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A Classification of Submanifolds of (κ,μ)-Paracontact Metric Space Forms

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Abstract

The aim of this paper is to study the invariant submanifolds of a (κ,μ)-paracontact metric space form. We characterize (κ,μ)-paracontact metric space form satisfying the curvature conditions $\nabla\sigma$=0, $R\cdot\sigma=0$, $R\cdot\nabla\sigma=0$ and $\widetilde{C}\cdot\sigma=0$. Finally, we see that these conditions are equivalent to $\sigma=0$.

Keywords: *(κ,μ)-paracontact metric space form, semi-parallel submanifold, 2-semi-parallel submanifold, concircular semi-parallel submanifold.*
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1. Introduction

In the modern differential geometry, the geometry of submanifolds has became a subject of growing interest for its significant applications in applied mathematics and physics. For instance, the notion of invariant submanifold is used to discuss properties of non-linear autonomous system. On the other hand, the notion of geodesics plays an important role in the theory of relativity. For totally geodesic submanifolds, the geodesics of the ambient manifolds remain geodesics in the submanifolds. Therefore, totally geodesic submanifolds are also very much important in physical sciences rather than the simplest submanifolds.

The notion of paracontact geometry was initiated by Kaneyuki and Williams in [7]. A systematic investigation on paracontact metric manifolds done by Zamkovoy in [9]. After then, Cappelletti-Montano et al [6] introduced a new type of paracontact geometry so-called paracontact metric (κ,μ) space, where κ and μ are constants. It is well known [5] that in contact geometry case $\kappa\leq 1$. But in paracontact geometry case, there is no restriction for κ. This is an advantage for paracontact metric manifolds.

In [3], Authors introduced 2-semiparallel surfaces as surfaces satisfying the integrability condition of differential system.

In [8], Özgür et. al studied minimal anti-invariant semiparallel submanifolds of a generalized Sasakian space form and show that the submanifolds are totally geodesics under certain conditions.

Also, in [4], we studied invariant semiparallel and 2-semiparallel submanifolds in a normal paracontact metric manifold. Necessary and sufficient conditions are given for the submanifold to be totally geodesic.

Motivated by the above studies, in this paper, we investigate the geodesic cases of an invariant submanifold of a (κ,μ)-paracontact metric manifold by means of the curvatures of the ambient manifold.

2. Paracontact Metric Manifolds

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ has an almost paracontact structure (φ,ξ,η) if admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\varphi^2=I-\eta\otimes\xi,\ \ \eta(\xi)=1,\ \ \varphi\xi=0\ \ and\ \ \eta\circ\varphi=0. \tag{1.1}$$

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If an almost paracontact manifold $\widetilde{M}^{2n+1}$ with (φ,ξ,η) structure admits a pseudo Riemannian metric g such that

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{1.2}$$

for all vector fields X, Y tangent to $\widetilde{M}^{2n+1}$, then we call $\widetilde{M}^{2n+1}$ has an almost paracontact metric structure.

If $g(\varphi X,Y)+d\eta(X,Y)=0$, then η is a paracontact form and the almost paracontact metric manifold $\widetilde{M}$ is said to be paracontact metric manifold[9].

The concircular curvature tensor, projective curvature tensor, Conformal curvature tensor and quasi-conformal curvature tensor of a Riemannian manifold (M^{2n+1},g) are, respectively, given by

$$\begin{aligned}
\widetilde{Z}(X,Y)Z &= R(X,Y)Z-\frac{\tau}{2n(2n+1)}\{g(Y,Z)X-g(X,Z)Y\} && (1.3)\\
P(X,Y)Z &= R(X,Y)Z-\frac{1}{2n}\{S(Y,Z)X-S(X,Z)Y\}, && (1.4)\\
C(X,Y)Z &= R(X,Y)Z-\frac{1}{2n-1}\{S(Y,Z)X-S(X,Z)Y\\
&+ g(Y,Z)QX-g(X,Z)QY\}\\
&+ \frac{\tau}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}, && (1.5)
\end{aligned}$$

for all $X,Y,Z\in\Gamma(TM)$, where R, Q and S denote, respectively, the Riemannian curvature tensor, Ricci operator and Ricci tensor of M and r denotes the scalar curvature of M.

