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Solitary Wave Solutions of the Coupled Konno-Oono Equation by Using the Functional Variable Method and the Two Variables $\left(\frac{G'}{G},\frac{1}{G}\right)$ -Expansion Method

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Abstract

In this work, we investigate solitary wave solutions of the coupled Konno-Oono equation with the aid of the functional variable method (FVM) and the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method. We obtain solitary wave solutions in form of trigonometric function, hyperbolic function and rational function solutions. We also draw two and three-dimensional graphics for some solutions with help of Mathematica 7.

Keywords: Functional variable method; Two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method; The coupled Konno-Oono equation; Solitary wave solution; Traveling wave solution.

Birleştirilmiş Konno-Oono Denkleminin Fonksiyonel Değişken Metodu ve İki Değişkenli $\left(\frac{G'}{G},\frac{1}{G}\right)$-Açılım Metodu Yardımıyla Solitary Dalga Çözümleri

Öz

Bu çalışmada, birleştirilmiş Konno-Oono Denkleminin fonksiyonel değişken metodu ve iki değişkenli $\left(\frac{G'}{G},\frac{1}{G}\right)$- açılım metodu yardımıyla solitary dalga çözümlerini araştırdık. Ele

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aldığımız denklemin trigonometrik ve hiperbolik ve rasyonel fonksiyon formunda solitary dalga çözümlerini elde ettik. Ayrıca Mathematica 7 yardımıyla bazı çözümlerin iki ve üç boyutlu grafiklerini çizdik.

Anahtar Kelimeler: Fonksiyonel değişken metodu; İki değişkenli $\left(\frac{G'}{G},\frac{1}{G}\right)$-açılım metodu; Birleştirilmiş Konno-Oono denklemi; Solitary Dalga Çözümleri; Hareket eden dalga çözümü.

1. Introduction

Nonlinear partial differential equations (NPDEs) occur in many areas of mathematical physics. These equations are used in mathematical modeling of physical phenomena. The solutions of these equations are very important for the scientists. These solutions offer a lot of information about the character of the nonlinear event. That's why the scientists have focused on these solutions in recent years. The scientists have studied on different methods to obtain these solutions. When the scientists study these methods, they usually reduce the nonlinear event to linear. In doing this reduction, they often consider a nonlinear auxiliary equation. Many analytical methods have emerged thanks to the different choices of auxiliary equations. Another aspect of these methods that differ from each other is the selected solution function. The balancing term is used when writing these solutions. The balancing term is obtained by comparing the nonlinear term with the linear term. The solutions obtained by these methods are generally hyperbolic, trigonometric and rational solutions. Many different versions of these methods have been presented [1-11].

In this study, we implement FVM [12] and the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method [13] to the coupled Konno-Oono equation [14]. There are many studies about the coupled Konno-Oono equation in the literature. For example, Manafian obtained some analytical solutions by the external trial equation method [15]. Bashar found traveling wave solutions by using tanh-function method and extended tanh-function method [16]. Torvattanabun presented the new exact solutions by extended simplest equation method [17]. Inan obtained multiple soliton solutions by improved tanh function method [18].

2. Methods

2.1. Analysis of functional variable method (FVM)

In this part, we present the FVM. Let's consider a given NLPDE for $u(x,t)$ as follows:

$$H(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\dots)=0, \tag{1}$$

where H is a polynomial of its arguments. Using the transformation $u(x,t) = U(\xi),\ \xi = x \pm ct$ where u is the wave speed, then, we have ordinary differential equation (ODE) like

$$Q(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, U_{\xi\xi\xi\xi}, \ldots) = 0, \tag{2}$$

U is considered as a functional variable in the form

$$U_\xi = F(U), \tag{3}$$

and other derivatives of U are

$$U_{\xi\xi} = \frac{1}{2}(F^2)', \tag{4}$$

$$U_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2}, \tag{5}$$

$$U_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)''F^2 + (F^2)''(F^2)'], \tag{6}$$

we write Eqn. (2) with respect to U, F and their derivatives as follows:

$$G(U, F, F', F'', F''', F^{(4)}, \ldots) = 0, \tag{7}$$

Eqn. (7) can be rewritten with respect to F by integration of Eqn. (7) and suitable solutions for the investigated problem can be found by using Eqn. (3).

