$

The proof is completed.

4. Proof of Theorem 1.3

We take as hypothesis that for each radius $r>0$ such that the intrinsic cylinder $\mathcal{Q}_{4r}(\theta)$ is contained in Ω_T, the right-hand Harnack estimate (1.6) holds and we show that the full Harnack estimate (1.10) comes as a consequence.

4.1. **Step 1.** Let us suppose that there exists a time $t_1 < t_0$ such that

$$u(x_0,t_1) = 2\gamma u(x_0,t_0), \tag{4.19}$$

where $\gamma, c > 0$ are the constants in (1.6). For such a time, it must hold

$$t_0 - t_1 > \theta_{t_1}^{\bar{p}-2} r^{\bar{p}} := c\, u(x_0,t_1)^{2-\bar{p}} r^{\bar{p}} = c\,\frac{u(x_0,t_0)^{2-\bar{p}}}{(2\gamma)^{\bar{p}-2}} r^{\bar{p}}, \tag{4.20}$$

owing last equality to (4.19). Indeed, if (4.20) were violated then $t_0 \in [t_1, t_1 + \theta_{t_1}^{\bar{p}-2} r^{\bar{p}}]$, and by applying (1.6) evaluated in (x_0,t_1) for a radius $r>0$ small enough, we would incur a contradiction

$$u(x_0,t_1) \leq \gamma u(x_0,t_0) \qquad \Longleftrightarrow \qquad 2\gamma u(x_0,t_0) \leq u(x_0,t_0).$$

So (4.20) holds, and we set t_2 to be the time

$$t_2 = t_0 - \theta_{t_1}^{\bar{p}-2} r^{\bar{p}}. \tag{4.21}$$

By (4.20), we deduce that $t_1 < t_2 < t_0$ and again by the right-hand Harnack estimate (1.6), we have that

$$u(x_0,t_0) = \frac{u(x_0,t_1)}{2\gamma} \leq u(x_0,t_2) < 2\gamma u(x_0,t_0), \tag{4.22}$$

where the last inequality comes from t_1 being the first time before t_0 respecting (4.19). The contradiction of (4.19) is, in our context $u(x_0,t_2) < 2\gamma u(x_0,t_0)$, because the converse inequality conflicts with our hypothesis (1.6). Now, let $r>0$ be fixed, as in (1.6), and consider the vector $\xi \in \mathbb{R}^N$ whose components are

$$\xi_i := \theta^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}}, \quad \theta = \left(\frac{c}{u(x_0,t_0)}\right). \tag{4.23}$$

Now for each vector of parameters $s \in [0,1]^N$ define $\xi_s = (s_1\xi_1, ..., s_N\xi_N)$. As s varies in $[0,1]^N$, the configuration $x_0 + \xi_s$ describes all points of $x_0 + \mathcal{K}_r(\theta)$. Consider $\bar{s} \in [0,1]^N$ such that the vector $\xi_{\bar{s}}$ satisfies $u(x_0 + \xi_{\bar{s}}, t_2) = 2\gamma u(x_0, t_0)$. We claim that such an $\bar{s}$ does not exist or that $\bar{s} \geq 1$: in either case the conclusion is that

$$\sup_{x_0+\mathcal{K}_r(\theta)} u(\cdot, t_2) \leq 2\gamma\, u(x_0, t_0). \tag{4.24}$$

Thus to establish the claim, assume that such vector $\bar{s}$ exists and that $\bar{s} < 1$. Apply the estimate (1.6) in the point (x_2, t_2) with $x_2 = x_0 + \xi_{\bar{s}}$ to get

$$u(x_2, t_2) \leq \gamma \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_2 + \theta_{t_2}^{\bar{p}-2} r^{\bar{p}}) = \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0), \quad \text{being} \quad \theta_{t_2} = \frac{c}{u(x_2, t_2)},$$

where last equality holds because of

$$\begin{aligned} t_2 + \theta_{t_2}^{\bar{p}-2} r^{\bar{p}} &= t_0 - \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{u(x_2, t_2)}\right)^{\bar{p}-2} \\ &= t_0 - \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} = t_0, \end{aligned} \tag{4.25}$$

being x_2 the point for which holds $u(x_2, t_2) = u(x_0 + \xi_{\bar{s}}, t_2) = 2\gamma\, u(x_0, t_0)$ by assumption. But since $\bar{s} < 1$, then $x_0 \in \{x_2 + \mathcal{K}_r(\theta_{t_2})\}$ and we arrive to the contradiction

$$2\gamma\, u(x_0, t_0) = u(x_2, t_2) \leq \gamma \inf_{x_2+\mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0) \leq \gamma\, u(x_0, t_0).$$

