Combination of Bipolar Soft Set and Soft Expert Set with Application in Decision Making

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Highlights
• A mathematical model has been proposed by linking the soft expert set with the bipolarity logic.
• Basic set operations have been studied on the bipolar soft expert set.
• A new algorithm has been proposed to deal with the problems involving uncertainty.

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Abstract
In this paper, we propose a novel concept of the bipolar soft expert set by combining the soft expert set and the bipolar soft set. Then, we define its basic operations such as complement, union, intersection, AND and OR for bipolar soft expert sets with illustrative examples. Then, using this set theory, an algorithm is proposed to express an uncertainty problem in the best way. Finally, we exemplify an uncertainty problem on how the proposed algorithm can be applied against uncertain situations that may be encountered in any field and we give its implementation in detail.

Keywords
Soft set
Soft expert set
Bipolar soft set
Bipolar soft expert set
Decision making

1. INTRODUCTION

One of the most important properties that must be addressed in order to perform data analysis in the most accurate way is uncertainty. However, the separations made in order to express the uncertainty correctly and thus to obtain the most ideal results are generally not so straightforward in this sense. Many mathematical models put forward to overcome this problem have been insufficient to be successful. There are many set types that have been brought to the literature in order to analyze the data in a near-ideal way. To give an example; the fuzzy set (briefly FS) [1], one of the pioneers of these set types, was proposed by Zadeh. In the following years, the rough set (briefly RS) [2] and intuitionistic fuzzy sets (briefly IFS) [3] can be expressed as remarkable theories in terms of decomposing uncertainty. However, there are some shortcomings in all of these theories. Molodtsov [4], who thinks that the main reason for these inadequacies is due to the lack of a parameterization tool, suggested soft set (briefly SS) theory. In addition to these, Molodtsov successfully applied it in many fields such as game theory, Riemann integration, smoothness of functions, theory of measurement and so on. The application area and diversity of the SS theory are rapidly increasing due to its success in expressing uncertainty [5-14].

Many versions of soft sets have been developed. One of these versions is the soft expert set (briefly SES) introduced by Alk hazaleh and Salleh [15]. This set type suggests that an expert group can be useful in the decision-making process. In this way, it is thought that more near-ideal results can be achieved in solving problems related to uncertainty. They also studied fuzzy SESs [16] by using SESs and fuzzy SSs. Then Enginoğlu and Dönmez [17] made some modifications to the SESs. Especially in recent years, interest in SS theory has been increased greatly, and many interesting applications of this theory have been expanded by embedding the ideas of mathematical models such as FS, IFS, interval-valued FS, N-SS, interval-valued fuzzy parameterized intuitionistic fuzzy SS [18-26].

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Another mathematical model introduced to the literature as a result of the effort to express uncertain situations in an ideal way is the bipolar soft set (briefly BSS) theory proposed by Sabir and Naz [27]. BSS is an extended model of SS. It is a mathematical model that has great advantages in dealing with uncertain information and is proposed by including the idea of bipolarity in the SSs. Due to these advantages, many researches have been done on BSS theory, which has managed to attract the attention of many kinds of research [28–31].

In this paper, we examined the extension of BSSs and SESs and introduced the concept of bipolar soft expert set (bipolar SES, briefly BSES). In other words, BSES theory, a new mathematical model, has been developed by including the idea of bipolarity in the SSs. Due to these advantages, many situations in an ideal way is the bipolar soft set (briefly BSS) theory, which has managed to attract the attention of many kinds of research [28–31].

In this section, we recall some basic concepts in SS, SES and BSS. Detailed explanations related to SS, SES and BSS can be found in [4,5,15,17,27].

Throughout this study, let U be an universe of objects and \(2^U\) denotes the power set of U. Also, let \(P\) be a set of parameters and \(K, L, M\) be non-empty subsets of \(P\).

**Definition 2.1.** [4] A pair \((\Gamma, K)\) is called an SS over \(U\), where \(\Gamma: K \rightarrow 2^U\) is a set valued mapping.

**Definition 2.2.** [5] Let \(P = \{p_1, p_2, \ldots, p_n\}\) be a set of parameters. The NOT set of \(P\) denoted by \(\neg P\) is defined by \(\neg P = \{-\neg p_1, -\neg p_2, \ldots, -\neg p_n\}\) where, \(-\neg p_i = \text{not } p_i\) for all \(i\).

Now we present some basic notations for SESs. Let \(E\) be a set of experts and \(O\) be a set of opinions, \(Z = P \times E \times O\) and \(K \subseteq Z\).

**Definition 2.3.** [15] A pair \((\Gamma, K)\) is called an SES over \(U\), where \(\Gamma\) is a mapping given by \(\Gamma: K \rightarrow 2^U\).

**Definition 2.4.** [15] For two SESs \((\Gamma, K)\) and \((\Lambda, L)\) over \(U\), \((\Gamma, K)\) is called a soft expert subset of \((\Lambda, L)\) if \(K \subseteq L\) and \(\forall p \in L, \Lambda(p) \subseteq \Gamma(p)\). This relationship is denoted by \((\Gamma, K) \subseteq (\Lambda, L)\).

**Definition 2.5.** [15] Two SESs \((\Gamma, K)\) and \((\Lambda, L)\) over \(U\) are said to be equal if \((\Gamma, K) \subseteq (\Lambda, L)\) and \((\Lambda, L) \subseteq (\Gamma, K)\).

**Definition 2.6.** [17] Let \(\alpha = (p, e, o) \in Z\). Then not \(\alpha\) and NOT \(Z\) are defined by \(\neg \alpha = (p, e, 1 - o)\) and \(\neg Z = \{-\neg \alpha : \alpha \in Z\}\), respectively. It can easily be seen that \(\neg Z = Z\) but usually \(\neg K \neq K\), for some \(K \subseteq Z\).

**Definition 2.7.** [17] The complement of an SES \((\Gamma, K)\), denoted by \((\Gamma, K)^c = (\Gamma^c, -\neg K)\), is defined by \((\Gamma, K)^c = (\Gamma^c, -\neg K)\) where \(\Gamma^c: \neg K \rightarrow 2^U\) is a mapping given by \(\Gamma^c(\neg \alpha) = U - \Gamma(\alpha)\), for all \(\neg \alpha \in \neg K\).

