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Research Article

Geodesics on the Cotangent Bundle with Vertical Rescaled Cheeger-Gromoll Metric

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Abstract
In this paper, we introduce the vertical rescaled Cheeger-Gromoll metric (deformation in the vertical bundle) on the cotangent bundle T^*M over a Riemannian manifold (M,g) and we investigate the Levi-Civita connection of this metric. We study the geodesics on the cotangent bundle with respect to the vertical rescaled Cheeger-Gromoll metric. Afterward, we establish the necessary and sufficient conditions under which a curve be geodesic respect to this metric. Finally, we also construct some examples of geodesics.

Keywords and 2010 Mathematics Subject Classification
Keywords: Cotangent bundles — vertical rescaled Cheeger-Gromoll metric — geodesics.
MSC: 53C20, 53C22, 58E10

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1. Introduction

In the field, one of the first works to deal with Riemannian metrics on cotangent bundles is that of Patterson, E.M., Walker, A.G. [5], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa, M. [7] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca, F. considered another class of metrics on cotangent bundles of Riemannian manifolds, that he called g-natural metrics [1]. Also, there are studies by other authors, Salimov, A.A., Ağca, F. [2, 6], Yano, K., Ishihara, S. [8], Ocak, F. [4], Gezer, A., Altunbas, M. [3] etc.

The main idea in this note consists in the deformation (in the vertical bundle) of the Cheeger-Gromoll metric on the cotangent bundle [2]. We introduce the vertical rescaled Cheeger-Gromoll metric on the cotangent bundle T^*M over a Riemannian manifold (M,g) and we investigate the Levi-Civita connection (Theorem 5). We study the geodesics on the cotangent bundle with respect to the vertical rescaled Cheeger-Gromoll metric. First, we establish necessary and sufficient conditions under which a curve be geodesic respect (Theorem 10 and Corollary 11). As well when is the horizontal lift is geodesic (Corollary 12). We also construct some examples of geodesics (Example 15 and Example 16). Finally, we also mention special cases (Theorem 17 and Corollary 18).

2. Preliminaries

Let (M^m,g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U,x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U),x^i,x^{\bar{i}} = p_i)_{i=\overline{1,m},\bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r,s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $(U, x^i)_{i=\overline{1,m}}$, $U \subset M$ of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the complete and horizontal lifts X^C, $X^H \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$X^C = X^i \frac{\partial}{\partial x^i} - p_h \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial x^{\bar{i}}}, \tag{2}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{3}$$

$$\omega^V = \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{4}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, (see [8] for more details).

From (3) and (4) we see that $(\frac{\partial}{\partial x^i})^H$ and $(dx^i)^V$ have respectively local expressions of the form

$$\tilde{e}_i = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} + p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{5}$$

$$\tilde{e}_{\bar{i}} = (dx^i)^V = \frac{\partial}{\partial x^{\bar{i}}}. \tag{6}$$

The set of vector fields $\{\tilde{e}_i\}$ on $\pi^{-1}(U)$ define a local frame for HT^*M over $\pi^{-1}(U)$ and the set of vector fields $\{\tilde{e}_{\bar{i}}\}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$. The set $\{\tilde{e}_\alpha\} = \{\tilde{e}_i, \tilde{e}_{\bar{i}}\}$ define a local frame on T^*M, adapted to the direct sum decomposition (1). The indices $\alpha, \beta, \ldots = \overline{1, 2m}$ indicate the indices with respect to the adapted frame.

Using (3), (4) we have,

$$X^H = X^i \tilde{e}_i\ ,\ X^H = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{7}$$

$$\omega^V = \omega_i \tilde{e}_{\bar{i}}\ ,\ \omega^V = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \tag{8}$$

with respect to the adapted frame $\{\tilde{e}_\alpha\}_{\alpha=\overline{1,2m}}$, (see [8] for more details).

