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GOMPERTZ-EXPONENTIAL DISTRIBUTION: RECORD VALUE THEORY AND APPLICATIONS IN RELIABILITY

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Abstract: The continuous probability distributions have wide applications in the field of transportation and reliability engineering. The continuous distributions are used to estimate how funds can be allocated to improve roads, railways, bridges, waterways, airports etc. and used to check the reliability/performance of a product. The Gompertz exponential (GoE) distribution is derived using Gompertz G generator. Some basic properties of the model have been derived. The parameters of the GoE distribution are estimated by maximum likelihood estimation method. The upper record values from the GoE distribution have also been introduced with various properties. Moreover, applications of the GoE distributions has been provided in the field of reliability to check the performance of some transportation related parts and the suggested model provides better fit than the existing well-known models. Finally, a simulation study is carried out. Random numbers of size 50 are generated 15 times for GoE distribution and upper records has been noted.

Key words: Gompertz family of distributions, Exponential, MLE, GoE, Reliability.

1. Introduction

In several real-life applications, the classical distributions do not appropriately work to some real data sets. Thus, researchers introduced many generators by introducing one or more parameters to generate new distributions. The new generated distributions are more flexible as compared to the classical distributions.

Some well-known generators are as follows: Marshall and Olkin [14] generated Marshal-Olkin family, Eugene et al [12] and Jones [13] introduced Beta G, Cordeiro and de Castro [10] developed Kumaraswamy G, Alexander et al [2] presented McDonald G, Zografos and Balakrishnan [22] introduced gamma G type 1, Risti ́c and Balakrishnan [19] introduced gamma G type 2, Torabi and Hedesh [21] developed gamma G type 3, Amini et al [6] developed log-gamma G, Cordeiro et al [11] developed exponentiated generalized G, Alzaatreh et al [4] and Alzaghal et al [5] introduced transformed transformer T-X and exponentiated T-X respectively, Bourguignon et al [7] developed Weibull G, Cordeiro et al [9] generated exponentiated half logistic family. Morad et al [15] introduced another generator for continuous distributions called the Gompertz G generator and presented several mathematical properties of it.

In this article the Gompertz family of distribution is considered to develop a new model. Alizadeh et al [3], and Abdal-Hameed, et al [1] used this generator in their work. The cumulative distribution function (cdf) and probability density function (pdf) of the Gompertz family of distributions is

$$F(x) = 1 - e^{\frac{\alpha}{\beta}[1-[1-G(x)]^{-\beta}]}, \quad \alpha > 0, \beta > 0 \tag{1.1}$$

$$f(x) = \alpha g(x)[1 - G(x)]^{-\beta-1} e^{\frac{\alpha}{\beta}[1-[1-G(x)]^{-\beta}]}, \quad \alpha > 0, \beta > 0 \tag{1.2}$$

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where α and β are extra shape parameters and the cdf in Eq. (1.1) and the pdf in Eq. (1.2) was developed using the following transformation:

$$F(x) = \int_0^{-log[1-G(x)]} w(t)dt,$$

$w(t)$ is the probability density function (pdf) of the Gompertz distribution, where t is a random variable. $G(x)$ and $g(x)$ are the cdf and pdf of the baseline distribution. The probability density function (pdf) of the exponential distribution is

$$f(x) = \theta e^{-\theta x}, \theta > 0, x > 0, \tag{1.3}$$

where θ is scale parameter.

The cumulative distribution function (cdf) of the exponential distribution is

$$F(x) = 1 - e^{-\theta x}, \theta > 0, x > 0. \tag{1.4}$$

If an observation is greater than (or less than) of all the values in the experiment, then this value is called a record values. Records values, extreme and lower both have wide application in the fields of studies such as in the science of climates, sports, engineering, medical fields, traffic and transportation, and other industry. Basically, Chandler [8] developed the initials of the record value theory and then later many works has been done it. Further work done by many researchers on almost every continuous probability distribution. The pdf of the sequence of upper record values $[X_{U(n)}, n > 1]$ is

$$f_n(x) = \frac{[R(x)]^{n-1}}{\Gamma(n)} f(x), -\infty < x < \infty, \tag{1.5}$$

where $R(x) = -\ln[1 - F(x)]$.

