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Research Article

A note on induced connections

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Abstract

In this note, we will exploit the classical bijective correspondence between sections of an associated vector bundle and equivariant functions on the underlying principal bundle to revisit a global formula for induced connections on associated vector bundles. Consequently, we give the expression of the curvature in terms of the curvature 2-form of a connection on a principal bundle.

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Introduction

In his seminal book [12], R. W. Sharpe asked the following question: why differential geometry is the geometry of a principal bundle endowed with a connection? S. S. Chern tried to answer the question in the preface to Sharpe's book. It is unarguably true that most of the geometric objects and structures can be formulated in the language of principal bundles and their associated vector bundles (cf. [3, 6–10, 14]). Furthermore, principal bundles, hence vector bundles, are the backbone of modern mathematical physics, especially gauge theory which serves to formulate the most fundamental physical theories (cf. [1, 2, 4, 5, 11]).

Given a principal G-bundle (P, π, M) and a G-module V, there is the associated vector bundle (E, π_E, M) with standard fiber V. Assume P is endowed with a connection Γ with connection form ω. The connection Γ induces a connection on the vector bundle E in the following way: first, Γ gives rise to a horizontal distribution on E, inducing parallel transport, and then a covariant derivative on E, see [7, 8] for detailed expositions of connection theory. More to the point, in a modern approach to connection theory, the authors of [8] construct the induced connection on associated vector bundles using the induced connection (as a vertical subbundle-valued 1-form), next they derive the connector (i.e. the connection map) and finally the covariant derivative as a directional derivative. Such schemes define the covariant derivative on an associated vector bundle from a principal connection via either the notion of parallel transport or the connector (cf. [7, 8]). It would be interesting to find a way to define covariant derivative on E directly from the connection.

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For our purpose, we shall use the bijective correspondence between sections of the associated bundle and G-equivariant V-valued functions on P (cf. [7, 13]), to realize the covariant derivative of sections of associated bundles as the directional derivative of the associated G-equivariant maps. It is worth mentioning that the bijection above is extensively used in literature both in differential geometry and physics. For example, this correspondence is essential for Wood's theory of harmonic sections (cf. [15]). As an application, we prove a formula that relates the curvature of the induced connection, as a covariant derivative, with the curvature 2-form of the connection on the principal bundle.

We believe that this approach is fruitful for differential geometric uses, and also for some special situations in gauge theory, which uses an approach by connection matrices.

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1. Preliminaries

Given a principal bundle (P, π, M) with structure group G, denote by $R : P \times G \longrightarrow P$ the right (principal) action of G on P, and by $R_p = R(p, .) : G \longrightarrow P$ the map defined by $R_p(g) = R(p, g)$, for $p \in P$. In what follows, we use a 'dot' for actions and it should be understood from the context which one we mean.

Let V be a finite dimensional vector space and $\varrho : G \longrightarrow GL(V)$ be a linear representation of G on V. Then, G acts on the product $P \times V$ as $(p, v).g = (p.g, g^{-1}.v)$, for all $p \in P$ and $v \in V$. Denote by $E := P \times_\varrho G$ the quotient space $(P \times V)/G$, whose elements are denoted by $[p, v]$, for $(p, v) \in P \times V$. The projection π induces a map $\pi_E : E \longrightarrow M$ given by $\pi_E([p, v]) = \pi(p)$. The triple (E, π_E, M) is a vector bundle with standard fiber V (cf. [7, 8]).

Sections of associated vector bundles may be realized as G-equivariant maps from the total space of the principal bundle into the standard fiber V. Indeed, let σ be a section of E, which can be expressed as $\sigma(\pi(p)) = [p, \hat{\sigma}(p)]$, for each $p \in P$, where $\hat{\sigma} : P \longrightarrow V$ is G-equivariant in the sense that $\hat{\sigma}(p.g) = g^{-1}.\hat{\sigma}(p)$, for all $p \in P$. The smoothness of $\hat{\sigma}$ follows from that of σ. The G-equivariant map $\hat{\sigma}$ could be defined differently. Indeed, every $p \in P$ induces a diffeomorphism $p : V \longrightarrow P_{\pi(p)}$, $p(v) = [p, v]$, called the *framing* (cf. [7]). Here $P_{\pi(p)}$ denotes the fiber of P over $\pi(p)$. So, we have $\hat{\sigma}(p) = p^{-1}(\sigma(\pi(p)))$, for all $p \in P$.