In particular if $\mathscr{P}$ be a local covector field constant on each fiber $T^*_x M$ i.e. ($\mathscr{P} = p = p_i dx^i$), the vertical lift $\mathscr{P}^V$ is called the canonical vertical vector field or Liouville vector field on T^*M.

Lemma 1. *[8] Let (M^m, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of (M^m, g) satisfies the following*

1. $[\omega^V, \theta^V] = 0$,
2. $[X^H, \theta^V] = (\nabla_X \theta)^V$,
3. $[X^H, Y^H] = [X, Y]^H + (pR(X, Y))^V$,

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Let (M, g) be a Riemannian manifold, we define the map

$$\begin{aligned} \sharp : \Im^0_1(M) &\to \Im^1_0(M) \\ \omega &\mapsto \sharp\omega \end{aligned}$$

by for all $X \in \Im^1_0(M)$, $g(\sharp\omega, X) = \omega(X)$, the map $\sharp$ is $C^\infty(M)$-isomorphism.
Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\sharp\omega = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T_x^*M by $g^{-1}(\omega,\theta) = g(\sharp\omega,\sharp\theta) = g^{ij}\omega_i\theta_j$. If ∇ be the Levi-Civita connection of (M,g) we have

$$\nabla_X(\sharp\omega) = \sharp(\nabla_X\omega), \tag{9}$$
$$Xg^{-1}(\omega,\theta) = g^{-1}(\nabla_X\omega,\theta)+g^{-1}(\omega,\nabla_X\theta), \tag{10}$$

for all $X \in \Im_0^1(M)$ and $\omega,\theta \in \Im_1^0(M)$.

From now on, we noted $\sharp\omega$ by $\tilde{\omega}$ for all $\omega \in \Im_1^0(M)$.

3. Vertical rescaled Cheeger-Gromoll metric

Definition 2. *Let (M^m,g) be a Riemannian manifold and $f : M \to]0,+\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M, we define a vertical rescaled Cheeger-Gromoll metric noted g^f by*

$$g^f(X^H,Y^H) = g(X,Y)^V = g(X,Y)\circ\pi, \tag{11}$$
$$g^f(X^H,\theta^V) = 0, \tag{12}$$
$$g^f(\omega^V,\theta^V) = \frac{f}{\alpha}(g^{-1}(\omega,\theta)+g^{-1}(\omega,p)g^{-1}(\theta,p)), \tag{13}$$

for all $X,Y \in \Im_0^1(M)$, $\omega,\theta \in \Im_1^0(M)$ where $\alpha = 1+\|p\|^2$ and $\|p\| = \sqrt{g^{-1}(p,p)}$ is the norm of p with respect to the metric g.

Note that, if $f = 1$, then g^f is the Cheeger-Gromoll metric [2].

Since any tensor field of type $(0,s)$ on T^*M where $s \geq 1$ is completely determined with the vector fields of type X^H and ω^V where $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$ (see [8]). In the particular case the metric g^f is tensor field of type $(0,2)$ on T^*M. It follows that g^f is completely determined by its formulas (11), (12) and (13).

By means of (2) and (3), the complete lift X^C of $X \in \Im_0^1(M)$ is given by

$$X^C = X^H-(p(\nabla X))^V, \tag{14}$$

where $p(\nabla X) = p_h(\nabla X)_i^h dx^i = p_h(\frac{\partial X^h}{\partial x^i}+\Gamma_{ij}^h X^j)dx^i$.

Taking account of (11), (12), (13) and (14), we obtain

$$g^f(X^C,Y^C) = \frac{f}{\alpha}(g^{-1}(p(\nabla X),p(\nabla Y))+g^{-1}(p(\nabla X),p)g^{-1}(p(\nabla Y),p))+g(X,Y)^V, \tag{15}$$

where

$$g^{-1}(p(\nabla X),p(\nabla Y)) = g^{ij}p_hp_k(\nabla X)_i^h(\nabla Y)_j^k,$$

$$g^{-1}(p(\nabla X),p) = g^{ij}p_hp_j(\nabla X)_i^h.$$

Since the tensor field $g^f \in \Im_2^0(T^*M)$ is completely determined also by its action on vector fields of type X^C and Y^C (see [8]), we say that the formula (15) is an alternative characterization of g^f.