**Definition 2.8.** [15] Let \((\Gamma, K)\) be an SES over \(U\). Then,

1. An agree-SES \((\Gamma, K)_1\) over \(U\) is a soft expert subset of \((\Gamma, K)\) defined as follows:

\[ (\Gamma, K)_1 = \{ \Gamma_1(\alpha) : \alpha \in P \times E \times \{1\} \} \]  \hspace{1cm} (1)

2. A disagree-SES \((\Gamma, K)_0\) over \(U\) is a soft expert subset of \((\Gamma, K)\) defined as follows:
\((\Gamma, K)_0 = \{\Gamma_0(\alpha) : \alpha \in P \times E \times \{0\}\}\)  \hspace{1cm} (2)

**Definition 2.9.** [15] If \((\Gamma, K)\) and \((\Lambda, L)\) are two SESs over \(U\) then \((\Gamma, K)\ AND \((\Lambda, L)\) denoted by \((\Gamma, K) \wedge (\Lambda, L)\), is defined by
\[
(\Gamma, K) \wedge (\Lambda, L) = (\Omega, K \times L)
\]
where \(\Omega(p^k, p^l) = \Gamma(p^k) \cap \Lambda(p^l), \forall (p^k, p^l) \in K \times L.\)

**Definition 2.10.** [15] If \((\Gamma, K)\) and \((\Lambda, L)\) are two SESs over \(U\) then \((\Gamma, K)\ OR \((\Lambda, L)\) denoted by \((\Gamma, K) \vee (\Lambda, L)\), is defined by
\[
(\Gamma, K) \vee (\Lambda, L) = (\Omega, K \times L)
\]
where \(\Omega(p^k, p^l) = \Gamma(p^k) \cup \Lambda(p^l), \forall (p^k, p^l) \in K \times L.\)

**Definition 2.11.** [15] The union of two SESs \((\Gamma, K)\) and \((\Lambda, L)\) over \(U\) denoted by \((\Gamma, K) \cup (\Lambda, L)\), is the SES \((\Omega, M)\) where \(M = K \cup L, \forall p \in M,\)
\[
\Omega(p) = \begin{cases} 
\Gamma(p) & \text{if } e \in K - L \\
\Lambda(p) & \text{if } e \in L - K \\
\Gamma(p) \cup \Lambda(p) & \text{if } e \in K \cap L 
\end{cases}
\]
\hspace{1cm} (5)

**Definition 2.12.** [15] The intersection of two SESs \((\Gamma, K)\) and \((\Lambda, L)\) over \(U\) denoted by \((\Gamma, K) \cap (\Lambda, L)\), is the SES \((\Omega, M)\) where \(M = K \cup L, \forall p \in M,\)
\[
\Omega(p) = \begin{cases} 
\Gamma(p) & \text{if } e \in K - L \\
\Lambda(p) & \text{if } e \in L - K \\
\Gamma(p) \cap \Lambda(p) & \text{if } e \in K \cap L 
\end{cases}
\]
\hspace{1cm} (6)

**Definition 2.13.** [27] A triplet \((\Gamma, \Lambda, K)\) is called a BSS over \(U\), where \(\Gamma\) and \(\Lambda\) are mappings, given by \(\Gamma : K \rightarrow 2^U\) and \(\Lambda : \neg L \rightarrow 2^U\) such that \(\Gamma(p) \cap \Lambda(\neg p) = \emptyset, \forall p \in K.\)

**Definition 2.14.** [27] For two BSSs \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) over \(U\), we say that \((\Gamma, \Lambda, K)\) is a bipolar soft subset of \((\Gamma_1, \Lambda_1, L)\) if
1. \(K \subseteq L\)
2. \(\Gamma(p) \subseteq \Gamma_1(p)\) and \(\Lambda_1(\neg p) \subseteq \Lambda(\neg p), \forall p \in K.\)

This relationship is denoted by \((\Gamma, \Lambda, K) \subseteq (\Gamma_1, \Lambda_1, L)\). They are said to be equal if \((\Gamma, \Lambda, K) \equiv (\Gamma_1, \Lambda_1, L)\) and \((\Gamma_1, \Lambda_1, L) \subseteq (\Gamma, \Lambda, K).\)

**Definition 2.15.** [27] The complement of a BSS \((\Gamma, \Lambda, K)\), denoted by \((\Gamma, \Lambda, K)^{\circ}\), is defined by \((\Gamma, \Lambda, K)^{\circ} = (\Gamma^c, \Lambda^c, K)\) where \(\Gamma^c\) and \(\Lambda^c\) are mappings given by \(\Gamma^c(p) = \Lambda(\neg p)\) and \(\Lambda^c(\neg p) = \Gamma(p), \forall p \in K.\)

**Definition 2.16.** [27] If \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) are two BSSs over \(U\) then "\((\Gamma, \Lambda, K)\ AND \((\Gamma_1, \Lambda_1, L)\)" denoted \((\Gamma, \Lambda, K) \overline{\cap} (\Gamma_1, \Lambda_1, L)\) is defined by
\[
(\Gamma, \Lambda, K) \overline{\cap} (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, K \times L)
\]
\hspace{1cm} (7)
where \(\Gamma_2(p^k, p^l) = \Gamma(p^k) \cap \Gamma_1(p^l)\) and \(\Lambda_2(\neg p^k, \neg p^l) = \Lambda(\neg p^k) \cup \Lambda_1(\neg p^l), \forall (p^k, p^l) \in K \times L.\)
Definition 2.17. [27] If \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\) are two BSSs over \(U\) then "\((Γ, Λ, K)\) OR \((Γ_1, Λ_1, L)\)" denoted \((Γ, Λ, K) \lor (Γ_1, Λ_1, L)\) is defined by

\[(Γ, Λ, K) \lor (Γ_1, Λ_1, L) = (Γ_2, Λ_2, K \times L)\]  

where \(Γ_2(p^k, p^l) = Γ(p^k) ∪ Γ_1(p^l)\) and \(Λ_2(\neg p^k, \neg p^l) = Λ(\neg p^k) ∧ Λ_1(\neg p^l)\), \(∀(p^k, p^l) ∈ K \times L\).

Definition 2.18. [27] Let \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\) be BSSs over \(U\). Then,

1. The extended union of \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\), denoted by \((Γ, Λ, K) \sqcup (Γ_1, Λ_1, L)\), is defined as the BSS \((Γ_2, Λ_2, M)\) over \(U\), where \(M = K \cup L\) and \(∀p ∈ M\),

\[Γ_2(e) = \begin{cases} Γ(p) & \text{if } p ∈ K \setminus L \\ Γ_1(p) & \text{if } p ∈ L \setminus K \\ Γ(e) \sqcup Γ_1(e) & \text{if } p ∈ K \cap L \end{cases}\]  

\[Λ_2(\neg e) = \begin{cases} Λ(\neg p) & \text{if } \neg p ∈ K \setminus L \\ Λ_1(\neg p) & \text{if } \neg p ∈ L \setminus K \\ Λ(\neg p) ∪ Λ_1(\neg p) & \text{if } \neg p ∈ K \cap L \end{cases}\]  

2. The extended intersection of \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\), denoted by \((Γ, Λ, K) \sqcap (Γ_1, Λ_1, L)\), is defined as the BSS \((Γ_2, Λ_2, M)\) over \(U\), where \(M = K \cup L\) and \(∀p ∈ M\),

\[Γ_2(e) = \begin{cases} Γ(p) & \text{if } p ∈ K \setminus L \\ Γ_1(p) & \text{if } p ∈ L \setminus K \\ Γ(e) \sqcap Γ_1(e) & \text{if } p ∈ K \cap L \end{cases}\]  

\[Λ_2(\neg e) = \begin{cases} Λ(\neg p) & \text{if } \neg p ∈ K \setminus L \\ Λ_1(\neg p) & \text{if } \neg p ∈ L \setminus K \\ Λ(\neg p) ∩ Λ_1(\neg p) & \text{if } \neg p ∈ K \cap L \end{cases}\]  