2. Development of the GoE Distribution

The cumulative distribution function of the GoE distribution is obtained by substituting Eq. (1.4) in Eq. (1.1),

$$F(x) = 1 - e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}; \quad \alpha, \beta, \theta > 0, x > 0, \tag{2.1}$$

$$f(x) = \alpha\theta e^{\beta\theta x} e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}; \quad \alpha, \beta, \theta > 0, x > 0, \tag{2.2}$$

where θ is scale parameter and α,β are shape parameters. The Gompertz Exponential distribution is graphically represented in Figures 1 and 2.

In Figure 1, the proposed pdf is positively skewed for various combinations of parameters.

2.1. Properties of the GoE Distribution

In this section, some GoE distribution properties have been derived. The graph for the reliability measures have also been presented and discussed.

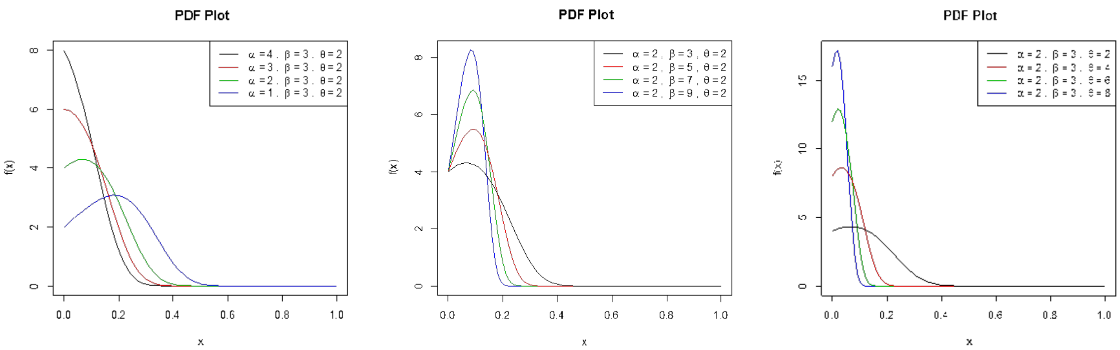
The mean of the GoE distribution is

$$E(X) = \alpha\theta \int_0^{\infty} x e^{\beta\theta x} e^{\frac{\alpha}{\beta}} (1 - e^{\beta\theta x}) dx. \tag{2.3}$$

The variance of the GoE distribution can be calculated by solving the integral in eq (2.3) and in eq (2.4)

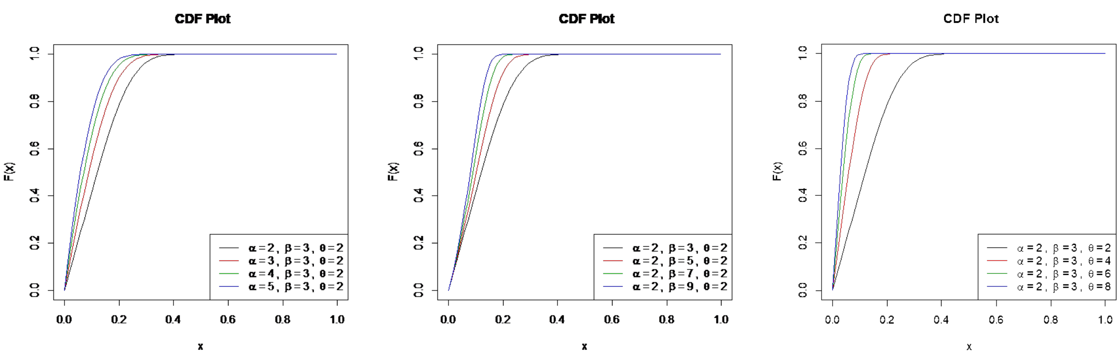
$$E(X^2) = \alpha\theta \int_0^{\infty} x^2 e^{\beta\theta x} e^{\frac{\alpha}{\beta}} (1 - e^{\beta\theta x}) dx. \tag{2.4}$$

The above integrals are unsolvable therefore the numerical values for the mean and variance have been calculated for different values of parameters and presented in Table 1.