Lemma 3. *[9] Let (M^m,g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function, we have the following:*

1. $X^H(\rho(r^2)) = 0$,
2. $\omega^V(\rho(r^2)) = 2\rho'(r^2)g^{-1}(\omega,p)$,
3. $X^H(g^{-1}(\theta,p)) = g^{-1}(\nabla_X\theta,p)$,
4. $\omega^V(g^{-1}(\theta,p)) = g^{-1}(\omega,\theta)$,

for all $X \in \Im_0^1(M)$ and $\omega,\theta \in \Im_1^0(M)$, where $r^2 = g^{-1}(p,p)$.

Lemma 4. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric, we have:*

$$
\begin{aligned}
1.\ X^H g^f(\theta^V,\eta^V) &= \frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)+g^f(\theta^V,(\nabla_X\eta)^V),\\
2.\ \omega^V g^f(\theta^V,\eta^V) &= -\frac{2}{\alpha}g^{-1}(\omega,p)g^f(\theta^V,\eta^V)+\frac{1}{\alpha}g^{-1}(\omega,\theta)g^f(\eta^V,\mathscr{P}^V)+\frac{1}{\alpha}g^{-1}(\omega,\eta)g^f(\theta^V,\mathscr{P}^V),\\
3.\ X^H g^f(\theta^V,\mathscr{P}^V) &= X(f)g^{-1}(\theta,p)+fg^{-1}(\nabla_X\theta,p),\\
4.\ \omega^V g^f(\theta^V,\mathscr{P}^V) &= fg^{-1}(\omega,\theta).
\end{aligned}
$$

for all $X\in\Im_0^1(M)$ and $\omega,\theta\in\Im_1^0(M)$, where $\mathscr{P}^V$ is the canonical vertical vector field on T^*M.

Proof. The results comes directly from Definition 2 and Lemma 3.

$$
\begin{aligned}
1.\ X^H g^f(\theta^V,\eta^V) &= X^H\big(\frac{f}{\alpha}(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p))\big)\\
&= X(f)\frac{1}{\alpha}(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p))+\frac{f}{\alpha}(g^{-1}(\nabla_X\theta,\eta)+g^{-1}(\theta,\nabla_X\eta)\\
&\quad +g^{-1}(\nabla_X\theta,p)g^{-1}(\eta,p))+g^{-1}(\theta,p)g^{-1}(\nabla_X\eta,p)\\
&= \frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)+g^f(\theta^V,(\nabla_X\eta)^V).\\
2.\ \omega^V g^f(\theta^V,\eta^V) &= \omega^V\big(\frac{f}{\alpha}(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p))\big)\\
&= -\frac{2f}{\alpha^2}g^{-1}(\omega,p)(g^{-1}(\theta,\eta)+g^{-1}(\theta,p)g^{-1}(\eta,p))\\
&\quad +\frac{f}{\alpha}(g^{-1}(\omega,\theta)g^{-1}(\eta,p)+g^{-1}(\theta,p))g^{-1}(\omega,\eta))\\
&= -\frac{2}{\alpha}g^{-1}(\omega,p)g^f(\theta^V,\eta^V)+\frac{1}{\alpha}g^{-1}(\omega,\theta)g^f(\eta^V,\mathscr{P}^V)+\frac{1}{\alpha}g^{-1}(\omega,\eta)g^f(\theta^V,\mathscr{P}^V).
\end{aligned}
$$

The other formulas are obtained by a similar calculation. ■

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of T^*M with vertical rescaled Cheeger-Gromoll metric g^f. This connection is characterized by the Koszul formula:

$$
2g^f(\widetilde{\nabla}_U V,W) = Ug^f(V,W)+Vg^f(W,U)-Wg^f(U,V)+g^f(W,[U,V])+g^f(V,[W,U])-g^f(U,[V,W]), \tag{16}
$$

for all $U,V,W\in\Im_0^1(T^*M)$.

