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Research Article

Approximation in weighted spaces of vector functions

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ABSTRACT. In this paper, we present the duality theory for general weighted space of vector functions. We mention that a characterization of the dual of a weighted space of vector functions in the particular case $V \subset C^+(X)$ is mentioned by J. B. Prolla in [6]. Also, we extend de Branges lemma in this new setting for convex cones of a weighted spaces of vector functions (Theorem 4.2). Using this theorem, we find various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3. We mention also that a brief version of this paper, in the particular case $V \subset C^+(X)$, is presented in [3], Chapter 2, subparagraph 2.5.

Keywords: Nachbin family, weighted space of vector functions, $p-$Radon measure, polar set, extreme point, convex cone, antialgebraic set with respect to a pair (M, C).

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Dedicated to Professor Francesco Altomare on the occasion of his 70th birthday.

1. INTRODUCTION

The weighted spaces of scalar functions was introduced and studied by L. Nachbin in [4] (see also [5]). We recall that if V is a Nachbin family of upper semi-continuous functions on the locally compact spaces X, then the weighted space associated to V, denoted by $CV_0(X)$, is the set of all continuous functions f on X such that the function $f \cdot v$ vanishes at infinity. Any weight $v \in V$ generate a seminorm $p_v : CV_0(X) \to \mathbb{R}_+$ defined by $p_v(f) = \sup\{v(x) \cdot |f(x)| : x \in X\}$. The locally convex topology defined by this family of seminorms is denoted by ω_V and it will be called the weighted topology on $CV_0(X)$. For some specific families of weights V, some different classes of continuous functions on a locally compact space are obtained, namely the functions with compact support, bounded functions, the functions vanishing at infinity, the rapidly decreasing functions at infinity and so on. A characterization of the dual space of the locally convex spaces $(CV_0(X), \omega_V)$ was obtained by W. H. Summers in [7]. More precisely, he showed that if $V \le C^+(X)$ then, the dual space $[CV_0(X)]^*$ is isomorphic with the space $V \cdot M_b(X)$, where $M_b(X)$ is the space of all bounded Radon measure on X. A similar result for weighted spaces of **vector functions**, in the particular case $V \subset C^+(X)$, is mentioned by J. B. Prolla in [6]. In Theorem 3.1 of this paper, we obtain a characterization of the dual of a weighted space of vector functions in the general case of the upper semi-continuous weights. The key to getting this result is a new result of Measure Theory, namely Proposition 2.1, in which it is proved that if $U : \mathsf{K}(X, E) \to \mathbb{R}$ is a $p-$Radon measure, then there exists a smallest

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positive Radon measure on X, denoted by $|U|$, such that

$$|U(f)| \leq \int p \circ f d|U|, \forall f \in \mathsf{K}(X,E).$$

Using two fundamental tools in functional analysis: Hahn-Banach and Krein –Milman theorems, in 1959, Louis de Branges [1] give a nice proof of Stone-Weierstrass theorem on algebras of real continuous functions on a compact Hausdorff space. Some generalizations of de Branges lemma for weighted space of scalar functions was obtained in [2]. In the last part of this paper, we present a generalization of de Branges lemma for a convex cone in a weighted spaces of **vector functions** (Theorem 4.2). Using this theorem, we obtain various approximations results for weighted spaces of vector functions: Theorems 4.2-4.6 as well as Corollary 4.3.

2. Weighted spaces of vector functions

Let X be a locally compact Hausdorff space, let E be a locally convex complete space endowed with a family P of seminorms of E. We denote by $C(X,E)$ the set of all continuous functions $f : X \to E$ and by $C_0(X,E)$ respectively $\mathsf{K}(X,E)$, the set of continuous functions vanishing at infinity, respectively having compact support. We recall that a function $f : X \to E$ ***vanishes at infinity*** if $\lim_{x\to\infty} f(x) = 0$, i.e., for any $p \in \mathsf{P}$ and any $\varepsilon > 0$, there exists a compact subset $K_{\varepsilon,p}$ of X such that

$$p[f(x)] < \varepsilon,\ \forall x \in X \backslash K_{\varepsilon,p}.$$

Further, we shall denote by $\mathcal{F}_0(X,E)$ the set of all functions $f : X \to E$ vanishing at infinity.

Definition 2.1. *A family V of upper semi-continuous, non-negative functions on X such that for any $v_1, v_2 \in V$ and any $\lambda \in \mathbb{R},\ \lambda > 0$ there exists $w \in V$ such that*

$$v_i(x) \leq \lambda \cdot w(x), \forall x \in X, i = 1,2$$

*will be called a **Nachbin family** on X. Any element of V will be called **a weight.***

If V is a Nachbin family of weights on X, we denote by

$$CV_0(X,E) = \{f \in C(X,E);\ v \cdot f \in C_0(X,E),\ \forall v \in V\}.$$

We endow this linear space with so called **the weighted topology** $\omega_{V,\mathsf{P}}$, given by the family of seminorms $\|\cdot\|_{v,p}$ or $\|\cdot\|_{p_v}$ defined by

$$\|f\|_{p_v} = \|f\|_{v,p} = \sup\{v(x) \cdot p[f(x)],\ \forall x \in X\},\ \forall f \in CV_0(X,E).$$