We denote the set of G-equivariant V-valued maps on P by $C^\infty_G(P, V)$, and by $\Gamma(E)$ the vector space of smooth sections of E. Hence we have a map $\Psi : \Gamma(E) \longrightarrow C^\infty_G(P, V)$, defined by $\Psi(\sigma) = \hat{\sigma}$.

Conversely, given a G-equivariant map $\varphi : P \longrightarrow V$, define a section $\sigma_\varphi(x) := [p, \varphi(p)]$, where p is any element in P with $\pi(p) = x$. It is not hard to see that σ_φ is a smooth section of E. Hence a map $\Phi : C^\infty_G(P, V) \longrightarrow \Gamma(E)$. Clearly, we have $\Psi \circ \Phi = Id_{C^\infty_G(P,V)}$ and $\Phi \circ \Psi = Id_{\Gamma(E)}$. Thus, Ψ is bijective and $\Psi^{-1} = \Phi$.

Besides the linear structure, the space $\Gamma(E)$ has a structure of $C^\infty(M)$-module. On the other hand, the pointwise addition and multiplication by real numbers induce a vector space structure on $C^\infty_G(P, V)$. Moreover, for every function $f \in C^\infty(M)$, denote by $f^v := f \circ \pi$ the *vertical lift* of f to P. For a G-equivariant map $\varphi : P \longrightarrow V$, define a multiplication by f as

$$(f.\varphi)(p) = f^v(p).\varphi(p),$$

for all $p \in P$. Clearly, $f.\varphi$ is again G-equivariant. It is a straightforward verification that $C^\infty_G(P, V)$ is a $C^\infty(M)$-module.

Let $\mathfrak{g}$ be the Lie algebra of the structure group G. Assume that P is endowed with a connection Γ with connection form $\omega \in \Omega^1(P, \mathfrak{g})$ and curvature form $\Omega \in \Omega^2(P, \mathfrak{g})$. The structure equation (cf. [7]) relates the connection form and the curvature form as follows

$$d\omega(X,Y) = \Omega(X,Y) - \frac{1}{2}[\omega(X),\omega(Y)], \tag{1.1}$$

for all $X, Y \in T_pP$ and $p \in P$.

At each point $p \in P$, denote by $\mathcal{V}_p$ the *vertical subspace*, that is the kernel $\ker(d_p\pi)$, and by $\mathcal{H}_p$ the horizontal subspace at p. Accordingly, denote by $\mathcal{V}$ and $\mathcal{H}$ the resulting subbundles, respectively. A vector (resp. vector field) is said to be *horizontal* (resp. *vertical*) if it lies in $\mathcal{H}$ (resp. $\mathcal{V}$). The decomposition

$$TP = \mathcal{H} \oplus \mathcal{V},$$

gives a decomposition of vectors and vector fields into horizontal and vertical parts.

Each $A \in \mathfrak{g}$ generates a vertical vector field on P defined by $A^*_p = d_eR_p(A)$, for each $p \in P$. The resulting vector field is called the *fundamental vector field* generated by A. It is well known that the map $\zeta_p : \mathfrak{g} \longrightarrow T_pP$ defined by $\zeta_p(A) = A^*_p$ is an isomorphism between $\mathfrak{g}$ and the vertical subspace $\mathcal{V}_p$. Further, the fundamental vector field A^* never vanishes unless $A = 0$ (cf. [7,8]).

On the other hand, for every $X \in \mathfrak{X}(M)$, there exists a unique horizontal vector field $X^h \in \mathfrak{X}(P)$ which is π-related to X, i.e. $\pi_* X^h = X \circ \pi$.