Theorem 5. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric, then we have:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H+\frac{1}{2}(pR(X,Y))^V,\\
2.\ \widetilde{\nabla}_{X^H}\theta^V &= (\nabla_X\theta)^V+\frac{1}{2f}X(f)\theta^V+\frac{f}{2\alpha}(R(\tilde{p},\tilde{\theta})X)^H,\\
3.\ \widetilde{\nabla}_{\omega^V}Y^H &= \frac{1}{2f}Y(f)\omega^V+\frac{f}{2\alpha}(R(\tilde{p},\tilde{\omega})Y)^H,\\
4.\ \widetilde{\nabla}_{\omega^V}\theta^V &= -\frac{1}{2f}g^f(\omega^V,\theta^V)(grad\, f)^H-\frac{1}{\alpha f}\big(g^f(\omega^V,\mathscr{P}^V)\theta^V+g^f(\theta^V,\mathscr{P}^V)\omega^V\big)\\
&\quad +\big(\frac{\alpha+1}{\alpha f}g^f(\omega^V,\theta^V)-\frac{1}{\alpha f^2}g^f(\omega^V,\mathscr{P}^V)g^f(\theta^V,\mathscr{P}^V)\big)\mathscr{P}^V,
\end{aligned}
$$

for all $X,Y\in\Im_0^1(M)$ and $\omega,\theta\in\Im_1^0(M)$, where $\mathscr{P}^V$ is the canonical vertical vector field on T^*M.

Proof. The proof of Theorem 5 follows directly from Kozsul formula (16), Lemma 1, Definition 2 and Lemma 4 (See similar calculations [9]). ■

4. Geodesics of vertical rescaled Cheeger-Gromoll metric

Let (M^m,g) be a Riemannian manifold and $\gamma : I \to M$ be a curve on (M^m,g) $(I \subset \mathbb{R})$. We define on T^*M the curve $C : I \to T^*M$ by $C(t) = (\gamma(t), \vartheta(t))$, for all $t \in I$ where $\vartheta(t) \in T^*_{\gamma(t)}M$ i.e. $\vartheta(t)$ is a covector field along $\gamma(t)$.

Definition 6. *Let (M^m,g) be a Riemannian manifold, $C(t) = (\gamma(t), \vartheta(t))$ be a curve on T^*M and ∇ be the Levi-Civita connection of (M^m,g). If $\nabla_{\dot\gamma}\vartheta = 0$ the curve $C(t)$ is said to be a horizontal lift of the curve $\gamma(t)$, where $\dot\gamma$ the tangent field along $\gamma(t)$.*

Lemma 7. *[9] Let (M^m,g) be a Riemannian manifold. If $\omega \in \Im^0_1(M)$ is a covector field on M and $(x,p) \in T^*M$ such that $\omega_x = p$, then we have:*

$$d_x\omega(X_x) = X^H_{(x,p)} + (\nabla_X \omega)^V_{(x,p)},$$

for all $X \in \Im^1_0(M)$.