A base of neighborhoods of the origin in $CV_0(X,E)$ is the family $(B_{v,p})_{v\in V, p\in\mathsf{P}}$ given by

$$B_{v,p} = \left\{f \in CV_0(X,E);\ \|f\|_{v,p} \leq 1\right\}.$$

Further, the space $CV_0(X,E)$ endowed with the weighted topology $\omega_{V,\mathsf{P}}$ will be called **the weighted space of vector functions.** As in the scalar case, one can see that $\mathsf{K}(X,E)$ is a dense subset of $CV_0(X,E)$ with respect to the weighted topology $\omega_{V,\mathsf{P}}$. For any $p \in \mathsf{P}$ and any $f \in \mathsf{K}(X,E)$, we denote

$$\|f\|_p = \sup_{x\in X} p[f(x)].$$

Obviously, $\|f\|_p < \infty$ since $p : E \to \mathbb{R}_+$ is a continuous function on the locally compact space E and $f(X) = f(K_f) \cup \{0\}$ is a compact subset of E, where K_f denotes the support of f. If we endow $\mathsf{K}(X,E)$ with the family of seminorms $\left(\|\cdot\|_p\right)_{p\in\mathsf{P}}$, then $\mathsf{K}(X,E)$ becomes a locally convex space and we shall denote by τ_{P} the topology given by these seminorms $\left(\|\cdot\|_p\right)_{p\in\mathsf{P}}$.

Definition 2.2. *A linear map $U : \mathsf{K}(X,E) \to \mathbb{R}$ is called a* $p-$***Radon measure,*** *where $p \in \mathsf{P}$, if for any compact subset $K \subset X$ there exists a positive number α_K such that for any $f \in \mathsf{K}(X,E)$, $f = 0$ on $X \backslash K$, we have*

$$|U(f)| \leq \alpha_K \cdot \|f\|_p \, .$$

If α_K does not depend of the compact K, then U is called a $p-$ ***bounded Radon measure.*** *The smallest $\alpha \in \mathbb{R}_+$, such that $|U(f)| \leq \alpha \cdot \|f\|_p$ will be denoted by $\|U\|_p$.*

Proposition 2.1. *If $U : \mathsf{K}(X,E) \to \mathbb{R}$ is a $p-$Radon measure, then there exists a smallest positive Radon measure on X, denoted by $|U|$, such that*

$$|U(f)| \leq \int p \circ f d\,|U| \, , \; \forall f \in \mathsf{K}(X,E).$$

Moreover, for any function $\varphi \in \mathsf{K}(X,\mathbb{R})$, the map $\varphi U : \mathsf{K}(X,E) \to \mathbb{R}$ given by

$$\varphi U(\psi) = U(\varphi \cdot \psi), \; \forall \psi \in \mathsf{K}(X,E)$$

is a $p-$ bounded Radon measure and we have
a) $\|\varphi U\|_p = |\varphi U|\,(1)$ and generally $\|U\|_p = |U|\,(1)$ if U is $p-$ bounded,
b) $|\varphi U| = |\varphi| \cdot |U| \, , \; \|\varphi U\|_p = |\varphi U|\,(1) = (|\varphi| \cdot |U|)\,(1) = \int |\varphi| \, d\,|U|$.

Proof. Passing to a factorization, we may suppose that p is a norm on X. We consider a relatively compact open subset D of the locally compact space X and for any $\varphi \in \mathsf{K}(X,\mathbb{R})$, $\varphi \geq 0$ and $supp\varphi \subset D$, we put by definition

$$|U|\,(\varphi) = \sup \{U(\psi); \; \psi \in \mathsf{K}(X,E), \; p \circ \psi \leq \varphi\} = \sup \{|U(\psi)|\,; \psi \in \mathsf{K}(X,E), \; p \circ \psi \leq \varphi\} \, .$$

Since $\overline{D}$ is compact and $\psi(x) = 0$, if $\varphi(x) = 0$, we deduce that $\psi = 0$ outside $\overline{D}$ and therefore there exists $\alpha \in \mathbb{R}_+$ such that $|U(\psi)| \leq \alpha \cdot \|\psi\|_p \leq \alpha \cdot \|\varphi\|$, where $\|\varphi\|$ is the uniform norm of φ on X. Hence $|U|\,(\varphi) \leq \alpha \cdot \|\varphi\|$ for all $\varphi \in \mathsf{K}(X,\mathbb{R})$, $\varphi \geq 0$ and $\mathrm{supp}\varphi \subset D$. We show now that for any $\varphi_i \in \mathsf{K}(X,\mathbb{R}), \; \varphi_i \geq 0, \; \mathrm{supp}\varphi_i \subset D, \; i = 1,2$, we have

$$|U|\,(\varphi_1 + \varphi_2) = |U|\,(\varphi_1) + |U|\,(\varphi_2).$$

The inequality $|U|\,(\varphi_1 + \varphi_2) \geq |U|\,(\varphi_1) + |U|\,(\varphi_2)$ follows just from the definition. Let $\psi \in \mathsf{K}(X,E)$, $p(\psi) \leq \varphi_1 + \varphi_2$. For any $n \in \mathbb{N}^*$, we consider the functions $\psi_i \in \mathsf{K}(X,E)$ given by

$$\psi_i = \frac{\varphi_i}{\varphi_1 + \varphi_2 + \frac{1}{n}} \cdot \psi, \; i = 1,2.$$