2.2. Analysis of two Variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method

In this part, we present analysis of the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [13]. Let's consider the following the second order linear ODE

$$G''(\zeta) + \lambda G(\zeta) = \mu, \tag{8}$$

where $\varphi = \frac{G'}{G}$ and $\psi = \frac{1}{G}$, then, we have

$$\varphi' = -\varphi^2 + \mu\psi - \lambda, \quad \psi' = -\varphi\psi. \tag{9}$$

Step 1. For $\lambda < 0$, the general solutions of Eqn. (8)

$$G(\zeta) = C_1 sinh(\sqrt{-\lambda}\,\zeta) + C_2 cosh(\sqrt{-\lambda}\,\zeta) + \frac{\mu}{\lambda}, \tag{10}$$

where C_1 and C_2 are arbitrary constants. Thus, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma+\mu^2}(\varphi^2 - 2\mu\psi + \lambda), \tag{11}$$

where $\sigma = C_1^2 - C_2^2$.

Step 2. For $\lambda > 0$, the general solutions of Eqn. (8)

$$G(\zeta) = C_1 sin(\zeta\sqrt{\lambda}) + C_2 cos(\zeta\sqrt{\lambda}) + \frac{\mu}{\lambda}, \tag{12}$$

and therefore,

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma-\mu^2}(\varphi^2 - 2\mu\psi + \lambda), \tag{13}$$

where $\sigma = C_1^2 + C_2^2$.

Step 3. For $\lambda = 0$, the solutions of Eqn. (8)

$$G(\zeta) = \frac{\mu}{2}\zeta^2 + C_1\zeta + C_2, \tag{14}$$

and therefore,

$$\psi^2 = \frac{1}{C_1^2-2\mu C_2}(\varphi^2 - 2\mu\psi). \tag{15}$$

Now, we show how this method is applied. Thus, let's consider an NLPDE is written by

$$H = (u, u_t, u_{x,} u_{xx}, u_{tt}, \dots). \tag{16}$$

If we use transformation $u(x,t) = u(\zeta),\ \ \zeta = x - Vt$ then, we yield a nonlinear ODE for $u(\zeta)$

$$H' = (u, u', u'' \dots) = 0. \tag{17}$$

It can be written the solutions of Eqn. (17) by a polynomial φ and ψ as:

$$u(\zeta) = \sum_{i=0}^{M} a_i\varphi^i + \sum_{i=1}^{M} b_i\varphi^{i-1}\psi, \tag{18}$$

where $a_i(i = 0,1,\dots,M)$ and $b_i(i = 1,\dots,M)$ are arbitrary constants to be found later. M is a positive integer that can be concluded through balancing the highest nonlinear terms in Eqn. (17) and the highest order derivative. Substitute Eqn. (18) into Eqn. (17), the Eqn. (17) is written depending on the φ and ψ polynomials. The coefficient of each powers of $\varphi^i\psi^j$ equals zero and it is obtained an equation system for $a_i, b_i, V, \mu, C_1, C_2$ and λ. We calculate this equation system by means of Mathematica 7. Therefore, we find the solutions in relation to the hyperbolic functions for $\lambda < 0$, the trigonometric functions for $\lambda > 0$ and finally the rational functions for $\lambda = 0$ respectively.

3. Application of FVM and Two Variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-Expansion Method to Coupled Konno-Oono Equation

Example 1. In this part, we present solitary wave solutions of coupled Konno-Oono equation by using the FVM.