The connection Γ on P induces a connection on E. Indeed, at each $e = [p, v] \in E$, the element $v \in V$ defines a map $i_v : P \longrightarrow E$ given by $i_v(p) = [p, v]$, for $p \in P$. The horizontal subspace $\mathcal{H}_p$ is mapped to $H_e = d_p i_v(\mathcal{H}_p)$. This defines a horizontal distribution on E for which H_e is independent of the choice of the representative of e and each H_e is complementary to the vertical subspace $V_e = \ker(d_e\pi_E)$. This gives a connection on the vector bundle E called the *induced connection* on E (cf. [7]).

In the context of vector bundles, for technical purposes, the covariant derivative is more suitable to work with. The induced connection gives rise to a covariant derivative as follows. First, the induced connection allows one to define parallel transport. Indeed, let $\gamma : [0,1] \longrightarrow M$ be a smooth curve in M and let $\gamma^* : [0,1] \longrightarrow P$ be any horizontal lift of γ to P with respect to Γ. This gives, for every $t_1, t_2 \in [0,1]$, the parallel transport along γ from $\pi_E^{-1}(\gamma(t_1))$ to $\pi_E^{-1}(\gamma(t_2))$, which is the isomorphism

$$\mathbb{P}_{\gamma,t_1,t_2} : \pi_E^{-1}(\gamma(t_1)) \longrightarrow \pi_E^{-1}(\gamma(t_2)), \quad \mathbb{P}_{\gamma,t_1,t_2}([\gamma^*(t_1), v]) = [\gamma^*(t_2), v],$$

for every $v \in V$.

Finally, the parallel transport induces the covariant derivative along the curve γ as follows

$$\nabla_{\dot\gamma(t)}\sigma = \lim_{h\to 0}\frac{1}{h}\Big(\mathbb{P}^{-1}_{\gamma,t,t+h}(\sigma(\gamma(t+h))) - \sigma(\gamma(t))\Big), \tag{1.2}$$

for every $\sigma \in \Gamma(E)$.

This induces a map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ which satisfies the axioms of a covariant derivative, see [7] for details. This way the induced connection gives rise to a covariant derivative which allows one to differentiate sections and hence to do differential geometry on associated vector bundles. As mentioned above, this definition passes necessarily through parallel transport. Further, this definition has no direct geometric interpretation with respect to the principal bundle. The main purpose of this note is to give a direct global formula for defining the covariant derivative, without passing via parallel transport.

2. Main results

We begin by exploring more algebraic properties of the maps Φ and Ψ. This will be crucial in forthcoming computations.

Lemma 2.1. *The map $\Phi : C^\infty_G(P, V) \longrightarrow \Gamma(E)$ (resp. $\Psi : \Gamma(E) \longrightarrow C^\infty_G(P, V)$) is an isomorphism of $C^\infty(M)$-modules.*

Proof. Additivity of Φ is clear. To prove the $C^\infty(M)$-linearity, let $f \in C^\infty(M)$ and $\varphi \in C^\infty_G(P, V)$. Then, for $x \in M$ and $p \in P$ with $\pi(p) = x$, we have

$$\begin{aligned}\Phi(f.\varphi)(x) &= [p, (f\varphi)(p)]\\ &= [p, f^v(p).\varphi(p)]\\ &= f^v(p)[p, \varphi(p)] = f^v(p).\Phi(\varphi)(p).\end{aligned}$$

Hence Φ is a bijective $C^\infty(M)$-module homomorphism, thus Φ is an isomorphism of $C^\infty(M)$-modules and $\Phi^{-1} = \Psi$. □

Now, we are in position to define a candidate for a covariant derivative using the $C^\infty(M)$-module isomorphisms Ψ and Φ. Precisely, for every $\sigma \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, set

$$\nabla_X\sigma := \Phi\big(X^h.\Psi(\sigma)\big), \tag{2.1}$$

where X^h is the horizontal lift of X to P w.r.t the connection Γ.