(a) (b) (c)

FIGURE 1. Pdf plots for various parametric values



(a) (b) (c)

FIGURE 2. Cdf plots for various parametric values

TABLE 1. Mean and variance for the Gompertz exponential distribution

$\theta=2,\ \beta=3$	α	Mean	Variance	CV
	1	0.192801	0.01232	57.57
	2	0.129332	0.007197	65.59
	3	0.099391	0.004897	70.41
	4	0.081352	0.003604	73.79
	5	0.069123	0.002786	76.36

$\theta=2,\ \alpha=2$	β	Mean	Variance	CV
	4	0.115364	0.005150	62.21
	5	0.104783	0.003903	59.62
	7	0.089546	0.002502	55.86
	9	0.078908	0.001509	49.23
	10	0.074667	0.001509	52.03

$\alpha=2,\ \beta=2$	θ	Mean	Variance	CV
	1	0.258665	0.030330	67.32
	3	0.086222	0.003199	65.60
	4	0.064666	0.001798	65.57
	5	0.051733	0.001152	65.61
	6	0.043111	0.000799	65.57

From Table 1, mean and variance are decreasing as α increasing at fixing β and θ. Similarly mean and variance are decreasing while fixing α and θ with increasing β and same trend is with fixing α and β and increasing θ. While CV is increasing with increasing α, CV is decreasing with increasing β and CV is not varying much with increasing θ.

Reliability function, hazard rate function, reversed hazard rate function and odd function are as follows respectively,

$$R(x) = e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}, \tag{2.5}$$

$$h(x) = \alpha\theta e^{\beta\theta x}. \tag{2.6}$$

In Eq. (2.6), for $\beta = 0$ the hazard rate function is constant as given below,

$$h(x) = \alpha\theta. \tag{2.7}$$

The hazard rate function given in Eq. (2.7) is constant due to putting $\beta = 0$, in Eq. (2.6). Therefore, we can say that if the shape parameter β is zero then GoE distribution showing the constant hazard rate.

$$r(x) = \frac{\alpha\theta e^{\beta\theta x} e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}}{1 - e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]},} \tag{2.8}$$

$$O(x) = \frac{1 - e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}}{e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}.} \tag{2.9}$$

The cumulative hazard function for GoE distribution is

$$H(x) = \frac{\alpha}{\beta}[e^{\beta\theta x} - 1]. \tag{2.10}$$

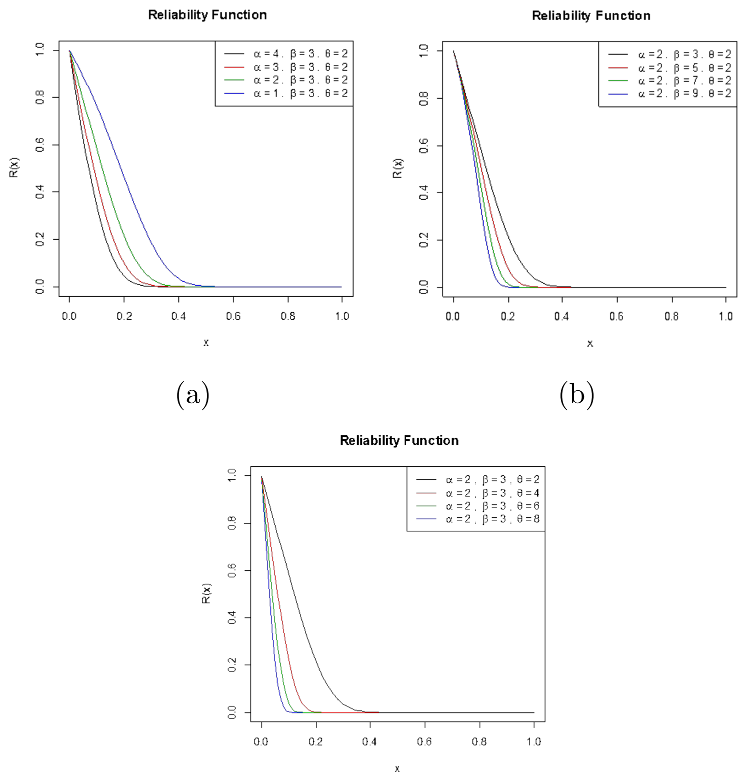
The Shannon entropy for GoE distribution is

$$S(x) = \frac{\theta(\alpha+\beta)}{\beta} - \ln(\alpha\theta) - \frac{\alpha^2\theta}{\beta^2} - \frac{\alpha e^{\frac{\alpha}{\beta}}}{\beta}\Psi_X(\frac{\alpha}{\beta}), \tag{2.11}$$

where $\Psi_X = \int_1^\infty \ln(x) e^{-\frac{\alpha}{\beta}x} dx$.