Let $(\widetilde{M},\varphi,\xi,\eta,g)$ be a paracontact metric manifold and we define a $(1,1)$-type tensor field by $h=\frac{1}{2}\ell_\xi\varphi$, where ℓ_ξ denote the Lie derivative operator along ξ. Then h is symmetric and satisfies the conditions

$$h\xi=0, h\varphi=-\varphi h,\ \ Trh=Tr\varphi h=0. \tag{1.6}$$

We denote the Levi-Civita connection on $\widetilde{M}$ by $\widetilde{\nabla}$, then we have the following relation

$$\widetilde{\nabla}_X\xi=-\varphi X+\varphi hX, \tag{1.7}$$

for all $X\in\Gamma(T\widetilde{M})$.

A paracontact metric manifold $(\widetilde{M},\varphi,\xi,\eta,g)$ is said to be a (κ,μ)-space form if its Riemannian curvature tensor R satisfies

$$R(X,Y)\xi=\kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}, \tag{1.8}$$

for all tangent to vector fields X,Y on $\widetilde{M}$, where κ and μ are arbitrary constants.

The geometric behavior of the (κ,μ)-paracontact metric manifold is different according as $\kappa<-1$, $\kappa=-1$ and $\kappa>-1$. In particular, for the case $\kappa<-1$ and $\kappa>-1$, (κ,μ)-nullity condition (1.8) determines the whole curvature tensor field completely.

For a (κ,μ)-paracontact metric manifold $(\widetilde{M},\varphi,\xi,\eta,g)$, we have the following case;

$$\begin{aligned}
h^2 &= (1+\kappa)\varphi^2, && (1.9)\\
(\widetilde{\nabla}_X\varphi)Y &= -g(X-hX,Y)\xi+\eta(Y)(X-hX), && (1.10)\\
S(X,Y) &= \{2(1-n)+n\mu\}g(X,Y)+\{2(n-1)+\mu\}g(hX,Y)\\
&+ \{2(n-1)+n(2\kappa-\mu)\}\eta(X)\eta(Y), && (1.11)\\
S(X,\xi) &= 2n\kappa\eta(X),\ \ Q\xi=2n\kappa\xi, && (1.12)\\
Q\varphi &- \varphi Q=2\{2(n-1)+\mu\}h\varphi, && (1.13)
\end{aligned}$$

for all $X,Y\in\Gamma(T\widetilde{M})$, where Q and S denote the Ricci operator and Ricci tensor of $\widetilde{M}$, respectively.

2. Invariant Submanifolds of a (κ,μ)-Paracontact Metric Manifold

Now, let M be an immersed submanifold of a (κ,μ)-paracontact metric manifold $\widetilde{M}$ and we denote the induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ by ∇ and $\nabla^{\perp}$, respectively. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_XY=\nabla_XY+\sigma(X,Y), \tag{2.1}$$

and

$$\widetilde{\nabla}_XV=-A_VX+\nabla^{\perp}_XV, \tag{2.2}$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$, where σ and A_V are said to be second fundamental form and the shape operator of M, respectively. They are related by

$$g(\sigma(X,Y),V)=g(A_VX,Y).$$

If $\widetilde{\nabla}_X \sigma = 0$, then the submanifold is said to be parallel of second fundamental form. The covariant derivatives of σ and A_V are defined by, respectively,

$$(\widetilde{\nabla}_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{2.3}$$

and

$$(\widetilde{\nabla}_X A)_V Y = \nabla_X A_V Y - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y. \tag{2.4}$$

For an immersed submanifold M of a paracontact metric manifold $(\widetilde{M},\varphi,\xi,\eta,g)$, if $\varphi(TM) \subseteq TM$, M is said to be an invariant submanifold. We note that an invariant submanifold inherits all properties of the ambient manifold. In the rest of our paper, we will assume that M is an invariant submanifold of (κ,μ)-paracontact metric manifold $\widetilde{M}$. By means of (1.6), we have $h\varphi X = -\varphi hX$ for any $X \in \Gamma(TM)$, This tells us that M is also invariant with respect to h. Thus, we have.