Obviously, we have successively

$$p(\psi_i) = \varphi_i \cdot \frac{p(\psi)}{\varphi_1 + \varphi_2 + \frac{1}{n}} \leq \varphi_i, \; i = 1,2,$$

$$\psi - (\psi_1 + \psi_2) = \frac{1}{n} \cdot \frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}},$$

$$p\left(\psi - (\psi_1 + \psi_2)\right) \leq \frac{1}{n} \cdot p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right),$$

$$\mathrm{supp}\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \subset D, \; p\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right) \leq 1, \; \left|U\left(\frac{\psi}{\varphi_1 + \varphi_2 + \frac{1}{n}}\right)\right| \leq \alpha,$$

$$|U(\psi) - U(\psi_1) - U(\psi_2)| \leq \frac{\alpha}{n}, \; U(\psi) \leq U(\psi_1) + U(\psi_2) + \frac{\alpha}{n},$$

$$U(\psi) \leq |U|\,(\varphi_1) + |U|\,(\varphi_2) + \frac{\alpha}{n}, \; \forall n \in \mathbb{N}^*,$$

$$U(\psi) \leq |U|\,(\varphi_1) + |U|\,(\varphi_2), \; |U|\,(\varphi_1 + \varphi_2) = \sup \{U(\psi); \; \psi \in \mathsf{K}(X,E), \; p(\psi) \leq \varphi_1 + \varphi_2\} \, ,$$

$$|U|(\varphi_1+\varphi_2) \le |U|(\varphi_1)+|U|(\varphi_2),\ |U|(\varphi_1+\varphi_2) = |U|(\varphi_1)+|U|(\varphi_2).$$

Obviously, we have

$$|U|(\lambda\cdot\varphi) = \lambda\cdot|U|(\varphi),\ \forall\lambda\in\mathbb{R}_+$$

and the map $|U| : \mathsf{K}^+(X,\mathbb{R}) \to \mathbb{R}_+$ is a positive Radon measure on X. Just from the definition, we have

$$|U(\psi)| \le |U|(p(\psi)),\ \forall\psi\in\mathsf{K}(X,E).$$

On the other hand, taking a positive Radon measure μ on X such that $|U(\psi)| \le \int p(\psi)d\mu$ then for any $\varphi\in\mathsf{K}(X,\mathbb{R}),\ \varphi\ge 0$, we have

$$\int\varphi d\mu \ge \int p(\psi)d\mu,\ \forall\psi\in\mathsf{K}(X,E),\ p(\psi)\le\varphi,$$

$$\int\varphi d\mu \ge |U(\psi)|,\ \forall\psi\in\mathsf{K}(X,E),\ p(\psi)\le\varphi,$$

$$\int\varphi d\mu \ge |U|(\varphi),\ |U|\le\mu \text{ on } \mathsf{K}^+(X,\mathbb{R}).$$

a) For any $\varphi\in\mathsf{K}(X,\mathbb{R})$, the map $\varphi U : \mathsf{K}(X,E)\to\mathbb{R}$ defined by $\varphi U(\psi) = U(\varphi\cdot\psi)$ is linear and we have

$$|\varphi U(\psi)| \le \alpha_K\cdot\|\varphi\cdot\psi\|_p \le \alpha_K\cdot\|\varphi\|\cdot\|\psi\|_p,$$

where $K = \mathrm{supp}\varphi$ and therefore φU is a $p-$ bounded Radon measure on $\mathsf{K}(X,E)$. Further, we have

$$\begin{aligned}|\varphi U|(1) &= \int 1d|\varphi U|\\ &= \sup\left\{\int hd|\varphi U|;\ 0\le h\le 1,\ h\in\mathsf{K}(X,\mathbb{R})\right\}\\ &= \sup\{(\varphi U)(\psi);\ \psi\in\mathsf{K}(X,\mathbb{R}),\ p(\psi)\le 1\}\\ &= \|\varphi U\|_p\end{aligned}$$

(In fact, for any $p-$ bounded Radon measure $U' : \mathsf{K}(X,E)\to\mathbb{R}$ we have, using the definition of $|U'|$:

$$\|U'\|_p = |U'|(1) = \int_X d|U'|),$$

b) The inequality $|\varphi U| \le |\varphi|\cdot|U|$ follows immediately. Indeed, if $h\in\mathsf{K}(X,\mathbb{R}),\ h\ge 0$ then,

$$\begin{aligned}|\varphi U|(h) &= \sup\{U(\varphi\cdot\psi);\ p(\psi)\le h\}\\ &\le \sup\{|U|(p(\varphi\cdot\psi);\ p(\psi)\le h\}\\ &= \sup\{(|\varphi|\cdot|U|)(p(\psi));\ p(\psi)\le h\}\\ &= (|\varphi|\cdot|U|)(h).\end{aligned}$$

Hence $|\varphi U|(h) \le |\varphi|\cdot|U|(h)$ for any $h\in\mathsf{K}(X,\mathbb{R}),\ h\ge 0$. For the converse inequality, we restrict ourself to the case $\varphi\ge 0$. Let us consider $\psi\in\mathsf{K}(X,E)$ such that $p(\psi)\le h\cdot\varphi$ and for any $n\in\mathbb{N}^*$, we consider the function $f_n\in\mathsf{K}(X,E)$ defined by

$$f_n = \frac{\psi}{\varphi+\frac{1}{n}}.$$

Obviously, $p(f_n)\le h$ and therefore

$$|\varphi U|(h) \ge U(\varphi\cdot f_n),\ p(\varphi\cdot f_n)\le h\cdot\varphi,\ p(\psi-\varphi\cdot f_n)\le\frac{1}{n}\cdot p(h).$$

Since $\psi = 0$ outside $K = \text{supp}\varphi$, we have

$$\psi - \varphi \cdot f_n = 0 \text{ on } X \backslash K,\, p\left(\psi - \varphi \cdot f_n\right) \leq \frac{1}{n} \cdot \|h\|\,,\ \left|U\left(\psi - \varphi \cdot f_n\right)\right| \leq \alpha_K \cdot \frac{1}{n} \cdot \|h\|$$

and therefore

$$|\varphi U|\,(h) \geq U(\varphi \cdot f_n) \geq U(\psi) - \alpha_K \cdot \|h\| \cdot \frac{1}{n},\ |\varphi U|\,(h) \geq U(\psi).$$

But

$$\left(\varphi\, |U|\right)(h) = |U|\,(\varphi \cdot h) = \sup\left\{U(\psi);\ \psi \in \mathsf{K}(X, E),\ p(\psi) \leq h \cdot \varphi\right\}.$$