$$\begin{cases} v_t + 2uu_x = 0, \\ u_{xt} - 2uv = 0, \end{cases} \tag{19}$$

where $u = u(x,t)$, $v = v(x,t)$. Using the transformation $u(x,t) = U(\xi),\ v(x,t) = V(\xi)$, and $\xi = x - wt$, we find that

$$\begin{cases} -wV' + 2UU' = 0, \\ -wU'' - 2UV = 0, \end{cases} \tag{20}$$

where the prime denotes the derivation with respect to ξ. Integrating Eqn. (20), we have

$$\begin{cases} -wV + U^2 + c = 0, \\ -wU'' - 2UV = 0, \end{cases} \tag{21}$$

where c is the integration constant. We have from first equation of Eqn. (21)

$$V = \frac{1}{w}(U^2 + c). \tag{22}$$

If we substitute Eqn. (22) in Eqn. (21), we obtain the following:

$$U'' = -\frac{2}{w^2}U^3 - \frac{2c}{w^2}U. \tag{23}$$

Substituting Eqn. (4) into Eqn. (23) leads to the following equation:

$$F(U) = U' = \sqrt{-\frac{1}{w^2}U^4 - \frac{2c}{w^2}U^2 + h_0}\,, \tag{24}$$

where h_0 is an integration constant. Let be $h_4 = -\frac{1}{w^2}$ and $h_2 = -\frac{2c}{w^2}$. We obtain bell shaped solitary wave solutions of Eqn. (19) by solving Eqn. (24) for $h_0 = 0$ and $h_2 > 0,\ (\mathrm{c} < 0), h_4 < 0$,

$$\begin{cases} u(x,t) = \sqrt{-2c}\,\mathrm{Sech}\left(\sqrt{\frac{-2c}{w^2}}(x-wt)\right), \\ v(x,t) = \frac{1}{w}\left(\left(\sqrt{-2c}\,\mathrm{Sech}\left(\sqrt{\frac{-2c}{w^2}}(x-wt)\right)\right)^2 + \mathrm{c}\right). \end{cases} \tag{25}$$

Example 2. In this part, we obtain solitary wave solutions of coupled Konno-Oono equation by using the two variables $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method. If we balance the highest order derivative with the nonlinear term in Eqn. (23), we write solution for Eqn. (23) as

$$U(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi). \tag{26}$$

Substituting Eqn. (26) in Eqn. (23), we have an algebraic system as follows:

$$\begin{gathered} \frac{2ca_0}{w^2} + \frac{2a_0^3}{w^2} = 0,\quad \frac{2cb_1}{w^2} + \lambda b_1 + \frac{6a_0^2 b_1}{w^2} = 0,\quad -\mu b_1 + \frac{6a_0 b_1^2}{w^2} = 0, \\ \frac{2ca_1}{w^2} + 2\lambda a_1 + \frac{6a_0^2 a_1}{w^2} = 0,\ 3\mu a_1 + \frac{12a_0 a_1 b_1}{w^2} = 0,\quad \frac{6a_1 b_1^2}{w^2} = 0, \\ 2b_1 + \frac{6a_1^2 b_1}{w^2} = 0,\ 2a_1 + \frac{2a_1^3}{w^2} = 0,\quad \frac{6a_0 a_1^2}{w^2} = 0,\quad \frac{2b_1^3}{w^2} = 0, \end{gathered} \tag{27}$$

If Eqn. (27) is solved by Mathematica 7, we obtain

$$\mu = 0,\ c = -w^2\lambda,\ a_0 = 0,\ a_1 = -iw,\ b_1 = 0,\ w \neq 0. \tag{28}$$

Case 1. For $\lambda < 0$,

$$\begin{cases} u(x,t) = iw\left(\frac{A_1\sqrt{-\lambda}\,\mathrm{Cosh}\left(\sqrt{-\lambda}\,(x-wt)\right) + A_2\sqrt{-\lambda}\,\mathrm{Sinh}\left(\sqrt{-\lambda}\,(x-wt)\right)}{A_1\mathrm{Sinh}\left(\sqrt{-\lambda}\,(x-wt)\right) + A_2\mathrm{Cosh}\left(\sqrt{-\lambda}\,(x-wt)\right)}\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\left(\frac{A_1\sqrt{-\lambda}\,\mathrm{Cosh}\left(\sqrt{-\lambda}\,(x-wt)\right) + A_2\sqrt{-\lambda}\,\mathrm{Sinh}\left(\sqrt{-\lambda}\,(x-wt)\right)}{A_1\mathrm{Sinh}\left(\sqrt{-\lambda}\,(x-wt)\right) + A_2\mathrm{Cosh}\left(\sqrt{-\lambda}\,(x-wt)\right)}\right)\right)^2 - w^2\lambda\right)' \end{cases} \tag{29}$$

where $\sigma = A_1^2 - A_2^2$.