Lemma 2.1. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}$. Then*

$$\nabla_X \xi \quad = \quad -\varphi X + \varphi hX \ \ \textit{and} \ \ \sigma(X,\xi) = 0.$$

Theorem 2.2. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}$. The second fundamental form σ of M is parallel if and only if M is a totally geodesic submanifold.*

Proof. Let us assume that the second fundamental form σ is parallel. Then, from (2.3) we have

$$\nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z) = 0, \tag{2.5}$$

for all $X,Y,Z \in \Gamma(TM)$. On the other hand, by using Gauss formula, taking into account of Lemma 2.1 and taking $Y = \xi$ in (2.5), we get $\sigma(\nabla_X \xi, Z) = 0$. Thus, we reach

$$\sigma(-\varphi X + \varphi hX, Z) = 0, \ \ that \ \ is, \ \ \sigma(\varphi X, Z) = \sigma(\varphi hX, Z). \tag{2.6}$$

Substituting X by hX in (2.6), we get

$$\begin{aligned} \sigma(\varphi hX, Z) &= \sigma(\varphi h^2 X, Z) = (1+\kappa)\sigma(\varphi^3 X, Z) \\ &= (1+\kappa)\sigma(\varphi X, Z) \end{aligned}$$

that is, $\kappa\sigma(\varphi X, Z) = 0$ for $\kappa \neq 0$. This proves our assertion. The converse is trivial. □

A submanifold is called semi-parallel if $R(X,Y)\sigma = 0$, for all $X,Y \in \Gamma(TM)$, R denotes the Riemannian curvarure tensor of $\widetilde{M}$ and it is defined by

$$\begin{aligned} (R(X,Y)\sigma)(Z,U) &= R^{\perp}(X,Y)\sigma(Z,U) - \sigma(R(X,Y)Z,U) \\ &- \sigma(Z,R(X,Y)U), \end{aligned} \tag{2.7}$$

for all $X,Y,Z,U \in \Gamma(TM)$.

Theorem 2.3. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}$. Then M is semi-parallel submanifold if and only if it is a totally geodesic submanifold, provided $\mu^2(1+\kappa) - \kappa^2 \neq 0$.*

Proof. We suppose that M is semiparallel. Then (2.7) leads to

$$R^{\perp}(X,Y)\sigma(Z,U) - \sigma(R(X,Y)Z,U) - \sigma(Z,R(X,Y)U) = 0, \tag{2.8}$$

for all $X,Y,Z,U \in \Gamma(TM)$. Choosing $U = \xi$ in (2.8) and we taking into account that Lemma 2.1, we have

$$\begin{aligned} \sigma(Z,R(X,Y)\xi) &= \sigma(Z,\kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)) \\ &= \eta(Y)\{\kappa\sigma(Z,X) + \mu\sigma(Z,hX)\} \\ &- \eta(X)\{\mu\sigma(Z,hY) + \kappa\sigma(Z,Y)\} = 0. \end{aligned}$$

Also, substituting Y by ξ, we have

$$\kappa\sigma(X,Z) + \mu\sigma(Z,hX) = 0. \tag{2.9}$$

Substituting X by hX in (2.9), we obtain

$$\begin{aligned} \kappa\sigma(hX,Z) + \mu\sigma(Z,h^2X) &= \kappa\sigma(hX,Z) + \mu\sigma(Z,(1+\kappa)(X - \eta(X)\xi)) \\ &= \kappa\sigma(hX,X) + \mu(1+\kappa)\sigma(Z,X) = 0. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we reach at

$$(\kappa^2 - (1+\kappa)\mu^2)\sigma(Z,X) = 0.$$

The converse follows from the definition and hence, the theorem is proved completely. □

A submanifold M is called 2-semiparallel if $R\cdot\widetilde{\nabla}\sigma=0$ and it is defined by

$$\begin{aligned}(R(X,Y)\nabla\sigma)(Z,U,V) &= R^{\perp}(X,Y)(\nabla_Z\sigma)(U,V)-(\nabla_{R(X,Y)Z}\sigma)(U,V)\\ &- (\nabla_Z\sigma)(R(X,Y)U,V)-(\nabla_Z\sigma)(U,R(X,Y)V,\end{aligned}$$

for all $X,Y,U,V,Z\in\Gamma(TM)$[4].