The graphs of the reliability function and hazard rate function of the GoE are presented in Figures 3 and 4. From Figure 3, the reliability function is monotonically decreasing with varying parametric values.

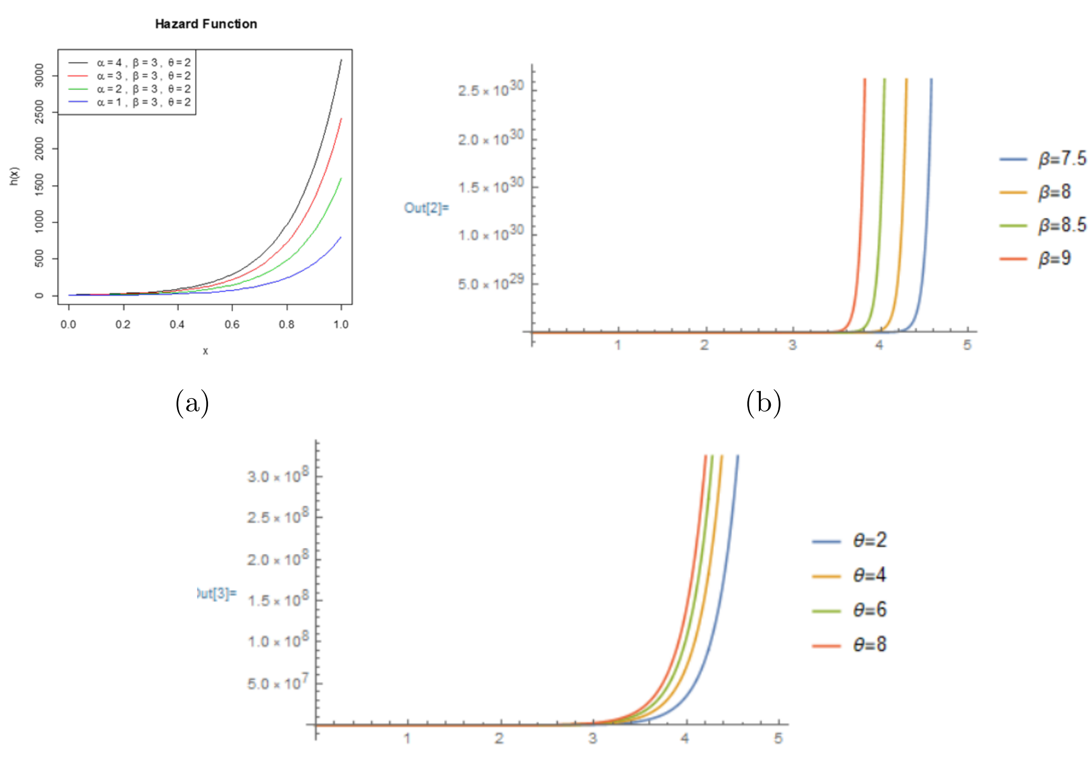
From Figure 4, we can see that the hazard rate function of the GoE distribution shows the increasing form of bath tub (IBT) shape or J-shaped. As the time passes on, or aging a person, there is greater chance of death and same concept with the products, as the life time of the product increases there more chances of failure that product.



(a) (b)

(c)

FIGURE 3. Reliability graphs for different parametric values



(a) (b)

(c)

FIGURE 4. Hazard rate graphs for different parametric values

2.2. Quantile Function and Median

The quantile function of the GoE distribution is

$$Q(u) = \frac{ln[1 - \frac{\beta}{\alpha} ln[1-u]]}{\beta\theta}. \tag{2.12}$$

The median of the GoE distribution is

$$median = \frac{ln[1 - \frac{\beta}{\alpha} ln[0.5]]}{\beta\theta}. \tag{2.13}$$