3. The restricted union of \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\), denoted by \((Γ, Λ, K) \uplus \# (Γ_1, Λ_1, L)\), is defined as the BSS \((Γ_2, Λ_2, M)\) over \(U\), where \(M = K \cap L\) is non-empty and \(∀p ∈ M\),

\[Γ_2(p) = Γ(p) ∪ Λ(p)\]  

\[Λ_2(\neg p) = Γ_1(\neg p) ∪ Λ_1(\neg p)\]  

4. The restricted intersection of \((Γ, Λ, K)\) and \((Γ_1, Λ_1, L)\), denoted by \((Γ, Λ, K) \cap \# (Γ_1, Λ_1, L)\), is defined as the BSS \((Γ_2, Λ_2, M)\) over \(U\), where \(M = K \cap L\) is non-empty and \(∀p ∈ M\),

\[Γ_2(p) = Γ(p) \cap Λ(p)\]  

\[Λ_2(\neg p) = Γ_1(\neg p) \cup Λ_1(\neg p)\]

3. BIPOLAR SOFT EXPERT SETS

In this section, we introduce a new mathematical model, bipolar soft expert set (briefly BSES), to express uncertainty problems in a more ideal way and give some basic operations such as complement, subset, equal, AND, OR, extended union, extended intersection, restricted union and restricted intersection. Then, some basic properties of these concepts are given.
Let $E$ be a set of experts, $O = \{0, 1\}$ be a set of opinions, $Z = P \times E \times O$ and $K, L, M \subseteq Z$.

**Remark 3.1.** For simplicity, in this paper we assume that there are two-valued opinions only in set $O$, that is, $O = \{0, 1\} = \{\text{disagree}, \text{agree}\}$, but multivalued opinions may be assumed as well.

**Definition 3.1.** A triplet $(\Gamma, \Lambda, K)$ is called a BSES over $U$, where $\Gamma$ and $\Lambda$ are mappings, given by $\Gamma: K \rightarrow 2^U$ and $\Lambda: \neg K \rightarrow 2^U$ such that $\Gamma(p, e, 1) \cap \Lambda(\neg p, e, 1) = \emptyset$ or $\Gamma(p, e, 0) \cap \Lambda(\neg p, e, 0) = \emptyset$ for all $(p, e, o) \in K$ and $(\neg p, e, o) \in \neg K$. Here:

- $\Gamma(p, e, 1)$: the set of objects that provide the parameter $p$ by expert $e$,
- $\Lambda(\neg p, e, 1)$: the set of objects that provide the parameter $\neg p$ by expert $e$,
- $\Gamma(p, e, 0)$: the set of objects that do not provide the parameter $p$ by expert $e$,
- $\Lambda(\neg p, e, 0)$: the set of objects that do not provide the parameter $\neg p$ by expert $e$.

Here $(\Gamma, K)$ and $(\Lambda, \neg K)$ are SESs, since $K \subseteq Z = P \times E \times O$.

**Example 3.1.** Since people’s needs and desires in general are different, it is a difficult task for a person to choose the right car. When choosing a car, the impact of many factors affects their decision-making. For this, a private company wants to get help from two experts in this field to increase its profits. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of hybrid cars under consideration, $P = \{p_1, p_2\} = \{\text{durability, fuel efficient}\}$ and $\neg P = \{\neg p_1, \neg p_2\} = \{\text{non-durable, fuel inefficient}\}$ be the set of parameters and $E = \{e_1, e_2\}$ be the set of experts company has consulted. Suppose that the opinions expressed by the experts about the cars in the private company are as follows:

- $\Gamma(p_1, e_1, 1) = \{u_2, u_3\}$, 
- $\Lambda(\neg p_1, e_1, 1) = \{u_4\}$,
- $\Gamma(p_1, e_2, 1) = \{u_3, u_5\}$, 
- $\Lambda(\neg p_1, e_2, 1) = \{u_1, u_4\}$,
- $\Gamma(p_2, e_1, 1) = \{u_2, u_3, u_5\}$, 
- $\Lambda(\neg p_2, e_1, 1) = \{u_1, u_4\}$,
- $\Gamma(p_2, e_2, 1) = \{u_2, u_3, u_5\}$, 
- $\Lambda(\neg p_2, e_2, 1) = \{u_1, u_4\}$,
- $\Lambda(\neg p_1, e_1, 0) = \{u_1, u_2, u_3, u_5\}$, 
- $\Gamma(p_1, e_2, 0) = \{u_3, u_4\}$,
- $\Lambda(\neg p_1, e_2, 0) = \{u_2, u_3, u_5\}$,
- $\Gamma(p_2, e_1, 0) = \{u_1, u_4\}$, 
- $\Lambda(\neg p_2, e_1, 0) = \{u_2, u_3, u_5\}$,
- $\Gamma(p_2, e_2, 0) = \{u_1, u_3, u_4\}$, 
- $\Lambda(\neg p_2, e_2, 0) = \{u_2, u_3, u_4, u_5\}$.

All these opinions expressed by experts can be expressed with the help of the BSES $(\Gamma, \Lambda, Z)$ as follows:

$$
(\Gamma, \Lambda, Z) = \left\{ \begin{array}{c}
(p_1, e_1, 1, \{u_2, u_3\}), (\neg p_1, e_1, 1, \{u_4\}), (p_1, e_2, 1, \{u_3, u_5\}), \\
(\neg p_2, e_2, 1, \{u_1, u_4\}), (p_2, e_1, 1, \{u_2, u_3, u_5\}), (\neg p_2, e_2, 1, \{u_1\}), \\
(p_2, e_1, 0, \{u_2, u_3, u_5\}), (\neg p_1, e_2, 0, \{u_2, u_3, u_5\}), (p_2, e_1, 0, \{u_1, u_3\}), \\
(\neg p_2, e_2, 0, \{u_2, u_3, u_5\}), (p_2, e_2, 0, \{u_1, u_3, u_4\}), (\neg p_2, e_2, 0, \{u_2, u_3, u_4, u_5\}) \end{array} \right\}
$$

Here $(\Gamma, \Lambda, Z)$ is a BSES over $U$.

**Definition 3.2.** For two BSESs $(\Gamma_1, \Lambda_1, K)$ and $(\Gamma_1, \Lambda_1, L)$ over $U$, we say that $(\Gamma, \Lambda, K)$ is a bipolar soft expert subset of $(\Gamma_1, \Lambda_1, L)$ if

1. $K \subseteq L$ and
2. $\Gamma(p, e, o) \subseteq \Gamma_1(p, e, o)$ and $\Lambda_1(\neg p, e, o) \subseteq \Lambda(\neg p, e, o)$ for all $(p, e, o) \in K \subseteq P \times E \times O$.

This relationship is denoted by $(\Gamma, \Lambda, K) \sqsubseteq (\Gamma_1, \Lambda_1, L)$.