If we specifically select $\mathrm{A}_1 = 0$, $\mathrm{A}_2 > 0$ in Eqn. (29), we have hyperbolic solitary wave solutions

$$\begin{cases} u(x,t) = iw\sqrt{-\lambda}\,\mathrm{Tanh}\left(\sqrt{-\lambda}\,(x-wt)\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\sqrt{-\lambda}\,\mathrm{Tanh}\left(\sqrt{-\lambda}\,(x-wt)\right)\right)^2 - w^2\lambda\right)' \end{cases} \tag{30}$$

If we specifically select $A_2 = 0$, $A_1 > 0$ in Eqn. (29), we obtain hyperbolic solitary wave solutions

$$\begin{cases} u(x,t) = iw\sqrt{-\lambda}\,\mathrm{Coth}\left(\sqrt{-\lambda}\,(x - wt)\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\sqrt{-\lambda}\,\mathrm{Coth}\left(\sqrt{-\lambda}\,(x - wt)\right)\right)^2 - w^2\lambda\right) \end{cases}. \tag{31}$$

Case 2. For $\lambda > 0$,

$$\begin{cases} u(x,t) = iw\left(\frac{A_1\sqrt{\lambda}\,\mathrm{Cos}\left(\sqrt{\lambda}\,(x-wt)\right)+A_2\sqrt{\lambda}\,\mathrm{Sin}\left(\sqrt{\lambda}\,(x-wt)\right)}{A_1\mathrm{Sin}\left(\sqrt{\lambda}\,(x-wt)\right)+A_2\mathrm{Cos}\left(\sqrt{\lambda}\,(x-wt)\right)}\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\left(\frac{A_1\sqrt{\lambda}\,\mathrm{Cos}\left(\sqrt{\lambda}\,(x-wt)\right)+A_2\sqrt{\lambda}\,\mathrm{Sin}\left(\sqrt{\lambda}\,(x-wt)\right)}{A_1\mathrm{Sin}\left(\sqrt{\lambda}\,(x-wt)\right)+A_2\mathrm{Cos}\left(\sqrt{\lambda}\,(x-wt)\right)}\right)\right)^2 - w^2\lambda\right) \end{cases}, \tag{32}$$

where $\sigma = A_1^2 + A_2^2$.

If we specifically select $A_1 = 0$, $A_2 > 0$ in Eqn. (32), we have trigonometric solitary wave solutions

$$\begin{cases} u(x,t) = iw\sqrt{\lambda}\,\mathrm{Tan}\left(\sqrt{\lambda}\,(x - wt)\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\sqrt{\lambda}\,\mathrm{Tan}\left(\sqrt{\lambda}\,(x - wt)\right)\right)^2 - w^2\lambda\right) \end{cases}. \tag{33}$$

If we specifically select $A_2 = 0$, $A_1 > 0$ in Eqn. (32), we obtain trigonometric solitary wave solutions

$$\begin{cases} u(x,t) = iw\sqrt{\lambda}\,\mathrm{Cot}\left(\sqrt{\lambda}\,(x - wt)\right) \\ v(x,t) = \frac{1}{w}\left(\left(iw\sqrt{\lambda}\,\mathrm{Cot}\left(\sqrt{\lambda}\,(x - wt)\right)\right)^2 - w^2\lambda\right) \end{cases}. \tag{34}$$

Case 3. For $\lambda = 0$, we obtain the rational solutions of Eqn. (19)

$$\begin{cases} u(x,t) = -iw\frac{A_1}{A_1(x-wt)+A_2} \\ v(x,t) = \frac{1}{w}\left(-iw\frac{A_1}{A_1(x-wt)+A_2}\right)^2 \end{cases}, \tag{35}$$