Mode of the GoE distribution

$$mode = \frac{ln[\frac{\beta}{\alpha}]}{\beta\theta}. \tag{2.14}$$

2.3. Order Statistics of the Gompertz Exponential Distribution

The probability density function of the r^{th} order statistics from GoE distribution is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \alpha\theta e^{\beta\theta x} e^{\frac{(n-r+1)\alpha}{\beta}(1-e^{\beta\theta x})} [1 - e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}]^{r-1}; \alpha, \beta, \theta, x > 0. \tag{2.15}$$

The probability density function of smallest and largest order statistics from GoE distribution is

$$f_{1:n}(x) = n\alpha\theta e^{\beta\theta x} e^{\frac{n\alpha}{\beta}(1-e^{\beta\theta x})}, ; \alpha, \beta, \theta, x > 0, \tag{2.16}$$

$$f_{n:n}(x) = n\alpha\theta e^{\beta\theta x} e^{\frac{\alpha}{\beta}(1-e^{\beta\theta x})} [1 - e^{\frac{\alpha}{\beta}[1-e^{\beta\theta x}]}]^{n-1}; \alpha, \beta, \theta, x > 0. \tag{2.17}$$

3. Record Values from GoE distribution

Using Eq. (2.1) and Eq. (2.2) in Eq. (1.5) the pdf of the upper record values from Gompertz exponential distribution (UR-GoED) is

$$f_n(x) = \frac{\alpha^n \theta}{\beta^{n-1}\Gamma(n)} e^{\beta\theta x} (e^{\beta\theta x} - 1)^{n-1} e^{\frac{\alpha}{\beta}(1-e^{\beta\theta x})}; \alpha, \beta, \theta, x > 0. \tag{3.1}$$

The cumulative distribution function of the UR-GoED is

$$F_n(x) = \frac{1}{\Gamma(n)} \gamma(n, \frac{\acute{x}\alpha}{\beta}). \tag{3.2}$$

The reliability function of the UR-GoED is

$$R_n(x) = \frac{1}{\Gamma(n)}\Gamma(n, \frac{\acute{x}\alpha}{\beta}). \tag{3.3}$$

The hazard rate function of the UR-GoED is

$$h_n(x) = \frac{\alpha^n \theta e^{\beta\theta x}(e^{\beta\theta x}-1)^{n-1} e^{\frac{\alpha}{\beta}(1-e^{\beta\theta x})}}{\beta^{n-1}\Gamma(n, \frac{\acute{x}\alpha}{\beta})}. \tag{3.4}$$

The mean of the UR-GoED is

$$E(X_{U(n)}) = \frac{1}{\theta\beta\Gamma(n)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\Gamma(k+n)}{k} \Big(\frac{\beta}{\alpha}\Big)^k. \tag{3.5}$$

The relation between pdf and cdf of the GoE distribution using Eq. (2.1) and Eq. (2.2), we get

$$f(x) = \alpha\theta e^{\beta\theta x}[1-F(x)]. \tag{3.6}$$

4. Parameter Estimation

The Maximum Likelihood Estimation (MLE) methodology is used to estimate the parameters of the GoE distribution. Let $x_1, x_2, x_3, ..., x_n$ be the random samples distributed GoE distribution.

$$L(x_1, x_2, ..., x_n; \alpha, \beta, \theta) = \prod_{i=1}^{n} \alpha\theta e^{\beta\theta x_i} e^{\frac{\alpha}{\beta}(1-e^{\beta\theta x_i})},$$

$$\ln L(\alpha, \beta, \theta) = n\log\theta + n\log\alpha + \beta\theta\sum_{i=1}^{n} x_i + \frac{\alpha}{\beta}\sum_{i=1}^{n}(1-e^{\beta\theta x_i}), \tag{4.1}$$

$$\frac{\partial L(\alpha, \beta, \theta)}{\partial\theta} = \frac{n}{\theta} + \beta\sum_{i=1}^{n} x_i - \alpha\sum_{i=1}^{n} x_i e^{\beta\theta x_i}, \tag{4.2}$$

$$\frac{\partial L(\alpha, \beta, \theta)}{\partial\alpha} = \frac{n}{\alpha} + \frac{1}{\beta}\sum_{i=1}^{n}(1-e^{\beta\theta x_i}), \tag{4.3}$$

$$\frac{\partial L(\alpha, \beta, \theta)}{\partial\beta} = \theta\sum_{i=1}^{n} x_i - \frac{\alpha\theta}{\beta}\sum_{i=1}^{n} x_i e^{\beta\theta x_i} - \frac{\alpha}{\beta^2}\sum_{i=1}^{n}(1-e^{\beta\theta x_i}). \tag{4.4}$$