**Definition 3.3.** [25] Two BSESs $(\Gamma, \Lambda, K)$ and $(\Gamma_1, \Lambda_1, L)$ over $U$ are said to be equal if $(\Gamma, \Lambda, K) \sqsubseteq (\Gamma_1, \Lambda_1, L)$ and $(\Gamma_1, \Lambda_1, L) \sqsubseteq (\Gamma, \Lambda, K)$.

**Example 3.2.** Consider Example 3.1 and suppose that the private company consults the same experts again after a certain period of time. Then,
Example 3.4. Consider Example 3.1. Then the agree-BSES \((\Gamma, \Lambda, K)_1\) over \(U\) is

\[
(\Gamma, \Lambda, K)_1 = \{ (\Gamma, e, 1, \Lambda) : p \in P, -p \in -P, e \in E \}.
\]

Definition 3.5. An disagree-BSES \((\Gamma, \Lambda, K)_0\) over \(U\) is a bipolar soft expert subset of \((\Gamma, \Lambda, K)\) defined as follows:

\[
(\Gamma, \Lambda, K)_0 = \{ (\Gamma, e, 0, \Lambda) : p \in P, -p \in -P, e \in E \}.
\]

Example 3.3. Consider Example 3.1. Then the agree-BSES \((\Gamma, \Lambda, K)_1\) over \(U\) is

\[
(\Gamma, \Lambda, K)_1 = \{ \{(p_1, e_1, 1, \{u_2, u_3\}), ((-p_1, e_1, 1, \{u_4\}), \{(p_2, e_2, 1, \{u_5\})\}, \{(-p_2, e_2, 1, \{u_1\})\} \}
\]

and the disagree-BSES \((\Gamma, \Lambda, K)_0\) over \(U\) is

\[
(\Gamma, \Lambda, K)_0 = \{ \{(p_1, e_1, 0, \{u_1, u_4, u_5\}), ((-p_1, e_2, 0, \{u_2, u_3, u_4\}), \{(p_2, e_1, 0, \{u_1, u_4\})\}, \{(-p_2, e_2, 0, \{u_2, u_3, u_4, u_5\})\} \}
\]

Definition 3.6. The complement of a BSES \((\Gamma, \Lambda, K)\) is denoted by \((\Gamma, \Lambda, K)^c\) and is defined by \((\Gamma, \Lambda, K)^c = (\Gamma^c, \Lambda^c, K)\), where \(\Gamma^c\) and \(\Lambda^c\) are mappings given by \(\Gamma^c(p, e, 1) = \Gamma(p, e, 0)\)

\[(\Gamma^c(p, e, 0) = \Gamma(p, e, 1)) \text{ and } \Lambda^c(-p, e, 0) = \Lambda(-p, e, 0) \text{ for all } p \in P, e \in E.\]

Proposition 3.1. If \((\Gamma, \Lambda, K)\) is a BSES over \(U\), then

\[
(1) \ ((\Gamma, \Lambda, K)^c)^c = (\Gamma, \Lambda, K), \]

(2) \((\Gamma, \Lambda, K)_1^c = (\Gamma, \Lambda, K)_0), \]

(3) \((\Gamma, \Lambda, K)_0^c = (\Gamma, \Lambda, K)_1).\]

Proof. The proof is straightforward.

Example 3.4. Consider the BSES \((\Gamma, \Lambda, Z)\) over \(U\) given in Example 3.1. Then, we obtain
\[(\Gamma, \Lambda, Z)^\hat{c} = \left\{ \left( (p_1, e_1, 0), (u_2, u_3) \right), \left( (\neg p_1, e_1, 0), (u_4) \right), \left( (p_2, e_2, 0), (u_3, u_5) \right), \left( (\neg p_2, e_2, 0), (u_1) \right), \left( (p_1, e_1, 1), (u_2, u_3, u_5) \right), \left( (\neg p_1, e_2, 1), (u_2, u_3, u_5) \right), \left( (p_2, e_1, 1), (u_1, u_4) \right), \left( (\neg p_2, e_1, 1), (u_2, u_3, u_5) \right) \right\}. \]

**Definition 3.7.** If \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) are two BSESs over \(U\) then "\((\Gamma, \Lambda, K)\) AND \((\Gamma_1, \Lambda_1, L)\)" denoted \((\Gamma, \Lambda, K) \land (\Gamma_1, \Lambda_1, L)\) is defined by

\[(\Gamma, \Lambda, K) \land (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, K \times L) \]

where

\[\Gamma_2 \left( (p^k, e^k, o^k), (p^l, e^l, o^l) \right) = \Gamma(p^k, e^k, o^k) \cap \Gamma_1(p^l, e^l, o^l)\]

and

\[\Lambda_2 \left( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \right) = \Lambda(\neg p^k, e^k, o^k) \cup \Lambda_1(\neg p^l, e^l, o^l)\]

for all \( (p^k, e^k, o^k), (p^l, e^l, o^l) \) \( \in K \times L \), \( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \) \( \in (\neg K) \times (\neg L) \) \( (p^k, p^l) \in P \), \( e^k \in X_k, \ e^l \in X_l \). \( o^k \in O_k, \ o^l \in O_l \).

**Definition 3.8.** If \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) are two BSESs over \(U\) then "\((\Gamma, \Lambda, K)\) OR \((\Gamma_1, \Lambda_1, L)\)" denoted \((\Gamma, \Lambda, K) \lor (\Gamma_1, \Lambda_1, L)\) is defined by

\[(\Gamma, \Lambda, K) \lor (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, K \times L) \]

where

\[\Gamma_2 \left( (p^k, e^k, o^k), (p^l, e^l, o^l) \right) = \Gamma(p^k, e^k, o^k) \cup \Gamma_1(p^l, e^l, o^l)\]

and

\[\Lambda_2 \left( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \right) = \Lambda(\neg p^k, e^k, o^k) \cap \Lambda_1(\neg p^l, e^l, o^l)\]

for all \( (p^k, e^k, o^k), (p^l, e^l, o^l) \) \( \in K \times L \), \( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \) \( \in (\neg K) \times (\neg L) \) \( (p^k, p^l) \in P \), \( e^k \in X_k, \ e^l \in X_l \). \( o^k \in O_k, \ o^l \in O_l \).