To make sense of this formula we need the following:

Lemma 2.2. *Let $X \in \mathfrak{X}(M)$ and $\varphi \in C^\infty_G(P, V)$, then the map $X^h.\varphi : P \longrightarrow V$ is G-equivariant.*

Proof. For every $g \in G$, we have

$$\begin{aligned}(X^h.\varphi) \circ R_g &= (X^h \circ R_g).\varphi\\ &= \Big((R_g)_* X^h\Big).\varphi\\ &= X^h.(\varphi \circ R_g)\\ &= X^h.(g^{-1}.\varphi)\\ &= g^{-1}.(X^h.\varphi).\end{aligned}$$

The second equality uses the right invariance of X^h and the last equation follows from the linearity of the map $g^{-1} : V \longrightarrow V$. □

Hence ∇ is well defined. Further, we have

Proposition 2.3. *The map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ is a covariant derivative.*

Proof. Since Ψ and Φ are $C^\infty(M)$-module isomorphisms, the map ∇ is $\mathbb{R}$-bilinear. It remains to prove the $C^\infty(M)$-linearity in X and the Leibnitz rule in σ. Let $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$. We have

$$\begin{aligned}\nabla_{fX}\sigma &= \Phi\big((fX)^h\Psi(\sigma)\big)\\ &= \Phi\big(f^vX^h.\Psi(\sigma)\big)\\ &= f\Phi\big(X^h.\Psi(\sigma)\big)\\ &= f\nabla_X\sigma.\end{aligned}$$

On the other hand, we have $\Psi(f\sigma) = f^v\Psi(\sigma)$, then

$$\begin{aligned}X^h.\Psi(f\sigma) &= X^h.(f^v.\Psi(\sigma))\\ &= (X^h.f^v)\Psi(\sigma) + f^vX^h.\Psi(\sigma)\\ &= (Xf)^v\Psi(\sigma) + f^vX^h.\Psi(\sigma)\end{aligned}$$

Applying Φ to both sides, we get

$$\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma.$$

□

The covariant derivative defined by (2.1) is exactly the covariant derivative coming from the induced connection. In fact, a similar formula was used as an intermediary lemma to prove that the map given by formula (1.2) is actually a covariant derivative, see [7,13] for details. One of the aims here is to show that the global formula (2.1) can be taken as the defining formula.

The global formula (2.1) allows the interpretation of covariant derivative of sections as the directional derivative of G-equivariant V-valued maps on P in the horizontal directions. In particular, this formula can be used to give the equations of parallelism of local sections. Precisely, given a coordinate system $(U, x^1, ..., x^n)$ on the base manifold ($\dim(M) = n$), the isomorphisms Ψ and Φ induce $C^\infty(U)$-module isomorphisms between $\Gamma(U, E)$ and $C^\infty_G(\pi^{-1}(U), V)$, which we denote also by Ψ and Φ, respectively. For the sake of simplicity, assume the standard fiber is taken to be $\mathbb{R}^k$. A local section $\sigma \in \Gamma(U, E)$ is parallel if and only if $X^h.\Psi(\sigma) = 0$, for every $X \in \mathfrak{X}(U)$. Equivalently, for any local coordinate system $\{x^i\}$, the section σ is parallel if and only if $(\frac{\partial}{\partial x^i})^h.\Psi(\sigma) = 0$, for all $i = 1, ..., n$.

Given a basis $\{e_p : p = 1, .., l\}$ for $\mathfrak{g}$, with $\dim(\mathfrak{g}) = l$, $\{\frac{\partial}{\partial x^i}, e^*_p\}_{i,p}$ is a local frame for TP. Denote by $\{x^i, y^p\}$ the local coordinate associated to the latter local frame. Hence the connection form is expressed on $\pi^{-1}(U)$ as

$$\omega = \omega_{ir} dx^i \otimes e^*_r + \omega_{rq} dy^r \otimes e^*_q, \tag{2.2}$$

where $\{\omega_{ir}, \omega_{rq}\}$ are smooth function on $\pi^{-1}(U)$. Thus, we have

$$\omega(\frac{\partial}{\partial x^i}) = \omega_{iq} e_q.$$

Hence, the horizontal lift takes the expression

$$(\frac{\partial}{\partial x^i})^h = \frac{\partial}{\partial x^i} - \omega_{iq} e^*_q. \tag{2.3}$$