5. Simulations

Random numbers of size 50 from GoE distribution are generated 15 times and the upper record are noted. That are considered the upper record values from GoE distribution. To simulate the random numbers, we use the quantile function given in Eq. (2.12). Simulations are done by using R-package. Then some descriptive measures are calculated from the upper records from GoE distribution.

The below table mentioned that how we can record the upper records (lower records) in real life situations and used them to forecast the results.

Table 2. Descriptive measures for UR-GoE distribution when $n = 15, \alpha = 0.05, \beta = 0.25, \theta = 0.3$

Mean	Median	G.M	H.M	Variance	S.D	M.D	C.V
41.4374	41.1112	41.2872	41.1407	12.7347	3.5686	2.9141	8.6120%

6. Model Validation and Application

In this section the proposed distribution GoE is applied on two real life data sets and compared with some well-known models. R software is used for the applications and the criterion used for model selection are AIC, CAIC, BIC, NLL and HQIC.

Firstly, we have used data of failure and service times for a particular windshield taken from Murthy et al. [16] Ramos et al. [18] also used this data. There are 147 observations in the data from which 84 are failed windshields, and 63 are service times of windshields that had not failed at the time of observation.

Secondly, this data is relating to the strengths of 1.5cm glass fibers which taken from Oguntunde et al. [17], and Bourguignon, Silva, and Cordeiro [7], Smith and Naylor [20] were also used this data.

Table 3. NLL and goodness of fit criterion for failure times of 84 Aircraft Windshield data

Models	Estimates	NLL	AIC	CAIC	BIC	HQIC
GoED (proposed)	$\hat{\alpha}$=0.0753 $\hat{\beta}$=0.7658 $\hat{\theta}$=1.0372	128.71	263.42	263.72	266.28	266.35
WL	$\hat{a}$=0.0118 $\hat{b}$=0.6462 $\hat{\alpha}$=5.9950 $\hat{\lambda}$=1.3790	127.837	263.675	264.175	273.446	267.605
EL	$\hat{\delta}$=0.0237 $\hat{\alpha}$=6.3074 $\hat{\lambda}$=3.7412	130.549	267.099	267.099	274.427	270.046
KGL	$\hat{a}$=2.9739 $\hat{b}$=19.877 $\hat{\alpha}$=2.2171 $\hat{\lambda}$=12.385	134.888	277.776	278.276	287.546	281.706
GL	$\hat{\alpha}$=5.7000 $\hat{\lambda}$=3.7700 $\hat{\beta}$=1.7e+5 $\hat{\sigma}$=5.1e+4	135.071	278.143	278.643	287.913	282.073
Exponentiated Lomax	$\hat{a}$=3.5417 $\hat{\alpha}$=11653 $\hat{\lambda}$=15521	141.405	288.811	289.107	296.139	291.758
Exponentiated LP	$\hat{\gamma}$=1e-10 $\hat{\sigma}$=3.5473 $\hat{\alpha}$=22063 $\hat{\lambda}$=29382	141.404	290.808	291.308	300.578	294.738
Lomax	$\hat{\alpha}$=30087 $\hat{\lambda}$=76941	164.989	333.978	334.124	338.863	335.943

*WL (Weibull Lomax), EL (exponential Lomax), KGL (Kumaraswamy-Generalized Lomax), GL (Gumbel-Lomax), Exponentiated LP (Lomax Poisson)

Table 4. NLL and goodness of fit criterion for service times of 63 Aircraft Windshield data