Theorem 2.4. *Let M be an invariant submanifold of a (κ,μ)-paracontact metric space form $\widetilde{M}$. Then M is 2-semi-parallel submanifold if and only if M is totally geodesic submanifold provided that $\mu^2(1+\kappa)-\kappa^2\neq 0$.*

Proof. Let us assume that M be 2-semi-parallel submanifold. Then we have

$$\begin{aligned}(R(X,Y)\nabla\sigma)(Z,U,V) &= R^{\perp}(X,Y)(\nabla_Z\sigma)(U,V)-(\nabla_{R(X,Y)Z}\sigma)(U,V)\\ &- (\nabla_Z\sigma)(R(X,Y)U,V)-(\nabla_Z\sigma)(U,R(X,Y)V)\\ &= 0,\end{aligned}$$

for all $X,Y,Z,U,V\in\Gamma(TM)$. This statement is also true for $X=U=\xi$, that is,

$$\begin{aligned}R^{\perp}(\xi,Y)(\nabla_Z\sigma)(\xi,V) &- (\nabla_{R(\xi,Y)Z}\sigma)(\xi,V)-(\nabla_Z\sigma)(R(\xi,Y)\xi,V)\\ &- (\nabla_Z\sigma)(\xi,R(\xi,Y)V)=0.\end{aligned}\tag{2.11}$$

Now, let's calculate each of these expressions.

$$\begin{aligned}R^{\perp}(\xi,Y)\{\nabla^{\perp}_Z\sigma(\xi,V) &- \sigma(\nabla_Z\xi,V)-\sigma(\xi,\nabla_ZV)\}\\ &= R^{\perp}(\xi,Y)\{-\sigma(-\varphi Z+\varphi hZ,V)\}\\ &= R^{\perp}(\xi,Y)\{\sigma(\varphi Z,V)-\sigma(\varphi hZ,V)\},\end{aligned}\tag{2.12}$$

$$\begin{aligned}(\nabla_{R(\xi,Y)Z}\sigma)(\xi,V) &= \nabla^{\perp}_{R(\xi,Y)Z}\sigma(\xi,V)-\sigma(\nabla_{R(\xi,Y)Z}\xi,V)\\ &- \sigma(\nabla_{R(\xi,Y)Z}V,\xi)\\ &= -\sigma(\nabla_{R(\xi,Y)Z}\xi,V)\\ &= -\sigma(-\varphi R(\xi,Y)Z+\varphi hR(\xi,Y)Z,V).\end{aligned}$$

From (1.8), we know that

$$\begin{aligned}R(\xi,Y)Z &= \kappa\{g(Y,Z)\xi-\eta(Z)Y\}+\mu\{g(hZ,Y)\xi\\ &- \eta(Z)hY\}.\end{aligned}\tag{2.13}$$

Also taking into account that (2.13), we obtain

$$\begin{aligned}(\nabla_{R(\xi,Y)Z}\sigma)(\xi,V) &= \sigma(\varphi R(\xi,Y)Z,V)-\sigma(\varphi hR(\xi,Y)Z,V)\\ &= -\kappa\eta(Z)\sigma(\varphi Z,V)+\mu(1+\kappa)\eta(Z)g(\varphi Z,V)\\ &= \eta(Z)\sigma(\varphi Y,V)(\mu(1+\kappa)\\ &- \kappa)+\eta(Z)(\kappa-\mu)(\varphi hY,V).\end{aligned}\tag{2.14}$$