If $\Psi(\sigma) = (\varphi^1, ..., \varphi^k) : \pi^{-1}(U) \longrightarrow \mathbb{R}^k$ is a G-equivariant map, then σ is parallel if and only if the φ^j's satsfies the following system of PDE's

$$\frac{\partial \varphi^j}{\partial x^i} - \omega_{iq} e^*_q.\varphi^j = 0, \quad i = 1, ..., n, j = 1, ..., k, \tag{2.4}$$

where

$$(e^*_q.\varphi^j)(p) = \frac{d}{dt}_{|t=0} \varphi^j(p.exp(te_q)).$$

In contrast with the classical second order ODE's that define parallelism on a curve, the system of PDE's (2.4) describes parallelsim of a local section on an open set.

An important application of the global formula (2.1) consists in establishing a direct link between the curvature of the covariant derivative ∇ and curvature 2-form Ω.

Given $X, Y \in \mathfrak{X}(M)$, the horizontal part of $[X^h, Y^h]$ is $[X, Y]^h$. Hence by the structure equation, we have

$$[X^h, Y^h] = [X, Y]^h - \Omega(X^h, Y^h)^*. \tag{2.5}$$

Theorem 2.4. *For $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$, we have*

$$R(X, Y)\sigma = -\Phi(\Omega(X^h, Y^h)^*.\Psi(\sigma)).$$

In particular, ∇ is flat if Γ is flat.

Proof. Using formula (2.1), we have

$$\begin{aligned} R(X,Y)\sigma &= \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma \\ &= \nabla_X(\Phi(Y^h.\Psi(\sigma))) - \nabla_Y(\Phi(X^h.\Psi(\sigma)) \\ &\quad - \nabla_{[X,Y]}\sigma \\ &= \Phi(X^hY^h.\Psi(\sigma)) - \Phi(Y^hX^h.\Psi(\sigma)) \\ &\quad - \Phi([X^h,Y^h].\Psi(\sigma)) \\ &= \Phi([X^h,Y^h].\Psi(\sigma)) - \Phi([X,Y]^h.\Psi(\sigma)) \\ &= \Phi(([X^h,Y^h] - [X,Y]^h).\Psi(\sigma)) \\ &= \Phi(\mathcal{V}([X^h,Y^h]).\Psi(\sigma)) \\ &= -\Phi(\Omega(X^h,Y^h)^*.\Psi(\sigma)), \end{aligned}$$

where $\mathcal{V}([X^h,Y^h])$ denotes the vertical part of $[X^h,Y^h]$. The last equality follows from equation (2.5). □

3. Some examples

To illustrate the relevance of Theorem 1, let us consider the following examples:

(1) **The tangent bundle:** assume M is an n-dimensional manifold. Let (LM,π,M) be the principal $GL_n(\mathbb{R})$-bundle of linear frames. The tangent bundle (TM,π_M,M) is the associated bundle to LM for the canonical representation of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$. In this case a similar formula to that of Theorem 1 was already given in [7].

(2) **Klein geometries:** Let G be a Lie group and H be a closed subgroup of G, then $(G,\pi,G/H)$ possesses a structure of principal H-bundle. The Maurer-Cartan form is the connection form of a flat connection on this principal bundle. In fact, it is the flat model of Cartan geometry, see [12] for details on Klein and Cartan geometries. So, by Theorem 1, every associated bundle to $(G,\pi,G/H)$, when endowed with the induced connection, is flat.

(3) **The case of Abelian structure groups:** If (P,π,M) is a principal bundle with Abelian structure group G, then the adjoint representation is trivial. In this case, a connection form ω on P and its curvature form Ω are invariant $\mathfrak{g}$-valued forms. In particular, there exists $F_\omega \in \Omega^2(M,\mathfrak{g})$ such that $\Omega = \pi^*F_\omega$ (see [7] for details). Consequently, if E is an associated bundle to P, then the formula of Theorem 1 becomes

$$R(X,Y)\sigma = -\Phi(F_\omega(X,Y)^*.\Psi(\sigma)),$$

for all $X,Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(E)$.

A classical case of extreme importance in Gauge theory is $G = U(1)$, which describes electromagnetism (cf. [4]). In this case the form F_ω is nothing but the Maxwell 2-form.

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