Lemma 8. *[9] Let (M^m,g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M,g). If $\gamma(t)$ is a curve on M and $C(t) = (\gamma(t), \vartheta(t))$ is a curve on T^*M, then*

$$\dot C = \dot\gamma^H + (\nabla_{\dot\gamma}\vartheta)^V. \tag{17}$$

Theorem 9. *Let (M^m,g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $C(t) = (\gamma(t), \vartheta(t))$ is curve on T^*M such $\vartheta(t)$ is a covector field along $\gamma(t)$, then*

$$\begin{aligned}\widetilde\nabla_{\dot C}\dot C \;=\;& \Big[\nabla_{\dot\gamma}\dot\gamma - \frac{1}{2\alpha}\big(g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big) grad\, f + \frac{f}{\alpha} R(\tilde\vartheta, \widetilde{\nabla_{\dot\gamma}\vartheta})\dot\gamma\Big]^H \\ &+ \Big[\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta + \big(\frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta - \frac{2}{\alpha} g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)\nabla_{\dot\gamma}\vartheta + \frac{1}{\alpha^2}\big((\alpha+1)g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)\vartheta\Big]^V,\end{aligned} \tag{18}$$

where ∇ (resp. $\widetilde\nabla$) denote the Levi-Civita connection of (M^m,g) (resp. (T^*M, g^f)).

Proof. Using Lemma 8 and Theorem 5 we obtain

$$\begin{aligned}\widetilde\nabla_{\dot C}\dot C \;=\;& \widetilde\nabla_{[\dot\gamma^H + (\nabla_{\dot\gamma}\vartheta)^V]}[\dot\gamma^H + (\nabla_{\dot\gamma}\vartheta)^V] \\ =\;& \widetilde\nabla_{\dot\gamma^H}\dot\gamma^H + \widetilde\nabla_{\dot\gamma^H}(\nabla_{\dot\gamma}\vartheta)^V + \widetilde\nabla_{(\nabla_{\dot\gamma}\vartheta)^V}\dot\gamma^H + \widetilde\nabla_{(\nabla_{\dot\gamma}\vartheta)^V}(\nabla_{\dot\gamma}\vartheta)^V \\ =\;& (\nabla_{\dot\gamma}\dot\gamma)^H + \frac{1}{2}(\vartheta R(\dot\gamma,\dot\gamma))^V + (\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{2f}\dot\gamma(f)(\nabla_{\dot\gamma}\vartheta)^V + \frac{f}{2\alpha}(R(\tilde\vartheta, \widetilde{\nabla_{\dot\gamma}\vartheta})\dot\gamma)^H + \frac{1}{2f}\dot\gamma(f)(\nabla_{\dot\gamma}\vartheta)^V \\ &+ \frac{f}{2\alpha}(R(\tilde\vartheta, \widetilde{\nabla_{\dot\gamma}\vartheta})\dot\gamma)^H - \frac{1}{2f}g^f((\nabla_{\dot\gamma}\vartheta)^V, (\nabla_{\dot\gamma}\vartheta)^V)(grad\, f)^H + \frac{2}{\alpha f}g^f((\nabla_{\dot\gamma}\vartheta)^V, \vartheta^V)(\nabla_{\dot\gamma}\vartheta)^V \\ &+ \big(\frac{\alpha+1}{\alpha^2}g^f((\nabla_{\dot\gamma}\vartheta)^V, (\nabla_{\dot\gamma}\vartheta)^V) - \frac{1}{2\alpha f^2}g^f((\nabla_{\dot\gamma}\vartheta)^V, \vartheta^V)^2\big)\vartheta^V \\ =\;& (\nabla_{\dot\gamma}\dot\gamma)^H - \frac{1}{2\alpha}\big(g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)(grad\, f)^H + \frac{f}{\alpha}(R(\tilde\vartheta, \widetilde{\nabla_{\dot\gamma}\vartheta})\dot\gamma)^H + (\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta)^V \\ &+ \big(\frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta - \frac{2}{\alpha}g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)(\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{\alpha^2}\big((\alpha+1)g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)\vartheta^V.\end{aligned}$$