Finally, the contradiction implies that (4.24) holds, which means that for each $r > 0$ such that $\mathcal{Q}_{4r}(\theta) \subseteq \Omega_T$ it holds

$$\sup_{x_0+\mathcal{K}_r(\theta)} u\left(\cdot\, , \left(\frac{c}{2\gamma\, u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}}\right) \leq 2\gamma\, u(x_0, t_0).$$

Let $\rho > 0$ be such that the right hand side of (1.10) holds, then by choosing $r = \rho(2\gamma)^{\frac{\bar{p}-2}{\bar{p}}}$ we obtain, by suitably redefining the constants, the full estimate (1.10).

4.2. **Step 2.** Suppose on the contrary that such a time $t < t_0$ for which holds true (4.19) does not exist. In this case, we have

$$u(x_0, t) < 2\gamma u(x_0, t_0), \quad \text{for all} \quad t \in [t_0 - \theta(4r)^{\bar{p}}, t_0], \tag{4.26}$$

because the converse inequality would be in conflict with the holding Harnack estimate. We establish by contradiction that this in turn implies

$$\sup_{x_0+\mathcal{K}_r(\theta)} u(\cdot, t_0 - \theta^{\bar{p}-2} r^{\bar{p}}) \leq 2\gamma^2 u(x_0, t_0). \tag{4.27}$$

If not, it simultaneously holds (4.26) and a fortiori

$$\sup_{x_0+\mathcal{K}_\rho(\theta)} u(\cdot, \bar{t}) > 2\gamma^2 u(x_0, t_0) > u(x_0, \bar{t}), \qquad \text{for} \quad \bar{t} = t_0 - \theta^{\bar{p}-2} r^{\bar{p}}. \tag{4.28}$$

Thus, by the proven continuity in space, there must exist by the intermediate value theorem a point $\bar{x} \in x_0 + \mathcal{K}_r(\theta)$ such that

$$u(\bar{x}, \bar{t}) = 2\gamma u(x_0, t_0). \tag{4.29}$$

We apply the Harnack estimate (1.6) centered in $(\bar{x}, \bar{t})$ to get

$$u(\bar{x}, \bar{t}) \leq \gamma \inf_{\bar{x}+\mathcal{K}_r(\theta_{\bar{t}})} u(\cdot, \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}}), \quad \text{where} \quad \theta_{\bar{t}} = \frac{c}{u(\bar{x}, \bar{t})}.$$

Now, as $\gamma > 1$ and $p_i > 2$ for each $i \in \{1, .., N\}$, we have

$$\begin{cases} \theta_{\bar{t}}^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} &= \left(\frac{2}{2\gamma u(x_0,t_0)}\right)^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} \geq \left(\frac{2}{2u(x_0,t_0)}\right)^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} = \theta^{\frac{p_i-\bar{p}}{p_i}} r^{\frac{\bar{p}}{p_i}} \quad \Rightarrow \quad x_0 \in \{\bar{x} + \mathcal{K}_r(\theta_{\bar{t}})\}, \\ \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}} &= t_0 - \left(\frac{c}{u(x_0,t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_0,t_0)}\right)^{\bar{p}-2} < t_0, \end{cases}$$

and thus, finally,

$$2\gamma^2 u(x_0, t_0) = u(\bar{x}, \bar{t}) \leq \gamma u(x_0, \bar{t} + \theta_{\bar{t}}^{\bar{p}-2} r^{\bar{p}}) < 2\gamma^2 u(x_0, t_0),$$

owing last inequality to (4.28) and establishing (4.27) by contradiction. Finally, the estimate (4.27), by possibly redefining the constants, is the desired left-hand estimate of (1.10).

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