Models	Estimates	NLL	AIC	CAIC	BIC	HQIC
GoED (proposed)	$\hat{\alpha}$=0.25528 $\hat{\beta}$=0.59955 $\hat{\theta}$=0.81140	98.27665	202.5533	202.9601	204.8396	205.0820
WL	$\hat{a}$=0.1276 $\hat{b}$=0.9204 $\hat{\alpha}$=3.9136 $\hat{\beta}$=3.0067	98.11712	204.2342	204.9239	212.8068	207.6059
McL	$\hat{a}$=1.3230 $\hat{b}$=53.7712 $\hat{c}$=5.7144 $\hat{\alpha}$=7.4371 $\hat{\beta}$=42.8972	98.5883	207.1766	208.2292	217.8923	211.3911
KwL	$\hat{a}$=1.6691 $\hat{b}$=60.5673 $\hat{\alpha}$=2.5649 $\hat{\beta}$=65.0640	100.8676	209.7353	210.4249	218.3078	213.1069
GL	$\hat{a}$=1.9073 $\hat{\alpha}$=35842.4330 $\hat{\beta}$=39197.5715	102.8332	211.6663	212.0731	218.0958	214.1951
BL	$\hat{a}$=1.9218 $\hat{b}$=31.2594 $\hat{\alpha}$=4.9684 $\hat{\beta}$=169.5719	102.9611	213.9223	214.6119	222.4948	217.2939
EL	$\hat{a}$=1.9145 $\hat{\alpha}$=22971.1536 $\hat{\beta}$=32881.9966	103.5498	213.0995	213.5063	219.5289	215.6282
Lomax Distribution	$\hat{\alpha}$=99269.7800 $\hat{\beta}$=207019.3700	109.2988	222.5976	222.7976	226.8839	224.2834

*WL (Weibull Lomax), McL (McDonald Lomax), KwL (Kumaraswamy Lomax),
GL (Gumbel-Lomax), BL (Beta Lomax), EL (Exponential Lomax)

Table 5. NLL and goodness of fit criterion for the strength of 1.5cm glass fibers

Models	Estimates	NLL	AIC	CAIC	BIC	HQIC
GoED (proposed)	$\hat{\alpha}$=0.007459 $\hat{\beta}$=2.762845 $\hat{\theta}$=1.298306	14.8237	35.64743	36.0542	37.9337	38.1761
GoWei.	$\hat{\alpha}$=0.228488761 $\hat{\beta}$=0.009628097 $\hat{\theta}$=0.794918813 $\hat{\lambda}$=5.612111282	15.18847	38.37694	39.06659	46.94948	41.74856
GL	$\hat{\alpha}$=0.004592168 $\hat{\beta}$=8.179090955 $\hat{\theta}$=0.506999370 $\hat{\lambda}$=1.515829085	14.50274	37.00548	37.69513	45.57802	40.3771

*GoWei (Gompertz Weibull), GL (Gompertz Lomax)

7. Conclusions

The proposed model named Gompertz exponential (GoE) distribution is derived in this article and shape of the model can be seen from Figure 1, it is having longer right tail means positively sewed distribution. Generally, positively skewed distributions are mostly preferred in lifetime data sets. Some basic properties of the GoE model have been derived including some reliability measures. Order statistics and upper record values have also been introduced for the proposed model. Parameters of the GoE distribution are estimated by the method of maximum likelihood estimation. A simulation study is carried out for the GoE model by generating random numbers. From the simulated data the upper records have been noted and some descriptive measures are calculated. The record values are used to learn how we can get record from any known model and used them to get results. Finally, the model is applied on three lifetime data sets (Failure times of 84 Aircraft Windshield, Service times of 63 Aircraft Windshield and strengths of 1.5cm glass fibers) and compared with other models.It can be seen from Table 3, 4 and 5 GoE distribution is better fitted as compared to the other well-known distributions. GoE distribution is more flexible as compared to GoWei, Gompertz Lomax, WL, EL, KGL, GL, Exponentiated LP, Lomax, McL, BL, KwL. The continuous probability distributions have great importance in the field of transportations (they are used to estimate how funds can be allocated for to improve roads, railways, bridges, waterways, airports etc.), reliability engineering (to check the reliability of a product or even to check the reliability of a system, failure chances etc.). The newly derived distribution GoE, is applied here in this article on the data sets mentioned previously and it can be seen not only the newly derived model is providing better approach on these data sets with comparison of other well-known existing models also it is showing application in the theory of reliability and transportation. The record values derived from GoE is a new approach that if we find maximum or minimum records in these fields (reliability, transportation, or others) then the record values from GoE distribution can be the best option to apply.

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