∎

Theorem 10. *Let (M^m,g) be a Riemannian manifold, (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $C(t) = (\gamma(t), \vartheta(t))$ a curve on T^*M such $\vartheta(t)$ is a covector field along $\gamma(t)$, then $C(t)$ is a geodesic on T^*M if and only if*

$$\nabla_{\dot\gamma}\dot\gamma = \frac{1}{2\alpha}\big(g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big) grad\, f - \frac{f}{\alpha}R(\tilde\vartheta, \widetilde{\nabla_{\dot\gamma}\vartheta})\dot\gamma, \tag{19}$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta = -\big(\frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta - \frac{2}{\alpha}g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)\nabla_{\dot\gamma}\vartheta - \frac{1}{\alpha^2}\big((\alpha+1)g^{-1}(\nabla_{\dot\gamma}\vartheta, \nabla_{\dot\gamma}\vartheta) + g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)^2\big)\vartheta. \tag{20}$$

Proof. The statement is a direct consequence of Theorem 9 and definition of geodesic. ∎

Corollary 11. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. Then the curve $C(t) = (\gamma(t), \widetilde{\dot\gamma(t)})$ is a geodesic on T^*M if and only if $\gamma(t)$ is a geodesic on (M^m, g).*

Proof. $\dot\gamma$ is the tangent field along $\gamma(t)$, i.e. $\dot\gamma(t) \in TM$, then $\widetilde{\dot\gamma}(t) = \widetilde{\dot\gamma(t)} \in T^*M$. From (9), we have $\nabla_{\dot\gamma}\widetilde{\dot\gamma} = \widetilde{\nabla_{\dot\gamma}\dot\gamma}$ and $\gamma(t)$ is a geodesic on M equivalent to $\nabla_{\dot\gamma}\dot\gamma = 0$. Using Theorem 10 we deduce the result. ■

Corollary 12. *Let (M^m, g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $C(t) = (\gamma(t), \vartheta(t))$ be the horizontal lift of the curve $\gamma(t)$. Then $C(t)$ is a geodesic on T^*M if and only if $\gamma(t)$ is a geodesic on (M^m, g).*

Proof. Let $C(t) = (\gamma(t), \vartheta(t))$ be the horizontal lift of the curve $\gamma(t)$, then $\nabla_{\dot\gamma}\vartheta = 0$. Using Theorem 10 we deduce the result. ■

Remark 13. *If $\gamma(t)$ is a geodesic on M locally we have:*

$$\nabla_{\dot\gamma}\dot\gamma = 0 \quad \Leftrightarrow \quad \frac{d^2\gamma^h}{dt^2} + \sum_{i,j=1}^{m} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^h_{ij} = 0, \quad h = \overline{1,m},$$

If $C(t) = (\gamma(t), \vartheta(t))$ is a horizontal lift of the curve $\gamma(t)$, locally we have:

$$\nabla_{\dot\gamma}\vartheta = 0 \quad \Leftrightarrow \quad \frac{d\vartheta_h}{dt} - \sum_{i,j=1}^{m} \Gamma^i_{jh}\frac{d\gamma^j}{dt}\vartheta_i = 0, \quad h = \overline{1,m}.$$

Remark 14. *Using Remark 13 we can construct an infinity of examples of geodesics on (T^*M, g^f).*

Example 15. *Let $\mathbb{R}$ equipped with the Riemannian metric*

$$g = e^x dx.$$

The Christoffel symbols of Riemannian connection are given by

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}.$$

The geodesics $\gamma(t)$ such that $\gamma(0) = a \in \mathbb{R}$, $\gamma'(0) = v \in \mathbb{R}$ satisfy the equation,

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{m} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^k_{ij} = 0 \Leftrightarrow \gamma'' + \frac{1}{2}(\gamma')^2 = 0.$$