From the preceding two lines, we get $|\varphi U|\,(h) \geq \left(\varphi\, |U|\right)(h)$ and finally $|\varphi U| = |\varphi| \cdot |U|$. □

Proposition 2.2. *Let $U : \mathsf{K}(X, E) \to E$ be a $p-$Radonn measure, $f : X \to \overline{\mathbb{R}}$ be an integrable function with respect to the positive Radon measure $|U|$ (i.e., $f \in \mathsf{L}^1\left(|U|\right)$) and let$\left(\varphi_n\right)_n$ be a sequence in $\mathsf{K}(X, \mathbb{R})$ such that $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$,$|U|$ $-$a.e. on X and such that*

$$\lim_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0.$$

Then, the sequence of $p-$ bounded Radon measures $\left(\varphi_n U\right)_n$ is convergent to a $p-$ bounded Radon measure (depending of f only), denoted by fU, i.e., $\lim\limits_{n\to\infty} \|fU - \varphi_n U\|_p = 0$. Moreover, we have

$$|fU| = |f| \cdot |U|\,.$$

Proof. Since $\lim\limits_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0$, we deduce that $\lim\limits_{n,m\to\infty} \int |\varphi_n - \varphi_m|\, d\,|U| = 0$ and therefore, using Proposition 2.1, we have

$$\lim_{n,m\to\infty} \|\varphi_n U - \varphi_m U\|_p = \lim_{n,m\to\infty} \int |\varphi_n - \varphi_m|\, d\,|U| = 0.$$

Hence for any $\psi \in \mathsf{K}(X, E)$, the sequence $\left(\varphi_n U(\psi)\right)_n$ of real numbers is convergent to a number denoted $fU(\psi)$ and for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$|\varphi_n U(\psi) - \varphi_m U(\psi)| \leq \|\varphi_n U - \varphi_m U\|_p \cdot \|\psi\|_p \leq \varepsilon \cdot \|\psi\|_p\,,\ \forall n, m \geq n_\varepsilon,$$

$$|fU(\psi) - \varphi_m U(\psi)| \leq \varepsilon \cdot \|\psi\|_p\,,\ \forall m \geq n_\varepsilon,$$

$$|fU(\psi)| \leq |\varphi_m U(\psi)| + \varepsilon \cdot \|\psi\|_p \leq \left(\|\varphi_m U\|_p + \varepsilon\right) \cdot \|\psi\|_p\,.$$

Hence fU is a $p-$ bounded Radon measure on $\mathsf{K}(X, E)$, $\lim\limits_{m\to\infty} \|fU - \varphi_m U\|_p = 0$ (Particularly if $f = 0\,|U|$ a.e., from the relation $\lim\limits_{n\to\infty} \int |f - \varphi_n|\, d\,|U| = 0$, we deduce $\lim\limits_{n\to\infty} \int |\varphi_n|\, d\,|U| = 0$ and therefore $\lim\limits_{n\to\infty} \|\varphi_n U\|_p = \lim\limits_{n\to\infty} \int |\varphi_n|\, d\,|U| = 0,\ \lim\limits_{n\to\infty} \left(\varphi_n U\right)(\psi) = 0,\ \forall \psi \in \mathsf{K}(X, E)$. This shows that the element fU, previously defined, depends only on f, does not depend on the choice of the sequence $\left(\varphi_n\right)_n$ tending to f). Let now $h \in \mathsf{K}(X, \mathbb{R}),\ 0 \leq h \leq 1$ and let $\psi \in \mathsf{K}(X, E)$ be such that $p(\psi) \leq h$. We have

$$|fU(\psi) - \varphi_n U(\psi)| \leq \|fU - \varphi_n U\|_p \cdot \|\psi\|_p \leq \|fU - \varphi_n U\|\,,\ \forall n \in \mathrm{N},$$

$$\left(\varphi_n U\right)(\psi) - \|fU - \varphi_n U\|_p \leq fU(\psi) \leq \varphi_n U(\psi) + \|fU - \varphi_n U\|_p\,,$$

$$|\varphi_n U|\,(h) - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq |\varphi_n U|\,(h) + \|fU - \varphi_n U\|_p\,.$$

Using Proposition 2.1 b), we deduce that

$$|\varphi_n| \cdot |U|\,(h) - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq |\varphi_n| \cdot |U|\,(h) + \|fU - \varphi_n U\|_p$$

$$\int |\varphi_n| \cdot hd\,|U| - \|fU - \varphi_n U\|_p \leq |fU|\,(h) \leq \int |\varphi_n| \cdot hd\,|U| + \|fU - \varphi_n U\|_p \,.$$

Passing to the limit on n, we get

$$\int |f| \cdot hd\,|U| \leq |fU|\,(h) \leq \int |f| \cdot hd\,|U|\,,$$

$$|fU|\,(h) = \int |f| \cdot hd\,|U| = |f| \cdot |U|\,(h).$$

The last equality holds for $0 \leq h \leq 1$ and therefore for all $h \in \mathsf{K}(X, \mathbb{R}),\ h \geq 0$, i.e.,

$$|fU| = |f| \cdot |U|\,.$$

□

3. On the dual of weighted spaces of vector functions

Let E, P, X and V as in the preceding section. For any $p \in \mathsf{P}$ and $v \in V$, let