**Proposition 3.2.** If \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) are two BSESs over \(U\) then

\[
\begin{align*}
(4) & \quad (\Gamma, \Lambda, K)^\hat{c} (\Gamma_1, \Lambda_1, L)^\hat{c} = (\Gamma, \Lambda, K)^\hat{c} \lor (\Gamma_1, \Lambda_1, L)^\hat{c}, \\
(5) & \quad (\Gamma, \Lambda, K)^\hat{c} (\Gamma_1, \Lambda_1, L)^\hat{c} = (\Gamma, \Lambda, K)^\hat{c} \land (\Gamma_1, \Lambda_1, L)^\hat{c}.
\end{align*}
\]

**Proof.**

(1) Suppose that \((\Gamma, \Lambda, K)\) \(\land (\Gamma_1, \Lambda_1, L)\) \(= (\Gamma_2, \Lambda_2, K \times L)\). Therefore, \((\Gamma_2, \Lambda_2, K \times L)^\hat{c} = (\Gamma_2^\hat{c}, \Lambda_2^\hat{c}, K \times L), \)

\[\Gamma_2^\hat{c} \left( (p^k, e^k, o^k), (p^l, e^l, o^l) \right) = \Gamma(p^k, e^k, o^k) \cap \Gamma_1(p^l, e^l, o^l) \hat{c} = \Gamma(p^k, e^k, o^k) \cup \Gamma_1(p^l, e^l, o^l) \hat{c} \]

and

\[\Lambda_2^\hat{c} \left( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \right) = \Lambda(\neg p^k, e^k, o^k) \cup \Lambda_1(\neg p^l, e^l, o^l) \hat{c} \]

for all \( (p^k, e^k, o^k), (p^l, e^l, o^l) \) \( \in K \times L \), \( (\neg p^k, e^k, o^k), (\neg p^l, e^l, o^l) \) \( \in (\neg K) \times (\neg L) \) \( (p^k, p^l) \in P \), \( e^k \in X_k, \ e^l \in X_l \). \( o^k \in O_k, \ o^l \in O_l \). Here, let \( o^k = 1 \) and \( o^l = 1 \). Then,

\[\Gamma^\hat{c} \left( (p^k, e^k, 1), (p^l, e^l, 1) \right) = \Gamma(p^k, e^k, 0) \cup \Gamma_1(p^l, e^l, 0) \hat{c} \]

and

\[\Lambda^\hat{c} \left( (\neg p^k, e^k, 1) \right) \cap \Lambda_1^\hat{c} \left( (\neg p^l, e^l, 1) \right) = \Lambda(\neg p^k, e^k, 0) \cap \Lambda_1(\neg p^l, e^l, 0) \hat{c} \]

On the other hand, let \((\Gamma, \Lambda, K)^\hat{c} \lor (\Gamma_1, \Lambda_1, L)^\hat{c} = (\Gamma^\hat{c}, \Lambda^\hat{c}, K) \lor (\Gamma_1^\hat{c}, \Lambda_1^\hat{c}, L) = (\Gamma_3, \Lambda_3, K \times L), \)

i.e.,
\[ \Gamma_3 \left( (p^k, e^k, 1), (p^l, e^l, 1) \right) = \Gamma^c(p^k, e^k, 1) \cup \Gamma_1^l(p^l, e^l, 1) = \Gamma(p^k, e^k, 0) \cup \Gamma_1(p^l, e^l, 0) \]

and

\[ \Lambda_3 \left( \neg \Gamma(p^k, e^k, 1), \neg \Gamma(p^l, e^l, 1) \right) = \Lambda^c(\neg \Gamma(p^k, e^k, 1)) \cap \Lambda_1^l(\neg \Gamma(p^l, e^l, 1)) = \Lambda(\neg \Gamma(p^k, e^k, 0)) \cap \Lambda_1(\neg \Gamma(p^l, e^l, 0)) \]

for all \( (p^k, e^k, o^k), (p^l, e^l, o^l) \in K \times L \). Similarly, it can be shown in the cases “\( o^k = 1 \) and \( o^l = 0 \)” and “\( o^k = 0 \) and \( o^l = 0 \)”.

(2) It is similar to the proof of (1).

**Definition 3.9.** Extended union of two BSESs \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) over \(U\) is the BSES \((\Gamma_2, \Lambda_2, M)\) over \(U\), where \(M = K \cup L\) and \(\forall p \in P, e \in E, o \in O\),

\[
\Gamma_2(p, e, o) = \begin{cases} 
\Gamma(p, e, o), & \text{if } (p, e, o) \in K - L \\
\Gamma_1(p, e, o), & \text{if } (p, e, o) \in L - K \\
\Gamma(p, e, o) \cup \Gamma_1(p, e, o), & \text{if } (p, e, o) \in K \cap L 
\end{cases}
\]  

(21)

\[
\Lambda_2(\neg p, e, o) = \begin{cases} 
\Lambda(\neg p, e, o), & \text{if } (\neg p, e, o) \in K - L \\
\Lambda_1(\neg p, e, o), & \text{if } (\neg p, e, o) \in L - K \\
\Lambda(\neg p, e, o) \cap \Lambda_1(\neg p, e, o), & \text{if } (\neg p, e, o) \in K \cap L 
\end{cases}
\]  

(22)

We denote it by \((\Gamma, \Lambda, K) \cup (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, M)\).

**Definition 3.10.** Extended intersection of two BSESs \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) over \(U\) is the BSES \((\Gamma_2, \Lambda_2, M)\) over \(U\), where \(M = K \cup L\) and \(\forall p \in P, e \in E, o \in O\),

\[
\Gamma_2(p, e, o) = \begin{cases} 
\Gamma(p, e, o), & \text{if } (p, e, o) \in K - L \\
\Gamma_1(p, e, o), & \text{if } (p, e, o) \in L - K \\
\Gamma(p, e, o) \cap \Gamma_1(p, e, o), & \text{if } (p, e, o) \in K \cap L 
\end{cases}
\]  

(23)

\[
\Lambda_2(\neg p, e, o) = \begin{cases} 
\Lambda(\neg p, e, o), & \text{if } (\neg p, e, o) \in K - L \\
\Lambda_1(\neg p, e, o), & \text{if } (\neg p, e, o) \in L - K \\
\Lambda(\neg p, e, o) \cup \Lambda_1(\neg p, e, o), & \text{if } (\neg p, e, o) \in K \cap L 
\end{cases}
\]  

(24)

We denote it by \((\Gamma, \Lambda, K) \cap (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, M)\).

**Definition 3.11.** Restricted union of two BSESs \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) over \(U\) is the BSES \((\Gamma_2, \Lambda_2, M)\) over \(U\), where \(M = K \cap L\) and \(\forall p \in P, e \in E, o \in O\),

\[
\Gamma_2(p, e, o) = \Gamma(p, e, o) \cup \Gamma_1(p, e, o) 
\]  

(25)

and

\[
\Lambda_2(\neg p, e, o) = \Lambda(\neg p, e, o) \cap \Lambda_1(\neg p, e, o). 
\]  

(26)

We denote it by \((\Gamma, \Lambda, K) \cup_R (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, M)\).

**Definition 3.12.** Restricted intersection of two BSESs \((\Gamma, \Lambda, K)\) and \((\Gamma_1, \Lambda_1, L)\) over \(U\) is the BSES \((\Gamma_2, \Lambda_2, M)\) over \(U\), where \(M = K \cap L\) and \(\forall p \in P, e \in E, o \in O\),

\[
\Gamma_2(p, e, o) = \Gamma(p, e, o) \cap \Gamma_1(p, e, o) 
\]  

(27)
and
\[ \Lambda_2(\neg p, e, o) = \Lambda(\neg p, e, o) \cup \Lambda_1(\neg p, e, o). \] (28)

We denote it by \( (\Gamma, \Lambda, K) \hat{\Gamma}_g (\Gamma_1, \Lambda_1, L) = (\Gamma_1, \Lambda_1, M) \).