Then $\gamma'(t) = \dfrac{2v}{2+vt}$ and $\dot\gamma(t) = \gamma'(t)\dfrac{d}{dx}$, hence $\gamma(t) = a + 2\ln(1 + \dfrac{vt}{2})$.
1) $\widetilde{\dot\gamma(t)} = \displaystyle\sum_{i,j=1}^{m} g_{ij}\dot\gamma^j(t)dx^i = g_{11}\gamma'(t)dx = \frac{v}{2+vt}dx,$
From Corollary 11, the curve $C_1(t) = (\gamma(t), \widetilde{\dot\gamma(t)})$ is a geodesic on $T^*\mathbb{R}$.
2) *If $C_2(t) = (\gamma(t), \vartheta(t))$ is horizontal lift of the curve $\gamma(t)$ and $\vartheta(t) = \vartheta_1(t)dx$, then*

$$\frac{d\vartheta_h}{dt} - \Gamma^i_{jh}\frac{d\gamma^j}{dt}\vartheta_i = 0 \Leftrightarrow \vartheta_1' - \frac{1}{2}\vartheta_1\gamma' = 0 \Leftrightarrow \vartheta_1(t) = k.\exp\left(\frac{1}{2}\gamma'(t)\right).$$

Then $\vartheta_1(t) = k.\exp(\dfrac{v}{2+vt})$ and $\vartheta(t) = k.\exp(\dfrac{v}{2+vt})dx$.
From Corollary 12, the curve $C_2(t) = (\gamma(t), \vartheta(t))$ is a geodesic on $T^*\mathbb{R}$.

Example 16. *Let $\mathbb{R}^2$ equipped with the Riemannian metric.*

$$g = x^2dx^2 + y^2dy^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^1_{11} = \frac{1}{x},\ \Gamma^2_{22} = \frac{1}{y}.$$

If $\gamma(t)=(x(t),y(t))$ is geodesics on $\mathbb{R}^2$, then

$$\frac{d^2\gamma^k}{dt^2}+\sum_{i,j=1}^{2}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^k_{ij}=0\Leftrightarrow\begin{cases}x''+\dfrac{(x')^2}{x}=0,\\ \\ y''+\dfrac{(y')^2}{x}=0.\end{cases}$$

Hence $\gamma(t)=(a\sqrt{t},b\sqrt{t})$ and $\dot{\gamma}(t)=\dfrac{a}{2\sqrt{t}}\dfrac{\partial}{\partial x}+\dfrac{b}{2\sqrt{t}}\dfrac{\partial}{\partial y}$ where $a,b\in\mathbb{R}$.
1) $\widetilde{\dot{\gamma}(t)}=\sum_{i,j=1}^{2}g_{ij}\dot{\gamma}^j(t)dx^i=\dfrac{a^3}{2}\sqrt{t}dx+\dfrac{b^3}{2}\sqrt{t}dy,$
From Corollary 11, the curve $C(t)=(\gamma(t),\widetilde{\dot{\gamma}(t)})$ is a geodesic on $T^*\mathbb{R}^2$.
2) *If $C(t)=(\gamma(t),\vartheta(t))$ is horizontal lift of the curve $\gamma(t)$ i.e. $\nabla_{\dot{\gamma}}\vartheta=0$, then*

$$\frac{d\vartheta_h}{dt}-\sum_{i,j=1}^{2m}\Gamma^i_{jh}\frac{d\gamma^j}{dt}\vartheta_i=0\Leftrightarrow\begin{cases}\vartheta_1'-\dfrac{x'}{x}\vartheta_1=0,\\ \\ \vartheta_2'-\dfrac{y'}{y}\vartheta_2=0.\end{cases}$$

Hence $\vartheta(t)=k_1a\sqrt{t}dx+k_2b\sqrt{t}dy$, where $k_1,k_2\in\mathbb{R}$.
From Corollary 12, the curve $C(t)=(\gamma(t),\vartheta(t))$ is a geodesic on T^*M.