$$B_{v,p} = \{f \in CV_0(X, E);\ p_v(f) \leq 1\}\,,$$

where $p_v(f) = \sup\{v(x) \cdot p[f(x)];\ \forall x \in X\} = \|f\|_{v,p}\,,\ \ \forall f \in CV_0(X, E)$. The linear vector space $CV_0(X, E)$ endowed with the family $(p_v)_{p \in \mathsf{P},\ v \in V}$ of seminorms is a locally convex space whose fundamental system of neighborhoods of the origin is just the family $(B_{v,p})_{v \in V,\ p \in \mathsf{P}}$. We recall that we have denoted by $\omega_{V,\mathsf{P}}$ the weighted topology on $CV_0(X, E)$ given by the family of seminorms $(p_v)_{p \in \mathsf{P},\ v \in V}$. It is no lost of generality if we suppose that for any real number $\alpha,\ \alpha > 0$, we have $\alpha \cdot p \in \mathsf{P},\ \alpha \cdot v \in V$ for any $p \in \mathsf{P}$ and any $v \in V$. So the dual of the locally convex space $(CV_0(X, E), \omega_{V,\mathsf{P}})$ is the set $\bigcup_{v \in V, p \in \mathsf{P}} B^0_{v,p}$, where

$$B^0_{v,p} = \{T : CV_0(X, E) \to \mathbb{R};\ T \text{ linear},\ T(f) \leq 1,\ \forall f \in B_{v,p}\}\,.$$

If we denote by $[CV_0(X, E)]^*$ this dual, then for any subset M of $CV_0(X, E)$ (respectively of $[CV_0(X, E)]^*$), we denote by M^0 the polar of M i.e.,

$$M^0 = \{T \in [CV_0(X, E)]^*;\ T(m) \leq 1,\ \forall m \in M\}$$

respectively

$$M^0 = \{f \in CV_0(X, E);\ m(f) \leq 1,\ \forall m \in M\}\,.$$

The map on $CV_0(X, E) \times [CV_0(X, E)]^* \to \mathbb{R}$, $(f, T) \to \langle f, T\rangle = T(f)$ is a natural duality between the linear space $CV_0(X, E)$ and $[CV_0(X, E)]^*$. The smallest topology on $[CV_0(X, E)]^*$ making continuous the maps

$$T \to \langle f, T\rangle : [CV_0(X, E)]^* \to \mathbb{R},\ \forall f \in CV_0(X, \mathbb{R})$$

is the ***weak topology*** on $[CV_0(X, E)]^*$. It is known (Alaoglu's Theorem) that for any $(p, v) \in \mathsf{P} \times V$, the set $B^0_{p,v}$ is a weakly compact subset of $[CV_0(X, E)]^*$. We know also that the topological space $[CV_0(X, E)]^*$ is a Hausdorff one with respect to this weak topology. Moreover, since $\mathsf{K}(X, E)$ is a dense subset of $CV_0(X, E)$ with respect to the weighted topology $\omega_{V,\mathsf{P}}$, we deduce that

1) any continuous linear functional $L : CV_0(X, E) \to \mathbb{R}$ is completely determined by its restriction to $\mathsf{K}(X, E)$,

2) the smallest topology on $[CV_0(X, E)]^*$ making continuous all linear functionals

$$T \to \langle f, T\rangle : [CV_0(X, E)]^* \to \mathbb{R},\ \forall f \in \mathsf{K}(X, \mathbb{R})$$

is also a Hausdorff one and therefore its restriction to $B^0_{p,v}$ coincides with the restriction to $B^0_{p,v}$ of the weak topology on $[CV_0(X, E)]^*$.

We conclude that any element of the dual of the locally convex space $(\mathsf{K}(X,E),\ \omega_{V,\mathsf{P}}\,|\mathsf{K}(X,E))$ may be uniquely extended to an element of $[CV_0(X,E)]^*$. The following assertion characterizes the elements of $[CV_0(X,E)]^*$ in terms of Radon measures on $\mathsf{K}(X,E)$. With the above notations, we have

Theorem 3.1. *For any $(p,v)\in\mathsf{P}\times V$, we have*
a) The restriction of any element $T\in B^0_{p,v}$ to $\mathsf{K}(X,E)$ is a $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is integrable with respect to the positive Radon measure $|T|$ on X.
Moreover, the following relation holds:

$$\int\frac{1}{v}d\,|T| = \|T\|_{p,v} = \sup\{T(f);\ f\in B_{p,v}\}\,,$$

b) For any $p-$Radon measure U on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|U|-$ integrable, there exists $T\in B^0_{p,v}$ such that U is the restriction of T to $\mathsf{K}(X,E)$.

Proof. a) Let $T\in B^0_{p,v}$ and let K be a compact subset of X. Since $v:X\to[0,\infty)$ is an upper semi-continuous function, its upper bound α_K on K is finite. Let $\varphi\in\mathsf{K}(X,E)$ such that $\varphi=0$ on $X\backslash K$. We have

$$\sup\{v(x)\cdot p(\varphi(x)):\ x\in X\}\le\alpha_K\cdot\sup\{p(\varphi(x)):\ x\in X\}=\alpha_K\cdot\|\varphi\|_p\,,$$

$$\frac{\varphi}{\alpha_K\cdot\|\varphi\|_p}\in B_{p,v},\ \left|T\left(\frac{\varphi}{\alpha_K\cdot\|\varphi\|_p}\right)\right|\le 1,\ |T(\varphi)|\le\alpha_K\cdot\|\varphi\|_p\,,$$

i.e., the restriction of T to $\mathsf{K}(X,E)$, denoted also by T, is a $p-$Radon measure. We have

$$\begin{aligned}\|T\|_{p,v} &= \sup\{T(f),\ f\in CV_0(X,E),\ p_v(f)\le 1\}\\ &= \sup\{T(f),\ f\in\mathsf{K}(X,E),\ p_v(f)\le 1\}\\ &= \sup\left\{T(f),\ f\in\mathsf{K}(X,E),\ p(f)\le\frac{1}{v}\right\}\\ &= \int\frac{1}{v}d\,|T|\,.\end{aligned}$$