**Proposition 3.3.** If \( (\Gamma_1, \Lambda_1, K), (\Gamma_2, \Lambda_2, L) \) and \( (\Gamma_3, \Lambda_3, M) \) are three BSEs over \( U \), then

1. \( (\Gamma_1, \Lambda_1, K) \star (\Gamma_2, \Lambda_2, L) = (\Gamma_2, \Lambda_2, L) \star (\Gamma_1, \Lambda_1, K) \),
2. \( (\Gamma_1, \Lambda_1, K) \star (\Gamma_2, \Lambda_2, L) \star (\Gamma_3, \Lambda_3, M) = (\Gamma_1, \Lambda_1, K) \star (\Gamma_2, \Lambda_2, L) \star (\Gamma_3, \Lambda_3, M) \)

For all \( \epsilon \in \{ \hat{\Gamma}_g, \hat{\Gamma}_g, \hat{\Gamma}_g \} \).

**Proof.** The proof is straightforward.

**Example 3.5.** Let \( U = \{ u_1, u_2, u_3, u_4, u_5 \} \) be the set of houses under consideration,

\[ P = \{ p_1 : \text{furnished}, \ p_2 : \text{in the green surroundings} \}, \ M = \{ u_5 : \text{pleasant} \} \]

be the set of parameters and \( E = \{ e_1, e_2 \} \) be a set of experts. Then

\[ \neg P = \{ \neg p_1 : \text{not furnished}, \ \neg p_2 : \text{not in the green surroundings} \}. \]

Suppose that \( K = \{ p_1, p_2 \} \) and \( L = \{ p_2, p_3 \} \). The BSEs \( (\Gamma, \Lambda, K) \) and \( (\Gamma_1, \Lambda_1, L) \) describe the "requirements of the houses" which Mr. A and Mrs. B are going to buy, respectively. Suppose that

\[
\begin{align*}
\Gamma(p_1, e_1, 1) &= \{ u_1, u_5 \}, \quad \Gamma(p_2, e_1, 1) = \{ u_1, u_4 \}, \quad \Lambda(\neg p_1, e_1, 1) = \{ u_2, u_3 \}, \\
\Lambda(\neg p_2, e_1, 1) &= \{ u_3, u_5 \}, \quad \Gamma(p_1, e_2, 1) = \{ u_1, u_2 \}, \quad \Gamma(p_2, e_2, 1) = \{ u_3, u_4 \}, \\
\Lambda(\neg p_1, e_2, 1) &= \{ u_4 \}, \quad \Lambda(\neg p_2, e_2, 1) = \{ u_3 \}, \quad \Gamma(p_1, e_1, 0) = \{ u_2, u_3, u_4 \}, \\
\Gamma(p_2, e_1, 0) &= \{ u_2, u_3, u_5 \}, \quad \Lambda(\neg p_1, e_1, 0) = \{ u_1, u_4, u_5 \}, \quad \Lambda(\neg p_2, e_1, 0) = \{ u_1, u_2, u_4 \}, \\
\Gamma(p_1, e_2, 0) &= \{ u_3, u_4, u_5 \}, \quad \Gamma(p_2, e_2, 0) = \{ u_1, u_2, u_3, u_5 \}, \quad \Lambda(\neg p_1, e_2, 0) = \{ u_1, u_2, u_3, u_5 \}, \\
\Lambda(\neg p_2, e_2, 0) &= \{ u_1, u_2, u_3, u_4 \}, \quad \Lambda(\neg p_2, e_2, 0) = \{ u_1, u_2, u_3, u_4 \}.
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_1(p_2, e_1, 1) &= \{ u_1, u_3, u_4 \}, \quad \Gamma_1(p_1, e_1, 1) = \{ u_3, u_4 \}, \quad \Lambda_1(\neg p_2, e_1, 1) = \{ u_5 \}, \\
\Lambda_1(\neg p_2, e_1, 1) &= \{ u_1, u_5 \}, \quad \Gamma_1(p_2, e_2, 1) = \{ u_3 \}, \quad \Gamma_1(p_3, e_2, 1) = \{ u_4, u_3 \}, \\
\Lambda_2(\neg p_3, e_2, 1) &= \{ u_3 \}, \quad \Lambda_2(\neg p_3, e_2, 1) = \{ u_5 \}, \quad \Gamma_1(p_1, e_1, 0) = \{ u_2, u_3 \}, \\
\Gamma_1(p_2, e_1, 0) &= \{ u_2, u_4, u_5 \}, \quad \Lambda_1(\neg p_2, e_1, 0) = \{ u_1, u_2, u_3, u_4 \}, \quad \Lambda_1(\neg p_2, e_1, 0) = \{ u_2, u_3, u_4 \}, \\
\Gamma_1(p_2, e_2, 0) &= \{ u_1, u_2, u_3, u_5 \}, \quad \Gamma_1(p_3, e_2, 0) = \{ u_1, u_2, u_3 \}, \quad \Lambda_1(\neg p_2, e_2, 0) = \{ u_1, u_3, u_4 \}, \\
\Lambda_1(\neg p_3, e_2, 0) &= \{ u_1, u_2, u_4, u_5 \}.
\end{align*}
\]

Now, we apply operations which are mentioned above on BSEs \( (\Gamma, \Lambda, K) \) and \( (\Gamma_1, \Lambda_1, L) \). Let

\( (\Gamma, \Lambda, K) \cup (\Gamma_1, \Lambda_1, L) = (\Gamma_2, \Lambda_2, K \cup U) \). Then

\[
\begin{align*}
\Gamma_2(p_1, e_1, 1) &= \{ u_1, u_5 \}, \quad \Gamma_2(p_2, e_1, 1) = \{ u_3, u_4 \}, \quad \Gamma_2(p_3, e_1, 1) = \{ u_3, u_4 \}, \\
\Gamma_2(p_1, e_2, 1) &= \{ u_1, u_2 \}, \quad \Gamma_2(p_2, e_2, 1) = \{ u_3, u_4 \}, \quad \Gamma_2(p_3, e_2, 1) = \{ u_3, u_4 \}, \\
\Gamma_2(p_1, e_1, 0) &= \{ u_2, u_3, u_4 \}, \quad \Gamma_2(p_2, e_1, 0) = \{ u_2, u_3, u_5 \}, \quad \Gamma_2(p_3, e_1, 0) = \{ u_1, u_2, u_3 \}, \\
\Gamma_2(p_1, e_2, 0) &= \{ u_3, u_4, u_5 \}, \quad \Gamma_2(p_2, e_2, 0) = \{ u_1, u_2, u_4, u_5 \}, \quad \Gamma_2(p_3, e_2, 0) = \{ u_1, u_2, u_3 \}.
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_2(\neg p_1, e_1, 1) &= \{ u_2, u_3 \}, \quad \Lambda_2(\neg p_2, e_1, 1) = \{ u_3, u_4 \}, \quad \Lambda_2(\neg p_3, e_1, 1) = \{ u_1, u_5 \}, \\
\Lambda_2(\neg p_1, e_2, 1) &= \{ u_4 \}, \quad \Lambda_2(\neg p_2, e_2, 1) = \{ u_2, u_5 \}, \quad \Lambda_2(\neg p_3, e_2, 1) = \{ u_3 \}, \\
\Lambda_2(\neg p_1, e_1, 0) &= \{ u_1, u_4, u_5 \}, \quad \Lambda_2(\neg p_2, e_1, 0) = \{ u_1, u_2, u_3, u_4 \}, \quad \Lambda_2(\neg p_3, e_1, 0) = \{ u_2, u_3, u_4 \}.
\end{align*}
\]
\[ \Lambda_2(-p_1, e_2, 0) = \{u_1, u_2, u_3, u_5\}, \quad \Lambda_2(-p_2, e_2, 0) = \{u_1, u_2, u_3, u_4\}, \quad \Lambda_2(-p_3, e_2, 0) = \{u_1, u_2, u_4, u_5\}. \]