Theorem 17. *Let (M^m,g) be a Riemannian manifold, (T^*M,g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric and $\gamma(t)$ be a geodesic on M. If $C(t)=(\gamma(t),\vartheta(t))$ is a geodesic on T^*M such that $\nabla_{\dot{\gamma}}\vartheta\neq 0$, then f is a constant along the curve $\gamma(t)$.*

Proof. Let $\gamma(t)$ be a geodesic on M , then $\nabla_{\dot{\gamma}}\dot{\gamma}=0$. From the equation (19) we obtain

$$\begin{aligned}g(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma})=0\quad&\Rightarrow\quad\frac{1}{2\alpha}\big(g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\nabla_{\dot{\gamma}}\vartheta)+g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\vartheta)^2\big)g(grad\,f,\dot{\gamma})-\frac{f}{\alpha}g(R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma},\dot{\gamma})=0\\&\Rightarrow\quad\frac{1}{2\alpha}\big(g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\nabla_{\dot{\gamma}}\vartheta)+g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\vartheta)^2\big)\dot{\gamma}(f)=0\\&\Rightarrow\quad\dot{\gamma}(f)=0.\end{aligned}$$

■

Corollary 18. *Let (M^m,g) be a Riemannian manifold and (T^*M,g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. Then the curve $C(t)=(\gamma(t),\vartheta(t))$ is a geodesic on T^*M such that $\|\vartheta\|$ is constant if and only if we have*

$$\nabla_{\dot{\gamma}}\dot{\gamma}=\frac{1}{2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\nabla_{\dot{\gamma}}\vartheta)grad\,f-\frac{f}{\alpha}R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma},\tag{21}$$

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta=-\frac{1}{f}\dot{\gamma}(f)\nabla_{\dot{\gamma}}\vartheta-\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\nabla_{\dot{\gamma}}\vartheta)\vartheta.\tag{22}$$

Proof. We have $0=\dot{\gamma}g^{-1}(\vartheta,\vartheta)=2g^{-1}(\nabla_{\dot{\gamma}}\vartheta,\vartheta)$, from the equations (19) and (20) we obtain the result. ■

5. Conclusions

In this work, we studied the geodesics on the cotangent bundle with respect to the vertical rescaled Cheeger-Gromoll metric and we gave the necessary and sufficient conditions under which a curve be geodesic respect to this metric. Also we can study the geodesics of an another metrics on the cotangent bundle by deformation in the vertical bundle or in the horizontal bundle.

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References

[1] Ağca, F. (2013). *g-Natural Metrics on the cotangent bundle*. Int. Electron. J. Geom. 6(1), 129-146.

[2] Ağca, F., & Salimov, A. A. (2013). *Some notes concerning Cheeger-Gromoll metrics*. Hacet. J. Math. Stat. 42(5), 533-549.

[3] Gezer, A., & Altunbas, M. (2016). *On the rescaled Riemannian metric of Cheeger Gromoll type on the cotangent bundle*. Hacet. J. Math. Stat. 45(2), 355-365.

[4] Ocak, F. (2019). *Notes about a new metric on the cotangent bundle*. Int. Electron. J. Geom. 12(2), 241-249.

[5] Patterson, E. M., & Walker, A. G. (1952). *Riemannian extensions*. Quart. J.Math. Oxford Ser. 3(1), 19-28.

[6] Salimov, A. A., & Ağca, F. (2011). *Some properties of Sasakian metrics in cotangent bundles*. Mediterr. J. Math. 8(2), 243-255.

[7] Sekizawa, M. (1987). *Natural transformations of affine connections on manifolds to metrics on cotangent bundles*. In: Proceedings of 14th Winter School on Abstract Analysis (Srni, 1986), Rend. Circ. Mat. Palermo, 14, 129-142.

[8] Yano, K., & Ishihara, S. (1973). *Tangent and cotangent bundles*. M. Dekker, New York.

[9] Zagane, A. (2020). *A new class of metrics on the cotangent bundle*. Bulletin of the Transilvania University of Brasov. Mathematics, Informatics, Physics. Series III, 13(1), 285-301.