b) Let U be a $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|U|-$ integrable. Then, we have

$$\begin{aligned}\infty>\int\frac{1}{v}d\,|U| &= \sup\left\{\int\varphi d\,|U|\,;\ \varphi\in\mathsf{K}(X,\mathbb{R}),\ 0\le\varphi\le\frac{1}{v}\right\}\\ &= \sup_{\varphi\le\frac{1}{v}}\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ p(\psi)\le\varphi\}\\ &= \sup\left\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ p(\psi)\le\frac{1}{v}\right\}\\ &= \sup\{U(\psi);\ \psi\in\mathsf{K}(X,E),\ v(x)\cdot p(\varphi(x))\le 1\}\\ &= \|U\|_{p,v}\,.\end{aligned}$$

□

Remark 3.1. *From the above considerations, we deduce that:*
The elements $T\in B^0_{p,v}$ are $p-$Radon measure on $\mathsf{K}(X,E)$ such that the function $\frac{1}{v}$ is $|T|-$ integrable and $\|T\|_{p,v}=\int\frac{1}{v}d\,|T|\le 1$.

Proposition 3.3. *Let T be a $p-$ Radon measure, $T \in B^0_{p,v}$. If $f \in CV_0(X,E)$, then*

$$|T(f)| \leq \int p(f)d\,|T|\,.$$

Proof. Let $(\psi_n)_n$ be a sequence in $\mathsf{K}(X,E)$ such that $\lim_{n\to\infty}\|f-\psi_n\|_{p,v} = 0$. We know that $|T(\psi_n)| \leq \int p(\psi_n)d\,|T|$ and $T(f) = \lim_{n\to\infty} T(\psi_n)$. On the other hand

$$p(f-\psi_n) \leq \frac{\|f-\psi_n\|_{p,v}}{v} \text{ on } X,$$

$$\int p(f-\psi_n)d\,|T| \leq \|f-\psi_n\|_{p,v}\cdot\int \frac{1}{v}d\,|T| \leq \|f-\psi_n\|_{p,v}$$

$$\int |p(f)-p(\psi_n)|\,d\,|T| \leq \int p(f-\psi_n)d\,|T| \leq \|f-\psi_n\|_{p,v}\,,$$

$$\int p(f)d\,|T| = \lim_{n\to\infty}\int p(\psi_n)d\,|T|\,.$$

Hence

$$|T(f)| = \lim_{n\to\infty}|T(\psi_n)| \leq \lim_{n\to\infty}\int p(\psi_n)d\,|T| = \int p(f)d\,|T|\,.$$

□

Corollary 3.1. *If $T \in B^0_{p,v}$ and $f \in CV_0(X,E)$ is such that $f=0$ on $\operatorname{supp}|T|$, then $T(f)=0$.*

4. Lemma de Branges and approximation results

In this section, we preserve all notations used in the preceding paragraphs. For any subset $A \subset CV_0(X,E)$, we denote by A^0 the polar of A, i.e.,

$$A^0 = \{T \in [CV_0(X,E)]^*;\ T(a) \leq 1,\ \forall a \in A\}\,.$$

If C is a convex cone of the real vector space $CV_0(X,E)$ then, one can see that

$$\mathsf{C}^0 = \{T \in [CV_0(X,E)]^*;\ T(c) \leq 0,\ \forall c \in \mathsf{C}\}\,.$$

Theorem 4.2. *Let C be a convex cone in $CV_0(X,E)$, $p \in \mathsf{P}$, $v \in V$ and let $L \in B^0_{p,v} \cap \mathsf{C}^0$, $L \neq 0$ be an extreme point of the convex and compact subset $B^0_{p,v} \cap \mathsf{C}^0$. If $h \in C(X,[0,1])$ is such for any $c \in \mathsf{C}$, we have $h\cdot c\,|\sigma(|L|) \in \mathsf{C}\,|\sigma(|L|)$ and $(1-h)\cdot c\,|\sigma(|L|) \in \mathsf{C}\,|\sigma(|L|)$, then h is constant on $\sigma(|L|)-$ the support of the positive Radon measure $|L|$ on X.*

Proof. Since $L \neq 0$ and L is an extreme point of the subset $B^0_{p,v}\cap\mathsf{C}^0$, we have $\|L\|_{p,v} = \int \frac{1}{v}d\,|L|\,.$ If h is an arbitrary element in$C(X,[0,1])$, then the map $hL : \mathsf{K}(X,E) \to \mathbb{R}$, given by $hL(\psi) = L(h\cdot\psi)$, is a $p-$Radon measure on $\mathsf{K}(X,E)$. It is not so difficult to show, using the definition, that $|hL| = |h|\cdot|L|$. Obviously, the function $\frac{1}{v}$ is $|h|\cdot|L|-$ integrable and using Remark 3.1 and the relations

$$\|hL\|_{p,v} = \int\frac{1}{v}d\,|hL| = \int\frac{h}{v}d\,|L| \leq \int\frac{1}{v}d\,|L| \leq 1,$$

we get $hL \in B^0_{p,v}$. Analogously, the map $(1-h)L : \mathsf{K}(X,E) \to \mathbb{R}$ given by $(1-h)L(\psi) = L((1-h)\cdot(\psi))$ is a $p-$Radon measure and

$$\|(1-h)L\|_{p,v} = \int\frac{1-h}{v}d\,|L| \leq \int\frac{1}{v}d\,|L| = 1,\ (1-h)L \in B^0_{p,v}.$$