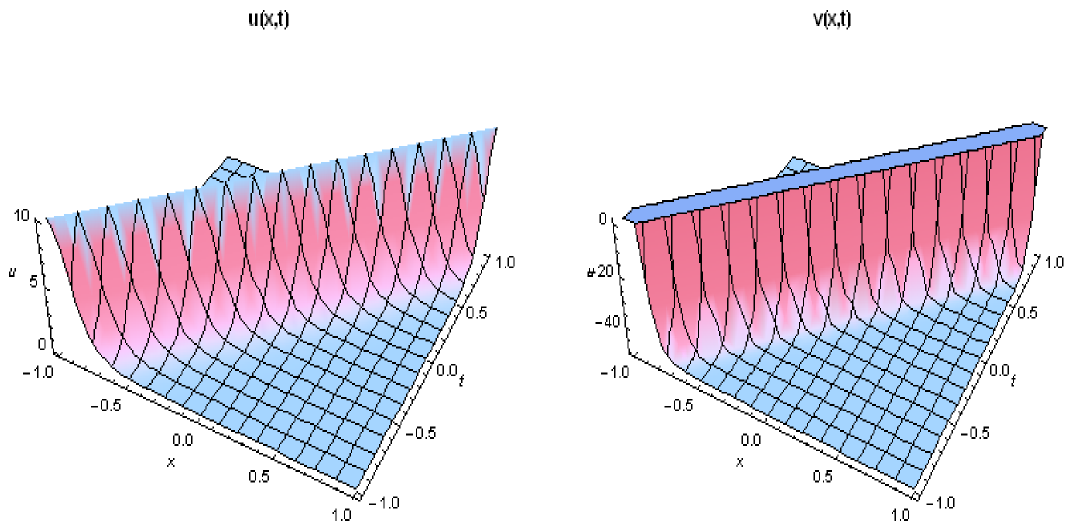

Figure 1: Above set of figures represent smooth bell shaped and singular bell shaped wave solutions of Eqn. (25) for $u(x,t)$ and $v(x,t)$, respectively $(c=-50, w=1)$

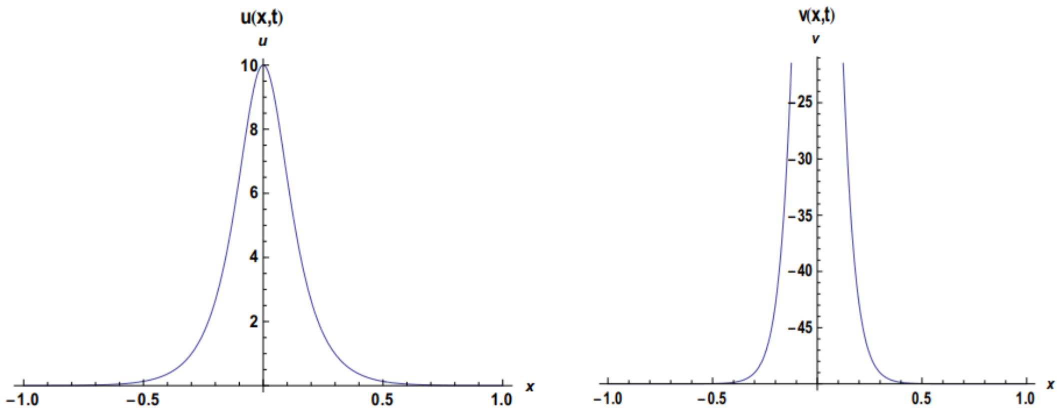

Figure 2: Above set of figures represent smooth bell shaped and singular bell shaped wave solutions of Eqn. (25) for $u(x,t)$ and $v(x,t)$, respectively $(c=-50,\ w=1,\ t=0)$