$$\begin{aligned}(\nabla_Z\sigma)(R(\xi,Y)\xi,V) &= (\nabla_Z\sigma)(\kappa(\eta(Y)\xi-Y)-\mu hY),V)\\ &= \kappa(\nabla_Z\sigma)(\eta(Y)\xi,V)-\kappa(\nabla_Z\sigma)(Y,V)\\ &- \mu(\nabla_Z\sigma)(hY,V)\\ &= \kappa\{\nabla^{\perp}_Z\sigma(\eta(Y)\xi,V)-\sigma(\nabla_Z\eta(Y)\xi,V)\\ &- \sigma(\nabla_ZV,\eta(Y)\xi)\}\\ &- \kappa(\nabla_Z\sigma)(Y,V)-\mu(\nabla_Z\sigma)(hY,V)\\ &= -\kappa\sigma(Z\eta(Y)\xi+\eta(Y)\nabla_Z\xi,V)-\kappa(\nabla_Z\sigma)(Y,V)\\ &- \mu(\nabla_Z\sigma)(hY,V)\\ &= -\kappa\sigma(\nabla_Z\xi,V)\eta(Y)-\kappa(\nabla_Z\sigma)(Y,V)\\ &- \mu(\nabla_Z\sigma)(hY,V)\\ &= -\kappa\eta(Y)\sigma(\varphi Z+\varphi hZ,V)-\kappa(\nabla_Z\sigma)(Y,V)\\ &- \mu(\nabla_Z\sigma)(hY,V)\\ &= \kappa\eta(Y)\sigma(\varphi Z,V)-\kappa\eta(Y)\sigma(\varphi hZ,V)\\ &- \kappa(\nabla_Z\sigma)(Y,V)-\mu(\nabla_Z\sigma)(hY,V).\end{aligned}\tag{2.15}$$

and finally,

$$\begin{aligned}(\nabla_Z\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_Z\sigma(R(\xi,Y)V,\xi)-\sigma(\nabla_Z R(\xi,Y)V,\xi)\\ &- \sigma(R(\xi,Y)V,\nabla_Z\xi)\\ &= -\sigma(-\varphi Z+\varphi hZ,\kappa(g(Y,V)\xi-\eta(V)Y)\\ &+ \mu(g(hV,Y)\xi-\eta(V)hY))\\ &= \kappa\eta(V)\{\sigma(\varphi hZ,Y)-\sigma(\varphi Z,Y)\}\\ &+ \mu\eta(Y)\{\sigma(\varphi hZ,hY)-\sigma(\varphi Z,hY)\}. \end{aligned} \tag{2.16}$$

Thus (2.12), (2.14), (2.15) and (2.16) statements put in (2.11), we have

$$\begin{aligned}R^{\perp}(\xi,Y)\sigma(\varphi Z &- \varphi hZ,V)-\eta(Z)\sigma(\varphi Y,V)(\mu(1+\kappa)-\kappa)\\ &- \eta(Z)(\kappa-\mu)\sigma(\varphi hY,V)+\kappa(\nabla_Z\sigma)(Y,V)\\ &+ \mu(\nabla_Z\sigma)(hY,V)-\kappa\eta(Y)\sigma(\varphi Z,V)+\kappa\eta(Y)\sigma(\varphi hZ,V)\\ &- \kappa\eta(V)\{\sigma(\varphi hZ,Y)-\sigma(\varphi Z,Y)\}\\ &- \mu\eta(V)\{\sigma(\varphi hZ,hY)-\sigma(\varphi Z,hY)\}=0,\end{aligned}$$

from which for $V=\xi$,

$$\begin{aligned}\kappa\{\sigma(\varphi hZ,Y)-\sigma(\varphi Z,Y)\} &+ \mu\{\sigma(\varphi hZ,hY)\\ &- \sigma(\varphi Z,hY)\}=0.\end{aligned} \tag{2.17}$$

If hY is written instead of Y in (2.17) and by making use of (1.9), we obtain

$$\begin{aligned}\kappa\{\sigma(\varphi hZ,hY)-\sigma(\varphi Z,hY)\} &+ \mu\{\sigma(\varphi hZ,h^2Z)-\sigma(\varphi Z,h^2Y)\}=0\\ \kappa\{\sigma(\varphi hZ,hY)-\sigma(\varphi Z,hY)\} &+ \mu(1+\kappa)\{\sigma(\varphi hZ,\varphi^2Y)\\ &- \sigma(\varphi Z,\varphi^2Y)\}=0,\end{aligned}$$

that is,

$$\begin{aligned}\kappa\{\sigma(\varphi hZ,hY)-\sigma(\varphi Z,hY)\} &+ \mu(1+\kappa)\{\sigma(\varphi hZ,\varphi Y)\\ &- \sigma(\varphi Z,\varphi Y)\}=0.\end{aligned} \tag{2.18}$$