If we denote $\alpha = \|hL\|_{p,v} = \int\frac{h}{v}d\,|L|\,,\ \ \beta = \|(1-h)L\|_{p,v} = \int\frac{1-h}{v}d\,|L|$, we have $\alpha+\beta = \int\frac{1}{v}d\,|L| = 1$. We remark also that the function $\frac{1}{v}$ is strictly positive on X. If $\alpha = 0$, then

$h = 0$ $|L|$ a.e. on $\sigma(|L|)$. Since the function h is continuous, it results that $h = 0$ on $\sigma(|L|)$, i.e., h is constant on $\sigma(|L|)$. Analogously, if $\beta = 0$, we obtain $h = 1$ on $\sigma(|L|)$, i.e., h is constant on $\sigma(|L|)$. We suppose further $\alpha \neq 0,\ \beta \neq 0$ and we denote

$$L_1 = \frac{1}{\alpha} \cdot hL,\ L_2 = \frac{1}{\beta} \cdot (1-h)L.$$

Obviously, $\|L_i\|_{p,v} = 1,\ i = 1,2$ and $\alpha \cdot L_1 + \beta \cdot L_2 = L$. We show now that $L_i \in \mathsf{C}^0,\ i = 1,2,$ if for any $c \in \mathsf{C}$ there exist $c_1, c_2 \in \mathsf{C}$ such that $h \cdot c = c_1,\ (1-h) \cdot c = c_2$ on $\sigma(|L|)$. Since the functions $h \cdot c,\ (1-h) \cdot c,\ c_1,\ c_2$ belong to $CV_0(X,E)$ and $h \cdot c = c_1$ on $\sigma(|L|)$, respectively $(1-h) \cdot c = c_2$ on $\sigma(|L|)$, using Corollary 3.1, we get

$$L(h \cdot c) = L(c_1) \leq 0,\ L((1-h) \cdot c) = L(c_2) \leq 0,$$

$$L_1(c) = \frac{1}{\alpha} \cdot L(h \cdot c) = \frac{1}{\alpha} \cdot L(c_1) \leq 0,\ L_2(c) = \frac{1}{\beta} \cdot L((1-h) \cdot c) = \frac{1}{\beta} \cdot L(c_2) \leq 0.$$

Hence L_1, L_2 belong to the set $B^0_{p,v} \cap \mathsf{C}^0$ and since $L = \alpha \cdot L_1 + \beta \cdot L_2$, we get $L_1 = L_2 = L$. Hence $|L_1| = |L|$, i.e., the measures $\frac{h}{\alpha} \cdot |L|$ and $|L|$ coincide and therefore $\frac{h}{\alpha} = 1$ almost everywhere on $\sigma(|L|)$. But h is continuous and hence $h = \alpha$ on $\sigma(|L|)$. □

Definition 4.3. *A subset $\mathsf{M} \subset C(X,[0,1])$ is called **complemented**, if for any $h \in \mathsf{M}$, the function $1-h$ belongs to M. If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone and $\mathsf{M} \subset C(X,[0,1])$ is a complemented family, then a subset $S \subset X$ is called **antialgebraic with respect to the pair** (M,C) (or simpler $(\mathsf{M},\mathsf{C})-$antialgebraic), if any $h \in \mathsf{M}$ such that the restriction to S of the functions $h \cdot c$ and $(1-h) \cdot c$ belong to the restriction of C to S (i.e., $h \cdot c\,|S \in \mathsf{C}\,|S\ ,\ \ (1-h) \cdot c\,|S\ \ \in \mathsf{C}\,|S$) for any $c \in \mathsf{C}$, is a constant function on S.*

We can reformulate de Branges lemma (Theorem 4.2) as follows:

Corollary 4.2. *For any extreme point L of $B^0_{p,v} \cap \mathsf{C}^0$, the support $\sigma(|L|)$ of the positive Radon measure $|L|$ on X is an antialgebraic subset with respect to the pair $(C(X,[0,1]),\mathsf{C})$. Further, we denote by S the family of all subsets of X antialgebraic with respect to the pair (M,C).*

The following assertions are almost obvious.
i) $\{x\} \in \mathsf{S},\ \forall x \in X$,
ii) $S_1, S_2 \in \mathsf{S},\ S_1 \cap S_2 \neq \phi \Rightarrow S_1 \cup S_2 \in \mathsf{S}$,
iii) $S \in \mathsf{S} \Rightarrow \bar{S} \in \mathsf{S}$,
iv) For any upper directed family $(S_\alpha)_{\alpha \in I}$ from S, we have $\bigcup\limits_{\alpha \in I} S_\alpha \in \mathsf{S}$.
If for any $x \in X$, we denote by $S_x = \cup\{S;\ S \in \mathsf{S}, x \in S\}$, then we have

$$S_x = \overline{S_x} \in \mathsf{S},\ S_x \cap S_y = \phi \ \text{ if } \ S_x \neq S_y.$$

The family $(S_x)_{x \in X}$ is a partition of X and for any $S \in \mathsf{S}$ there exists $x \in X$ such that $S \subset S_x$. For the general theory of duality, we have for any convex cone C, $\mathsf{C} \subset CV_0(X,E)$, the closure $\bar{\mathsf{C}}$ in $CV_0(X,E)$ with respect to the weighted topology $\omega_{\mathsf{P},V}$ coincides with the bipolar of C i.e., $\bar{\mathsf{C}} = \mathsf{C}^{00}$. In the our special case, we have the following general approximation theorem.

Theorem 4.3. *If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone, then the closure of C in $(CV_0(X,E),\ \omega_{\mathsf{P},V})$ is given by*

$$\overline{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|\sigma(|L|) \in \overline{\mathsf{C}\,|\sigma(|L|)},\ \forall L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right),\ \forall v \in V,\ \forall p \in \mathsf{P} \right\}.$$

Proof. We show only that for any function $g \in CV_0(X,E) \backslash \bar{\mathsf{C}}$ there exist $p \in \mathsf{P}$, $v \in V$ and $L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right)$ such that $g\,|\sigma(|L|) \notin \overline{\mathsf{C}\,|\sigma(\mu)}$. Indeed, using Hahn-Banach separation theorem, there exists $T \in [CV_0(X,E)]^*$ such that $T \in \mathsf{C}^0$ and $T(g) > 0$. Let $p \in \mathsf{P}$ and $v \in V$ be such that $|T(f)| \leq \|f\|_{p,v}$, $\forall f \in CV_0(X,E)$ i.e., $|T|\left(\frac{1}{v}\right) \leq 1$. Hence $T \in B^0_{p,v} \cap \mathsf{C}^0$. Since $B^0_{p,v} \cap \mathsf{C}^0$ is a compact convex subset of $[CV_0(X,E)]^*$ with respect to the weak topology and $T(g) > 0$, it follows from Krein-Milman theorem that there exists $L \in Ext\left(B^0_{p,v} \cap \mathsf{C}^0\right)$ such that $L(g) > 0$. Since $L \in \mathsf{C}^0$, we deduce that $\int \varphi d\,|L| \leq 0$ for any $\varphi \in \overline{\mathsf{C}\,|\sigma\,(|L|)}$. Hence $g\,\Big|\sigma(|L|) \notin \overline{\mathsf{C}\,|\sigma(|L|)}$. □

Let now $\mathsf{M} \subset C(X,[0,1])$ be a complemented family and for any $x \in X$ let S_x be the greatest $(\mathsf{M},\mathsf{C})-$ antialgebraic subset of X containing x.

Theorem 4.4. *If $\mathsf{C} \subset CV_0(X,E)$ is a convex cone, then the closure of C in $(CV_0(X,E),\ \omega_{\mathsf{P},V})$ is given by*

$$\overline{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|S_x \in \overline{\mathsf{C}\,|S_x},\ \forall x \in X \right\}.$$

Proof. For any $p \in \mathsf{P}$,$v \in V$ and any extreme point L of the compact convex subset $B^0_{p,v} \cap \mathsf{C}^0$, the support $\sigma\,(|L|)$ is a $(\mathsf{M},\mathsf{C})-$ antialgebraic subset of X. If we choose a point $x \in \sigma\,(|L|)$, then $\sigma\,(|L|) \subset S_x$, and therefore if $f\,|S_x \in \overline{\mathsf{C}\,|S_x}$, we have also $f\,|\sigma(L) \in \overline{\mathsf{C}\,|\sigma(L)}$. Further, we may use Theorem 4.3. □

Theorem 4.5. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family and the convex cone $\mathsf{C} \subset CV_0(X,E)$ is stable with respect to the multiplication of M (i.e., $c \cdot m \in \mathsf{C}, \forall c \in \mathsf{C}, m \in \mathsf{M}$), then we have*

$$\overline{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f\,|\ [x]_{\mathsf{M}} \in \overline{\mathsf{C}\,|\ [x]_{\mathsf{M}}},\ \forall x \in X \right\},$$

where for any $x \in X$ we denote $[x]_{\mathsf{M}} = \{y \in X;\ m(y) = m(x),\ \forall m \in \mathsf{M}\}$.

Proof. Using just the definitions and previous notations, we deduce that for any $x \in X$ we have $[x]_{\mathsf{M}} = S_x$. Further, we use Theorem 4.4. □

The following assertion needs to define so called **"section in C" by the points of X**, namely to consider the following convex cone $\mathsf{C}(x)$ in E given by

$$\mathsf{C}(x) = \{c(x);\ c \in \mathsf{C}\}$$

and also its closure $\overline{\mathsf{C}(x)}$ in E. Certainly the starting convex cone C in $CV_0(X,E)$ may be a linear subspace and in this case $\mathsf{C}(x)$ is a linear subspace in E.

Theorem 4.6. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family and the convex cone $\mathsf{C} \subset CV_0(X,E)$ is stable with respect to the multiplication with elements of M and M separates the points of X, i.e., for any $x,y \in X$ there exists $m \in \mathsf{M}$ such that $m(x) \neq m(y)$, then we have*

$$\overline{\mathsf{C}} = \left\{ f \in CV_0(X,E);\ f(x) \in \overline{\mathsf{C}(x)},\ \forall x \in X \right\}.$$

Indeed, in this case, for any $x \in X$, we have $[x]_{\mathsf{M}} = \{x\}$ and we close the proof applying Theorem 4.5.

Corollary 4.3. *If $\mathsf{M} \subset C(X,[0,1])$ is a complemented family, separating the points of X and $\mathsf{W} \subset CV_0(X,E)$ is a linear subspace which is stable with respect to the multiplication with elements of M and for any $x \in X$ the section $\mathsf{W}(x)$ is a dense subset of the locally convex space (E,P), then*

$$\overline{\mathsf{W}} = CV_0(X,E).$$

Remark 4.2. *For the scalar case $E = \mathbb{R}$, the density of $\mathsf{W}(x)$ in $\mathbb{R}$ is automatically fulfilled unless the case where $\mathsf{W}(x) = \{0\}$ for the points x of a closed subset $F \subset X$. In this case, we have*

$$\overline{\mathsf{W}} = \{f \in CV_0(X);\ f = 0 \text{ on } F\}\,.$$

Even this assertion may be drown from Theorem 4.6 as a particular case where there exists $F \subset X$ such that the section of C by x is trivial for all $x \in F$ i.e., $\mathsf{C}(x) = \{0_E\}\,,\ \forall x \in F$. Anyway Theorem 4.6 may be used in different manners to obtain density results.

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