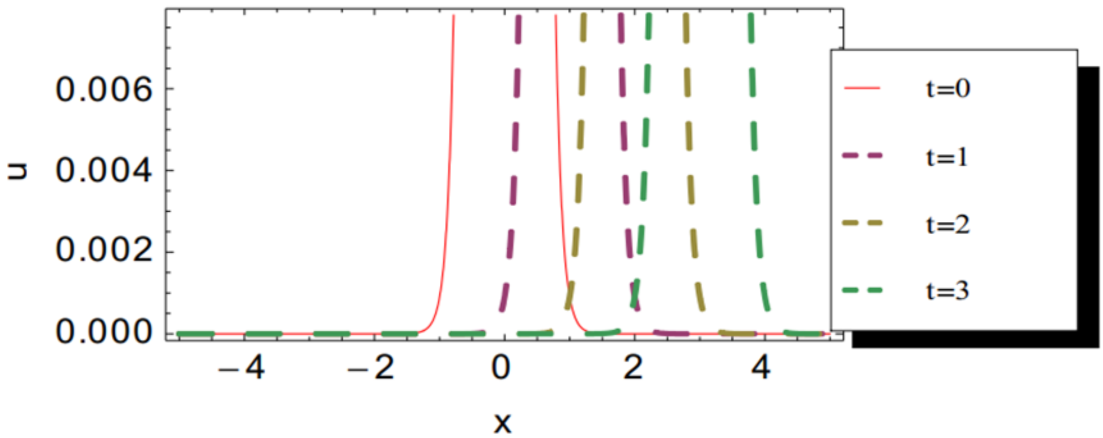

Figure 3: Above figure represent rightward traveling wave solutions of Eqn. (25) for $u(x,t)$ $(c=-50,$ $w=1)$

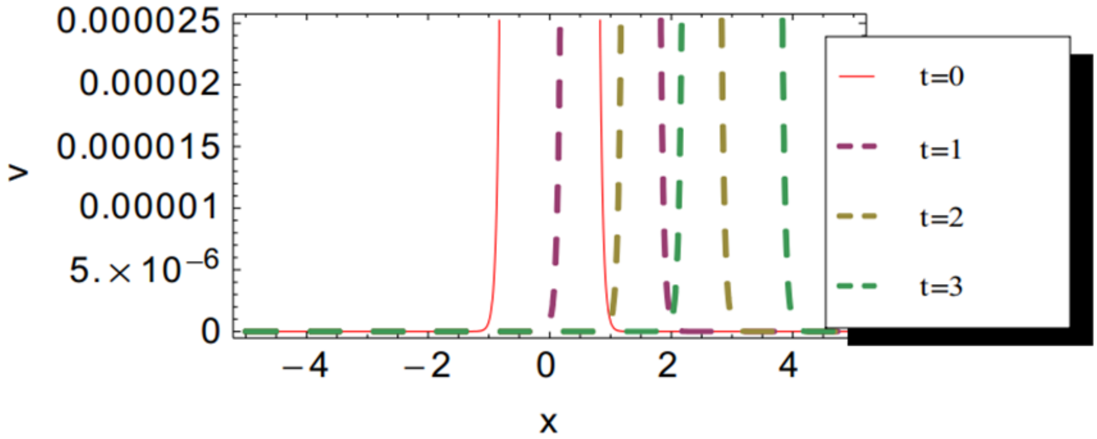

Figure 4: Above figure represent rightward traveling wave solutions of Eqn. (25) for $v(x,t)$ ($c=-50$, $w=1$)

4. Conclusion

In this study, we obtain solitary wave solutions of the coupled Konno-Oono equation by using the FVM and the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method. These methods used in this study can be used to obtain the solutions of many other NPDEs. While we only obtain hyperbolic solution by using FVM for the Eqn. (19), we find hyperbolic, trigonometric and rational solutions by the two variables $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method. We have smooth bell shaped and singular bell shaped wave solutions of Eqn. (19) for $u(x,t)$ and $v(x,t)$, respectively in Fig. 1. We draw traveling wave shape in 2D for $u(x,t)$ and $v(x,t)$ at $t=0$ in Fig. 2. We show rightward traveling wave solutions of Eqn. (25) for $u(x,t)$ and $v(x,t)$ at $t=0, t=1, t=2, t=3$, respectively in Fig. 3-4. We checked all the obtained solutions and they indeed satisfied Eqn. (19).

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