Let \((\Gamma, \Lambda, K) \cap (\Gamma_1, \Lambda_1, L) = (\Gamma_3, \Lambda_3, K \cup L)\). Then

\[
\begin{align*}
\Gamma_3(p_1, e_1, 1) &= \{u_1, u_5\}, \quad \Gamma_3(p_2, e_1, 1) = \{u_3, u_4\}, \quad \Gamma_3(p_3, e_1, 1) = \{u_3, u_4\}, \\
\Gamma_3(p_1, e_2, 1) &= \{u_1, u_2\}, \quad \Gamma_3(p_2, e_2, 1) = \{u_3\}, \quad \Gamma_3(p_3, e_2, 1) = \{u_4, u_5\}, \\
\Gamma_3(p_1, e_1, 0) &= \{u_2, u_3, u_4\}, \quad \Gamma_3(p_2, e_1, 0) = \{u_2, u_5\}, \quad \Gamma_3(p_3, e_1, 0) = \{u_1, u_2, u_3\}, \\
\Gamma_3(p_1, e_2, 0) &= \{u_3, u_4, u_5\}, \quad \Gamma_3(p_2, e_2, 0) = \{u_1, u_2, u_5\}, \quad \Gamma_3(p_3, e_2, 0) = \{u_1, u_2, u_3\}.
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_3(-p_1, e_1, 1) &= \{u_2, u_3\}, \quad \Lambda_3(-p_2, e_1, 1) = \{u_5\}, \quad \Lambda_3(-p_3, e_1, 1) = \{u_1, u_5\}, \\
\Lambda_3(-p_1, e_2, 1) &= \{u_4\}, \quad \Lambda_3(-p_2, e_2, 1) = \{u_5\}, \quad \Lambda_3(-p_3, e_2, 1) = \{u_3\}, \\
\Lambda_3(-p_1, e_1, 0) &= \{u_1, u_4, u_5\}, \quad \Lambda_3(-p_2, e_1, 0) = \{u_2, u_4, u_5\}, \quad \Lambda_3(-p_3, e_1, 0) = \{u_2, u_3, u_4\}, \\
\Lambda_3(-p_1, e_2, 0) &= \{u_1, u_2, u_3, u_5\}, \quad \Lambda_3(-p_2, e_2, 0) = \{u_1, u_3, u_4\}, \\
\Lambda_3(-p_3, e_2, 0) &= \{u_1, u_2, u_4, u_5\}.
\end{align*}
\]

Let \((\Gamma, \Lambda, K) \cup (\Gamma_1, \Lambda_1, L) = (\Gamma_4, \Lambda_4, K \cap L)\). Then

\[
\begin{align*}
\Gamma_4(p_1, e_1, 1) &= \{u_1, u_3, u_4\}, \quad \Gamma_4(p_2, e_1, 1) = \{u_3, u_4\}, \\
\Gamma_4(p_2, e_1, 0) &= \{u_2, u_5\}, \quad \Gamma_4(p_2, e_2, 0) = \{u_1, u_2, u_4\}.
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_4(-p_1, e_1, 1) &= \{u_3, u_5\}, \quad \Lambda_4(-p_2, e_1, 1) = \{u_2, u_5\}, \\
\Lambda_4(-p_1, e_1, 0) &= \{u_1, u_2, u_4\}, \quad \Lambda_4(-p_2, e_1, 0) = \{u_1, u_3, u_4\}.
\end{align*}
\]

Let \((\Gamma, \Lambda, K) \cap (\Gamma_1, \Lambda_1, L) = (\Gamma_5, \Lambda_5, K \cap L)\). Then

\[
\begin{align*}
\Gamma_5(p_2, e_1, 1) &= \{u_1, u_4\}, \quad \Gamma_5(p_2, e_1, 0) = \{u_2, u_3, u_5\}, \\
\Gamma_5(p_2, e_2, 1) &= \{u_3\}, \quad \Gamma_5(p_2, e_2, 0) = \{u_1, u_2, u_4, u_5\}.
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_5(-p_2, e_1, 1) &= \{u_5\}, \quad \Lambda_5(-p_2, e_1, 0) = \{u_1, u_2, u_3, u_4\}, \\
\Lambda_5(-p_2, e_2, 1) &= \{u_5\}.
\end{align*}
\]

Let \((\Gamma, \Lambda, K) \cup (\Gamma_2, \Lambda_2, L) = (\Gamma_6, \Lambda_6, K \times L)\). Then

\[
\begin{align*}
\Gamma_6((p_1, e_1, 1), (p_2, e_1, 1)) &= \{u_1, u_3, u_4, u_5\}, \quad \Gamma_6((p_1, e_1, 1), (e_3, x_1, 1)) = \{u_1, u_3, u_4, u_5\}, \\
\Gamma_6((p_2, e_1, 1), (p_2, e_1, 1)) &= \{u_1, u_3, u_4\}, \quad \Gamma_6((p_2, e_1, 1), (p_2, e_1, 1)) = \{u_1, u_3, u_4\}, \\
\Gamma_6((p_1, e_1, 0), (p_2, e_1, 0)) &= U, \quad \Gamma_6((p_1, e_1, 0), (p_2, e_1, 0)) = U, \\
\Gamma_6((p_2, e_1, 0), (p_2, e_1, 0)) &= \{u_2, u_3, u_5\}, \quad \Gamma_6((p_2, e_1, 0), (p_2, e_1, 0)) = \{u_1, u_2, u_3, u_5\}.
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_6(-p_1, e_1, 1), (-p_2, e_1, 1) &= \{u_2, u_3, u_5\}, \quad \Lambda_6(-p_1, e_1, 1), (-p_3, e_1, 1) = \{u_2, u_3, u_5\}, \\
\Lambda_6(-p_1, e_1, 1), (-p_2, e_1, 1) &= \{u_3, u_5\}, \quad \Lambda_6(-p_2, e_1, 1), (-p_3, e_1, 1) = \{u_1, u_3, u_5\}, \\
\Lambda_6(-p_1, e_1, 0), (-p_2, e_1, 0) &= U, \quad \Lambda_6(-p_1, e_1, 0), (-p_3, e_1, 0) = U, \\
\Lambda_6(-p_2, e_1, 0), (-p_2, e_1, 0) &= U, \quad \Lambda_6(-p_2, e_1, 0), (-p_3, e_1, 0) = \{u_1, u_2, u_3, u_5\}.
\end{align*}
\]

and so on. Let \((\Gamma, \Lambda, K) \cap (\Gamma_1, \Lambda_1, L) = (\Gamma_7, \Lambda_7, K \times L)\). Then
The following algorithm may be followed by the company to fill the position.