From the common solutions of (2.17) and (2.18), we conclude that

$$(\kappa^2-\mu^2(1+\kappa))(\sigma(\varphi hZ,Y)-\sigma(\varphi Z,Y))=0.$$

Since $\kappa^2-\mu^2(1+\kappa)\neq 0$, we have

$$\sigma(\varphi hZ,Y)-\sigma(\varphi Z,Y)=0. \tag{2.19}$$

Here, if hZ is taken instead of Z in (2.19) and by using (1.9), we reach at

$$\begin{aligned}\sigma(\varphi h^2Z,Y)-\sigma(\varphi hZ,Y) &= (1+\kappa)\sigma(\varphi^3Z,Y)-\sigma(\varphi hZ,Y)=0\\ (1+\kappa)\sigma(\varphi Z,Y) &- \sigma(\varphi hZ,Y)=0\end{aligned} \tag{2.20}$$

(2.19) and (2.20) prove our assertion. The converse is obvious. □

Now, we will consider the concircular curvature tensor $\widetilde{Z}$ of (κ,μ)-paracontact space form for later use. From (1.3) and (1.8), we have

$$\begin{aligned}\widetilde{Z}(X,Y)\xi &= R(X,Y)\xi-\frac{\tau}{2n(2n+1)}\{\eta(Y)X-\eta(X)Y\}\\ &= \kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}\\ &- \frac{\tau}{2n(2n+1)}\{\eta(Y)X-\eta(X)Y\}\\ &= (\kappa-\frac{\tau}{2n(2n+1)})\{\eta(Y)X-\eta(X)Y\}\\ &+ \mu\{\eta(Y)hX-\eta(X)hY\},\end{aligned} \tag{2.21}$$

for all $X,Y\in\Gamma(TM)$.

For the submanifold M of a (κ,μ)-paracontact metric space form M^{2n+1}, if $\widetilde{Z}(X,Y)\sigma=0$, then M is said to be concircular semi-parallel submanifold. Thus we have the following Theorem.

Theorem 2.5. *Let M be an invariant of a (κ,μ)-paracontact metric space form $\widetilde{M}$. Then M is a concircular semi-parallel if and only if M is either totally geodesic submanifold or the scalar curvature satisfies $\tau=2n(2n+1)(\kappa-\mu^2(1+\kappa))$.*

Proof. Let M be concircular semi-parallel submanifold, that is,

$$\widetilde{Z}(X,Y)\sigma = 0, \tag{2.22}$$

implies that

$$R^{\perp}(X,Y)\sigma(Z,U) - \sigma(\widetilde{Z}(X,Y)Z,U) - \sigma(Z,\widetilde{Z}(X,Y)U) = 0, \tag{2.23}$$

for all $X,Y,Z,U \in \Gamma(TM)$. For $U = \xi$ in (2.23), we have

$$\begin{aligned}
\sigma(\widetilde{Z}(X,Y)\xi,Z) &= (\kappa - \frac{\tau}{2n(2n+1)})\sigma(\eta(Y)X - \eta(X)Y,Z) \\
&+ \mu\sigma(\eta(Y)hX - \eta(X)hY,Z) = 0.
\end{aligned}$$

This reduce for $Y = \xi$,

$$(\kappa - \frac{\tau}{2n(2n+1)})\sigma(X,Z) + \mu\sigma(hX,Z) = 0. \tag{2.24}$$

Substituting X by hX in (2.24), we get

$$\begin{aligned}
(\kappa - \frac{\tau}{2n(2n+1)})\sigma(hX,Z) &+ \mu\sigma(h^2X,Z) = (\kappa - \frac{\tau}{2n(2n+1)})\sigma(hX,Z) \\
&+ \mu(1+\kappa)\sigma(\varphi^2X,Z) = 0,
\end{aligned}$$

that is,

$$(\kappa - \frac{\tau}{2n(2n+1)})\sigma(hX,Z) + \mu(1+\kappa)\sigma(X,Z) = 0. \tag{2.25}$$

From (2.24) and (2.25), we conclude that

$$(\mu^2(1+\kappa) - \kappa + \frac{\tau}{2n(2n+1)})\sigma(X,U) = 0.$$

This leads to $\sigma = 0$ or $\tau = 2n(2n+1)(\kappa - \mu^2(1+\kappa))$. □

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