**Algorithm**

1. Input the BSES (Γ, Λ, P).
2. Find an agree-BSES and a disagree-BSES.
3. Find $A_1 = \sum u_{ij}$ for agree-BSES (for $p_m, e_n$).
4. Find $B_j = \sum u_{ij}$ for agree-BSES (for $-p_m, e_n$).
5. Find $C_j = \sum u_{ij}$ for disagree-BSES (for $p_m, e_n$).
6. Find $D_j = \sum u_{ij}$ for disagree-BSES (for $-p_m, e_n$).
7. Find $(A_j - B_j) - (C_j - D_j) = S_j$.
8. Find $k$, for which $S_k = \max S_j$.

In Tables 1 and 2, we present the agree-BSES (for $(p_m, e_n)$) and disagree-BSES (for $(p_m, e_n)$) and disagree-BSES (for $(p_m, e_n)$), respectively, such that if "$u_{ij} \in \Gamma(p, e, o) \cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)"$ then $u_{ij} = 1$ otherwise $u_{ij} = 0$, and if "$u_{ij} \in \Gamma(p, e, o) \cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)\cup \Lambda(-p, e, o)"$ then $u_{ij} = 1$ otherwise $u_{ij} = 0$ where $u_{ij}$ are the entries in Tables 1 and 2, $(m, n) \in \mathbb{Z}^+$. 

The following algorithm may be followed by the company to fill the position.
Table 1. Agree-BSES (for \((p_m, e_n)\) and \((\neg p_m, e_n)\))

<table>
<thead>
<tr>
<th>U</th>
<th>(u_1)</th>
<th>(u_2)</th>
<th>(u_3)</th>
<th>(u_4)</th>
<th>(u_5)</th>
<th>(\neg p_1, e_1)</th>
<th>(p_1, e_2)</th>
<th>(p_2, e_1)</th>
<th>(p_2, e_2)</th>
<th>(\neg p_2, e_1)</th>
<th>(\neg p_2, e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p_1, e_1))</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((p_1, e_2))</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(\neg p_1, e_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((p_2, e_1))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>(\neg p_2, e_1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>((p_2, e_2))</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\neg p_2, e_2)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ A_j = \sum_{i} u_{ij} \]

\[ B_j = \sum_{i} u_{ij} \]

Table 2. Disagree-BSES (for \((p_m, e_n)\) and \((\neg p_m, e_n)\))

<table>
<thead>
<tr>
<th>U</th>
<th>(u_1)</th>
<th>(u_2)</th>
<th>(u_3)</th>
<th>(u_4)</th>
<th>(u_5)</th>
<th>(\neg p_1, e_1)</th>
<th>(p_1, e_2)</th>
<th>(p_2, e_1)</th>
<th>(p_2, e_2)</th>
<th>(\neg p_2, e_1)</th>
<th>(\neg p_2, e_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p_1, e_1))</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(\neg p_1, e_1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>((p_1, e_2))</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(\neg p_1, e_2)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((p_2, e_1))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(\neg p_2, e_1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>((p_2, e_2))</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>(\neg p_2, e_2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ C_j = \sum_{i} u_{ij} \]

\[ D_j = \sum_{i} u_{ij} \]

If the k value obtained here has more than one value, the company can choose any of them with their own option. Now let's use the proposed algorithm to find the candidate that best meets the parameters the private company wants, as given in Table 3:

Table 3. Evaluation result

<table>
<thead>
<tr>
<th>(A_j)</th>
<th>(B_j)</th>
<th>(C_j)</th>
<th>(D_j)</th>
<th>(S_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1 = 2)</td>
<td>(B_1 = 1)</td>
<td>(C_1 = 2)</td>
<td>(D_1 = 2)</td>
<td>(S_1 = 2)</td>
</tr>
<tr>
<td>(A_2 = 2)</td>
<td>(B_2 = 2)</td>
<td>(C_2 = 3)</td>
<td>(D_2 = 3)</td>
<td>(S_2 = 1)</td>
</tr>
<tr>
<td>(A_3 = 3)</td>
<td>(B_3 = 3)</td>
<td>(C_3 = 4)</td>
<td>(D_3 = 1)</td>
<td>(S_3 = 5)</td>
</tr>
<tr>
<td>(A_4 = 1)</td>
<td>(B_4 = 2)</td>
<td>(C_4 = 3)</td>
<td>(D_4 = 2)</td>
<td>(S_4 = -2)</td>
</tr>
<tr>
<td>(A_5 = 3)</td>
<td>(B_5 = 0)</td>
<td>(C_5 = 1)</td>
<td>(D_5 = 4)</td>
<td>(S_5 = 6)</td>
</tr>
</tbody>
</table>

Here, it is recommended the private company hire the candidate \(u_5\) for \(\text{max } S_k = S_5\).

A comparison: Bipolar soft expert sets are more successful than soft expert sets in terms of a better expression of complex data that may be encountered in an uncertainty problem. Indeed, when we consider two decision making algorithms for soft expert sets available in the literature and our proposed approach in this paper to solve the above-mentioned problem, the ranking among objects is obtained as follows:

For Algorithm [15],

\[ u_1 = u_2 = u_3 < u_4 < u_5 \]

For Algorithm [17],

\[ u_1 = u_2 = u_3 < u_4 = u_5 \]

For the proposed algorithm,

\[ u_3 < u_4 < u_2 < u_1 < u_5. \]

Therefore, it is necessary to use the proposed algorithm in order to make the best separation between the objects. In this case, we emphasize that bipolar soft expert sets should be preferred in order to better express complex data.

5. CONCLUSION

In this paper, we introduce bipolar soft expert sets by using soft expert sets and bipolar soft sets. We also study basic operations such as subset, equal, complement, OR, AND, extended union, extended
intersection, restricted union, restricted intersection. Thereafter, the basic properties of these operations are proven along with several examples to illustrate those properties. Finally, an algorithm based on bipolar soft expert sets was developed in order to better express a decision-making problem and an uncertainty problem was discussed to illustrate how this algorithm can be applied.

The researchers who will benefit from this paper in the future may be able to achieve more impressive results in applying different mathematical models to their decision-making problems regarding uncertainty situations. Thanks to the success of bipolar soft set and soft expert set especially in the area of uncertainty, we think that bipolar soft expert set we are working on will be an important research contribution.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES


