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IMPLEMENTATION OF COMPUTATION FORMULAS FOR CERTAIN CLASSES OF APOSTOL-TYPE POLYNOMIALS AND SOME PROPERTIES ASSOCIATED WITH THESE POLYNOMIALS

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Abstract. The main purpose of this paper is to present various identities and computation formulas for certain classes of Apostol-type numbers and polynomials. The results of this paper contain not only the λ-Apostol-Daehee numbers and polynomials, but also Simsek numbers and polynomials, the Stirling numbers of the first kind, the Daehee numbers, and the Chu-Vandermonde identity. Furthermore, we derive an infinite series representation for the λ-Apostol-Daehee polynomials. By using functional equations containing the generating functions for the Cauchy numbers and the Riemann integrals of the generating functions for the λ-Apostol-Daehee numbers and polynomials, we also derive some identities and formulas for these numbers and polynomials. Moreover, we give implementation of a computation formula for the λ-Apostol-Daehee polynomials in Mathematica by Wolfram language. By this implementation, we also present some plots of these polynomials in order to investigate their behaviour in some randomly selected special cases of their parameters. Finally, we conclude the paper with some comments and observations on our results.

1. Introduction

Let $\mathbb{N}$ and $\mathbb{C}$ denote respectively the set of natural numbers and the set of complex numbers and let $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$. For $z \in \mathbb{C}$, here assuming that

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$\log z$ denotes the principal branch of the many-valued function with the imaginary part $\mathrm{Im}(\log z)$ constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi,$$

it is considered to be $\log e = 1$ throughout this paper. Furthermore, it is assumed that

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N} \end{cases}$$

and also

$$\binom{\alpha}{n} = \frac{(\alpha)_n}{n!}$$

where $\alpha \in \mathbb{C}$, $n \in \mathbb{N}_0$ and $(\alpha)_n = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)$ with $(\alpha)_0 = 1$.

In recent years, many studies on Apostol-type numbers and polynomials have been carried out by some researchers (see [4]-[34]). Among others, in this paper, we are mainly dealt with the λ-Apostol-Daehee numbers $\mathfrak{D}_n(\lambda)$ and polynomials $\mathfrak{D}_n(x;\lambda)$ introduced and investigated by Simsek [27, 28] respectively as in the following generating functions:

$$G_{\mathfrak{D}}(t;\lambda) := \frac{\log\lambda + \log(1+\lambda t)}{\lambda(1+\lambda t) - 1} = \sum_{n=0}^{\infty} \mathfrak{D}_n(\lambda)\frac{t^n}{n!} \tag{1}$$

and

$$G_{\mathfrak{D}}(t,x;\lambda) := G_{\mathfrak{D}}(t;\lambda)(1+\lambda t)^x = \sum_{n=0}^{\infty} \mathfrak{D}_n(x;\lambda)\frac{t^n}{n!} \tag{2}$$

where λ is an arbitrary (real or complex) parameter not equal to 1 and satisfying $|\lambda t| < 1$ and $|\lambda^2 t| < |\lambda - 1|$ (*cf.* [27, 28]; and also see [32]).

First few values of the numbers $\mathfrak{D}_n(\lambda)$ are given as follows:

$$\begin{aligned}
\mathfrak{D}_0(\lambda) &= \frac{\log\lambda}{\lambda-1}, \\
\mathfrak{D}_1(\lambda) &= -\frac{\lambda^2\log\lambda}{(\lambda-1)^2} + \frac{\lambda}{\lambda-1}, \\
\mathfrak{D}_2(\lambda) &= \frac{2\lambda^4\log\lambda}{(\lambda-1)^3} + \frac{\lambda^2(1-3\lambda)}{(\lambda-1)^2}, \\
\mathfrak{D}_3(\lambda) &= -\frac{6\lambda^6\log\lambda}{(\lambda-1)^4} + \frac{\lambda^3\left(11\lambda^2 - 7\lambda + 2\right)}{(\lambda-1)^3},
\end{aligned}$$

and so on (*cf.* [14, 27, 28, 32]).

Another family of Apostol-type numbers and polynomials is the family of the numbers $Y_n(\lambda)$ (so-called Simsek numbers) and the polynomials $Y_n(x;\lambda)$ (so-called

Simsek polynomials) defined respectively by the following generating functions:

$$F(t;\lambda) := \frac{2}{\lambda(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!} \tag{3}$$

and

$$F(t,x;\lambda) := F(t;\lambda)(1+\lambda t)^x = \sum_{n=0}^{\infty} Y_n(x;\lambda)\frac{t^n}{n!} \tag{4}$$

where λ is an arbitrary (real or complex) parameter not equal to 1 and satisfying $|\lambda t| < 1$ and $|\lambda^2 t| < |\lambda - 1|$ (*cf.* [29]).

For $n \in \mathbb{N}_0$, the numbers $Y_n(\lambda)$ are computed by the following explicit formula:

$$Y_n(\lambda) = 2(-1)^n \frac{n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n, \tag{5}$$

by which, one may easily compute first few values of the numbers $Y_n(\lambda)$ as below:

$$Y_0(\lambda) = \frac{2}{\lambda-1}, \quad Y_1(\lambda) = -\frac{2\lambda^2}{(\lambda-1)^2}, \quad Y_2(\lambda) = \frac{4\lambda^4}{(\lambda-1)^3},$$
$$Y_3(\lambda) = -\frac{12\lambda^6}{(\lambda-1)^4}, \quad Y_4(\lambda) = \frac{48\lambda^8}{(\lambda-1)^5},$$

and so on (*cf.* [29]; and also see [34]).

Observe that the combination of (3) with (4) yields the relation between the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ given, for $n \in \mathbb{N}_0$, by

$$Y_n(x;\lambda) = \sum_{j=0}^{n} \binom{n}{j} Y_j(\lambda)\lambda^{n-j}(x)_{n-j}, \tag{6}$$

by which, one may easily compute first few values of the polynomials $Y_n(x;\lambda)$ as below:

$$\begin{aligned}
Y_0(x;\lambda) &= \frac{2}{\lambda-1}, \\
Y_1(x;\lambda) &= \frac{2\lambda}{\lambda-1}x - \frac{2\lambda^2}{(\lambda-1)^2}, \\
Y_2(x;\lambda) &= \frac{2\lambda^2}{\lambda-1}x^2 - \frac{6\lambda^3-2\lambda^2}{(\lambda-1)^2}x + \frac{4\lambda^4}{(\lambda-1)^3}, \\
Y_3(x;\lambda) &= \frac{2\lambda^3}{\lambda-1}x^3 - \frac{12\lambda^4-6\lambda^3}{(\lambda-1)^2}x^2 + \frac{22\lambda^5-14\lambda^4+4\lambda^3}{(\lambda-1)^3}x - \frac{12\lambda^6}{(\lambda-1)^4},
\end{aligned}$$

and so on (*cf.* [29]; and also see [34]).

Another family of Apostol-type numbers and polynomials is the family of the numbers $Y_n^{(-k)}(\lambda)$ (so-called negative higher-order Simsek numbers) and the polynomials $Q_n(x;\lambda,k)$ (so-called negative higher-order Simsek polynomials) defined respectively by the following generating functions:

$$\mathcal{G}_Y(t,k;\lambda) := 2^{-k}(\lambda(1+\lambda t)-1)^k = \sum_{n=0}^{\infty} Y_n^{(-k)}(\lambda)\frac{t^n}{n!} \tag{7}$$

and

$$\mathcal{G}_Q(t,x,k;\lambda) := \mathcal{G}_Y(t,k;\lambda)(1+\lambda t)^x = \sum_{n=0}^{\infty} Q_n(x;\lambda,k)\frac{t^n}{n!}, \tag{8}$$

(*cf.* [15]).

For $n \in \mathbb{N}_0$, the numbers $Y_n^{(-k)}(\lambda)$ are computed by the following explicit formula:

$$Y_n^{(-k)}(\lambda) = \begin{cases} 2^{-k}n!\binom{k}{n}\lambda^{2n}(\lambda-1)^{k-n} & \text{if} \quad n \le k \\ 0 & \text{if} \quad n > k \end{cases} \tag{9}$$

by which, one may easily compute the values of the numbers $Y_n^{(-k)}(\lambda)$ as below:

$$\begin{aligned}
Y_0^{(-k)}(\lambda) &= 2^{-k}(\lambda-1)^k, \\
Y_1^{(-k)}(\lambda) &= 2^{-k}\binom{k}{1}\lambda^2(\lambda-1)^{k-1}, \\
Y_2^{(-k)}(\lambda) &= 2^{-k}2!\binom{k}{2}\lambda^4(\lambda-1)^{k-2}, \\
&\vdots \\
Y_j^{(-k)}(\lambda) &= 2^{-k}j!\binom{k}{j}\lambda^{2j}(\lambda-1)^{k-j} \quad \textit{for} \quad j \le k, \\
&\vdots \\
Y_k^{(-k)}(\lambda) &= 2^{-k}k!\lambda^{2k},
\end{aligned}$$

(*cf.* [15]).

The combination of (7) with (8) yields the relation between the numbers $Y_n^{(-k)}(\lambda)$ and the polynomials $Q_n(x;\lambda,k)$ given, for $k,n \in \mathbb{N}_0$, by

$$Q_n(x;\lambda,k) = \sum_{j=0}^{n}\binom{n}{j}Y_j^{(-k)}(\lambda)\lambda^{n-j}(x)_{n-j}, \tag{10}$$

by which, one may easily compute first few values of the polynomials $Q_n(x;\lambda,k)$ as below:

$$\begin{aligned}
Q_0(x;\lambda,k) &= 2^{-k}(\lambda-1)^k, \\
Q_1(x;\lambda,k) &= 2^{-k}(\lambda-1)^k\lambda x + 2^{-k}k\lambda^2(\lambda-1)^{k-1}, \\
Q_2(x;\lambda,k) &= 2^{-k}(\lambda-1)^k\lambda^2x^2 + \left(-2^{-k}(\lambda-1)^k\lambda^2 + 2^{-k+1}k\lambda^3(\lambda-1)^{k-1}\right)x \\
&\quad +2^{-k}k(k-1)\lambda^4(\lambda-1)^{k-1},
\end{aligned}$$

and so on (*cf.* [15]).

Let $x \in [0,1]$. The Bernstein basis functions, $B_k^n(x)$, are defined as below:

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad (k = 0, 1, \ldots, n; \quad n \in \mathbb{N}_0) \tag{11}$$

and their generating functions are given by

$$\frac{(xt)^k e^{(1-x)t}}{k!} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}, \tag{12}$$

so that the Bernstein basis functions have relationships with a large number of concepts including the Bezier curves, the binomial distribution, the Poisson distribution, the Catalan numbers, and etc.; see, for details, [1, 10, 9, 16, 21, 23, 24, 25, 31] and also the references cited therein.

It is concluded with the help of (9) and (11) that there exists the following relation between the numbers $Y_n^{(-k)}(\lambda)$ and the Bernstein basis functions:

$$Y_n^{(-k)}(\lambda) = \frac{(-1)^{k-n} n!}{2^k} \lambda^n B_n^k(\lambda) \tag{13}$$

where $n, k \in \mathbb{N}_0$ and $\lambda \in [0,1]$ (*cf.* [15]).

Actually, the numbers $Y_n^{(-k)}(\lambda)$ have other relations than its relation to the Bernstein basis functions. Among others, the numbers $Y_n^{(-k)}(\lambda)$ have relationships with the Poisson–Charlier polynomials, the Bell polynomials (i.e., exponential polynomials) and other kinds of combinatorial numbers. To see the relations mentioned above, the interested readers may glance at the paper [15].

The Stirling numbers of the first kind, $S_1(n,k)$, are defined, for $n, k \in \mathbb{N}_0$, by the following recurrence relation:

$$S_1(n+1,k) = -nS_1(n,k) + S_1(n,k-1)$$

with the side conditions $S_1(0,0) = 1$, $S_1(0,k) = 0$ if $k > 0$, $S_1(n,0) = 0$ if $n > 0$, $S_1(n,k) = 0$ if $k > n$; and these numbers are also given by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \tag{14}$$

and

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!} \tag{15}$$

(*cf.* [2, 3, 5, 8, 18, 22]; and the references cited therein).

The Cauchy numbers (or the Bernoulli numbers of the second kind), $b_n(0)$, are defined by

$$b_n(0) = \int_0^1 (x)_n \, \mathrm{d}x \tag{16}$$

and

$$G_C(t) := \frac{t}{\log(t+1)} = \sum_{n=0}^{\infty} b_n(0) \frac{t^n}{n!} \tag{17}$$

(*cf.* [20, p. 116], [13], [17], [18]).

The combination of (14) with (16) yields the relation between the Cauchy numbers and the Stirling numbers of the first kind given by

$$b_n(0) = \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}, \tag{18}$$

(*cf.* [5, p. 294], [17, p. 1908], [20, p. 114]).

The Daehee numbers, D_n, are defined by the following generating function:

$$F_D(t) := \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{19}$$

and these numbers are computed, for $n \in \mathbb{N}_0$, by the following explicit formula:

$$D_n = \frac{(-1)^n n!}{n+1} \tag{20}$$

(*cf.* [7,11,27,28]).

The well-known Chu-Vandermonde identity is given by

$$\binom{x+y}{n} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} \tag{21}$$

(*cf.* [5,6,30]).

The outline of this paper may briefly given as follows:

In Section 2, we present various identities and computation formulas containing not only the λ-Apostol-Daehee numbers and polynomials, but also Simsek numbers and polynomials, the Stirling numbers of the first kind, the Daehee numbers and the Chu-Vandermonde identity. Besides, we derive an infinite series representation for the λ-Apostol-Daehee polynomials.

In Section 3, by using functional equations containing the generating functions for the Cauchy numbers and the Riemann integrals of the generating functions for the λ-Apostol-Daehee numbers and polynomials, we also derive some identities and formulas for these numbers and polynomials.

In Section 4, we give Mathematica implementation of a formula which computes the λ-Apostol-Daehee polynomials in terms of the Simsek polynomials. By this implementation, some plots of the λ-Apostol-Daehee polynomials are presented for some randomly selected special cases.

In Section 5, we conclude the paper with some comments and observations on our results.

2. Identities containing Apostol-type numbers and polynomials

In this section, by using the techniques of generating functions and their functional equations, we derive some identities involving not only the Chu-Vandermonde identity, but also some special numbers and polynomials such as the numbers $\mathfrak{D}_n(\lambda)$, the polynomials $\mathfrak{D}_n(x;\lambda)$, the numbers $Y_n(\lambda)$, the polynomials $Y_n(x;\lambda)$, the numbers $S_1(n,m)$ and the numbers D_n. In addition, we get computation formulas for not only the numbers $\mathfrak{D}_n(\lambda)$, but also the polynomials $\mathfrak{D}_n(x;\lambda)$. Moreover, we derive an infinite series representation for the polynomials $\mathfrak{D}_m(x;\lambda)$ in terms of the polynomials $Q_m(x;\lambda,n)$.

By (2), we have

$$(1+\lambda t)^x = \frac{\lambda(1+\lambda t)-1}{\log\lambda+\log(1+\lambda t)}\sum_{n=0}^{\infty}\mathfrak{D}_n(x;\lambda)\frac{t^n}{n!}$$

and

$$(1+\lambda t)^y = \frac{\lambda(1+\lambda t)-1}{\log\lambda+\log(1+\lambda t)}\sum_{n=0}^{\infty}\mathfrak{D}_n(y;\lambda)\frac{t^n}{n!}.$$

Multiplying the above two equations each other, we get

$$(1+\lambda t)^{x+y} = \left(\frac{\lambda(1+\lambda t)-1}{\log\lambda+\log(1+\lambda t)}\right)^2\sum_{n=0}^{\infty}\mathfrak{D}_n(x;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{D}_n(y;\lambda)\frac{t^n}{n!}.$$

For $|\lambda t|<1$, with the application of the Binomial theorem and the Cauchy product rule to the above equation, we get

$$\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^n t^n = \frac{(\lambda(1+\lambda t)-1)^2}{(\log\lambda)^2\left(1+\frac{\log(1+\lambda t)}{\log\lambda}\right)^2}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{D}_j(x;\lambda)\mathfrak{D}_{n-j}(y;\lambda)\frac{t^n}{n!}.$$

Then, assuming that $|\frac{\log(1+\lambda t)}{\log\lambda}|<1$ and using negative binomial series expansion in the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^n t^n &= \frac{(\lambda(1+\lambda t)-1)^2}{(\log\lambda)^2}\sum_{n=0}^{\infty}\binom{-2}{n}\frac{(\log(1+\lambda t))^n}{(\log\lambda)^n}\\ &\times\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{D}_j(x;\lambda)\mathfrak{D}_{n-j}(y;\lambda)\frac{t^n}{n!}.\end{aligned}$$

By combining (15) with the above equation, after some elementary calculations, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^n t^n &= \frac{(\lambda(1+\lambda t)-1)^2}{(\log\lambda)^2}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{-2}{n}\frac{n!S_1(m,n)}{(\log\lambda)^n}\frac{t^m}{m!}\\ &\times\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathfrak{D}_j(x;\lambda)\mathfrak{D}_{n-j}(y;\lambda)\frac{t^n}{n!}.\end{aligned}$$

By applying the Cauchy product rule to the above equation, we get

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{x+y}{m}\lambda^m t^m &= \frac{\left(\lambda^4 t^2+2\lambda^2(\lambda-1)t+(\lambda-1)^2\right)}{(\log\lambda)^2}\\ &\times\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}\sum_{n=0}^{k}\binom{-2}{n}\frac{n!S_1(k,n)}{(\log\lambda)^n}\\ &\times\sum_{j=0}^{m-k}\binom{m-k}{j}\mathfrak{D}_j(x;\lambda)\,\mathfrak{D}_{m-k-j}(y;\lambda)\frac{t^m}{m!}.\end{aligned}$$

After some elementary calculations and by comparing the coefficients of t^m on both sides of the final equation, we arrive at the following theorem:

Theorem 1. *Let $m\in\mathbb{N}\setminus\{1\}$ and $\lambda\neq 1$. Then we have*

$$\begin{aligned}m!\binom{x+y}{m} &= \frac{\lambda^{4-m}}{(\log\lambda)^2}m(m-1)A(m-2)\\ &+\frac{2\lambda^{2-m}(\lambda-1)}{(\log\lambda)^2}mA(m-1)\\ &+\frac{\lambda^{-m}(\lambda-1)^2}{(\log\lambda)^2}A(m),\end{aligned}\tag{22}$$

where

$$A(m)=\sum_{k=0}^{m}\sum_{n=0}^{k}\sum_{j=0}^{m-k}\binom{m}{k}\binom{-2}{n}\binom{m-k}{j}\frac{n!S_1(k,n)\,\mathfrak{D}_j(x;\lambda)\,\mathfrak{D}_{m-k-j}(y;\lambda)}{(\log\lambda)^n}.$$

Combining (22) with (21) yields the following corollary:

Corollary 2. *Let $m\in\mathbb{N}\setminus\{1\}$ and $\lambda\neq 1$. Then we have*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(x)_k(y)_{m-k} &= \frac{\lambda^{4-m}}{(\log\lambda)^2}m(m-1)A(m-2)\\ &+\frac{2\lambda^{2-m}(\lambda-1)}{(\log\lambda)^2}mA(m-1)\\ &+\frac{\lambda^{-m}(\lambda-1)^2}{(\log\lambda)^2}A(m)\end{aligned}\tag{23}$$

where $A(m)$ is as given in the Theorem 1.

By the combination of (2) with (4) and (19), we get the following functional equation:

$$G_{\mathfrak{D}}(t,x;\lambda)=\left(\frac{\log\lambda}{2}+\frac{\lambda tF_D(\lambda t)}{2}\right)F(t,x;\lambda).\tag{24}$$

which yields

$$\sum_{n=0}^{\infty}\mathfrak{D}_n\left(x;\lambda\right)\frac{t^n}{n!}=\left(\frac{\log\lambda}{2}+\frac{\lambda t}{2}\sum_{n=0}^{\infty}\lambda^n D_n\frac{t^n}{n!}\right)\sum_{n=0}^{\infty}Y_n\left(x;\lambda\right)\frac{t^n}{n!}.$$

By applying the Cauchy product rule to the above equation, after some elementary calculations, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{D}_n\left(x;\lambda\right)\frac{t^n}{n!} &= \frac{\log\lambda}{2}\sum_{n=0}^{\infty}Y_n\left(x;\lambda\right)\frac{t^n}{n!}\\ &\quad+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{j=0}^{n-1}n\binom{n-1}{j}\lambda^{n-j}D_{n-j-1}Y_j\left(x;\lambda\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 3. *Let $n\in\mathbb{N}$ and $\lambda\neq 1$. Then we have*

$$\mathfrak{D}_n\left(x;\lambda\right)=\frac{\log\lambda}{2}Y_n\left(x;\lambda\right)+\frac{1}{2}\sum_{j=0}^{n-1}n\binom{n-1}{j}\lambda^{n-j}D_{n-j-1}Y_j\left(x;\lambda\right). \tag{25}$$

Using (20) in (25), we get a relation between the λ-Apostol-Daehee polynomials and the Simsek polynomials by the following corollary:

Corollary 4. *Let $n\in\mathbb{N}$ and $\lambda\neq 1$. Then we have*

$$\mathfrak{D}_n\left(x;\lambda\right)=\frac{\log\lambda}{2}Y_n\left(x;\lambda\right)-\frac{n!}{2}\sum_{j=0}^{n-1}\left(-1\right)^{n-j}\frac{\lambda^{n-j}Y_j\left(x;\lambda\right)}{j!\left(n-j\right)}. \tag{26}$$

Substituting $x=0$ into (26), we also get a relation between the λ-Apostol-Daehee numbers and the Simsek numbers by the following corollary:

Corollary 5. *Let $n\in\mathbb{N}$ and $\lambda\neq 1$. Then we have*

$$\mathfrak{D}_n\left(\lambda\right)=\frac{\log\lambda}{2}Y_n\left(\lambda\right)+\frac{n!}{2}\sum_{j=0}^{n-1}\left(-1\right)^{n-j-1}\frac{\lambda^{n-j}Y_j\left(\lambda\right)}{j!\left(n-j\right)}. \tag{27}$$

Combining (5) with (27), we get a computation formula for the numbers $\mathfrak{D}_n\left(\lambda\right)$ by the following corollary:

Corollary 6. *Let $n\in\mathbb{N}$ and $\lambda\neq 1$. Then we have*

$$\mathfrak{D}_n\left(\lambda\right)=\frac{(-1)^n n!}{\lambda-1}\left(\left(\frac{\lambda^2}{\lambda-1}\right)^n\log\lambda-\lambda^n\sum_{j=0}^{n-1}\frac{1}{n-j}\left(\frac{\lambda}{\lambda-1}\right)^j\right). \tag{28}$$

Remark 7. *The computation formula* (28)*, obtained by reduction from the equation* (25)*, may also be obtained with the help of the application of the binomial theorem on the generating function for the numbers* $\mathfrak{D}_n(\lambda)$*. In the meanwhile, for another form of this formula, the interested readers may see the paper* [14, Theorem 8, p. 492] *in which other methods and generating function families used in order to achieve the aforementioned formula.*

By (28), we obtain a finite sum whose value is computed by the numbers $\mathfrak{D}_n(\lambda)$ as in the following corollary:

Corollary 8. *Let* $n \in \mathbb{N}$ *and* $\lambda \neq 1$*. Then we have*

$$\sum_{j=0}^{n-1} \frac{1}{n-j}\left(\frac{\lambda}{\lambda-1}\right)^{j} = (-1)^{n+1}\frac{(\lambda-1)\,\mathfrak{D}_n(\lambda)}{n!\lambda^n} + \left(\frac{\lambda}{\lambda-1}\right)^{n}\log\lambda. \tag{29}$$

By using the Taylor series expansion of the function $\log(1+(\lambda(1+\lambda t)-1))$, assuming that $|\lambda(1+\lambda t)-1|<1$, in the equation (2), and then by making some simplifications, we get

$$\sum_{m=0}^{\infty}\mathfrak{D}_m(x;\lambda)\frac{t^m}{m!} = (1+\lambda t)^x \sum_{n=0}^{\infty}(-1)^n\frac{(\lambda(1+\lambda t)-1)^n}{n+1}.$$

By combining (8) with the above equation, we get

$$\sum_{m=0}^{\infty}\mathfrak{D}_m(x;\lambda)\frac{t^m}{m!} = \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{n+1}\sum_{m=0}^{\infty}Q_m(x;\lambda,n)\frac{t^m}{m!}.$$

which yields

$$\sum_{m=0}^{\infty}\mathfrak{D}_m(x;\lambda)\frac{t^m}{m!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n\frac{2^nQ_m(x;\lambda,n)}{n+1}\frac{t^m}{m!}.$$

By assuming that $|\lambda-1|<1$ and comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields a relation between the numbers $\mathfrak{D}_m(x;\lambda)$ and the polynomials $Q_m(x;\lambda,n)$ as in the following theorem:

Theorem 9. *Let* $m \in \mathbb{N}_0$*. If* $|\lambda-1|<1$*, then we have the following infinite series representation for the polynomials* $\mathfrak{D}_m(x;\lambda)$*:*

$$\mathfrak{D}_m(x;\lambda) = \sum_{n=0}^{\infty}(-1)^n\frac{2^nQ_m(x;\lambda,n)}{n+1}. \tag{30}$$

3. Further identities derived from integral formulas and Cauchy numbers

In this section, by using functional equations involving the generating functions for the Cauchy numbers and the integrals of the generating functions for the numbers $\mathfrak{D}_n(\lambda)$ and the polynomials $\mathfrak{D}_n(x;\lambda)$, we derive some identities and formulas.

Integrating both-sides of the equation (2), with respect to the variable x, from 0 to 1, we get the following integral formula:

$$\int_0^1 G_{\mathfrak{D}}(t,x;\lambda)\,\mathrm{d}x = \int_0^1 \frac{\log\lambda+\log(1+\lambda t)}{\lambda(1+\lambda t)-1}(1+\lambda t)^x\,\mathrm{d}x \tag{31}$$

$$= \frac{\lambda t(\log\lambda+\log(1+\lambda t))}{(\lambda(1+\lambda t)-1)\log(1+\lambda t)}, \tag{32}$$

which, by (1) and (17), yields the following functional equation:

$$\int_0^1 G_{\mathfrak{D}}(t,x;\lambda)\,\mathrm{d}x = G_{\mathfrak{D}}(t;\lambda)\,G_C(\lambda t). \tag{33}$$

Combining the above equation with (1), (2) and (17) yields

$$\int_0^1 \sum_{n=0}^{\infty}\mathfrak{D}_n(x;\lambda)\frac{t^n}{n!}\mathrm{d}x = \sum_{n=0}^{\infty}\mathfrak{D}_n(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\lambda^n b_n(0)\frac{t^n}{n!}. \tag{34}$$

By applying the Cauchy product rule to the right-hand side of the above equation, after some elementary calculations, we get

$$\sum_{n=0}^{\infty}\int_0^1 \mathfrak{D}_n(x;\lambda)\,\mathrm{d}x\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\lambda^m b_m(0)\,\mathfrak{D}_{n-m}(\lambda)\frac{t^n}{n!}. \tag{35}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 10. *Let $n\in\mathbb{N}_0$ and $\lambda\neq 1$. Then we have*

$$\int_0^1 \mathfrak{D}_n(x;\lambda)\,\mathrm{d}x = \sum_{m=0}^{n}\binom{n}{m}\lambda^m b_m(0)\,\mathfrak{D}_{n-m}(\lambda). \tag{36}$$

Remark 11. *By using the generating function for the k-th order λ-Apostol-Daehee polynomials, Choi* [4, Theorem 5, p.1854] *gave the following integral formula:*

$$\int_\alpha^{\alpha+1} \mathfrak{D}_n^{(k)}(x;\lambda)\,\mathrm{d}x = \sum_{m=0}^{n} m!\binom{n}{m}\lambda^m\mathfrak{D}_{n-m}^{(k)}(\alpha;\lambda)\,p_m.$$

If we substitute $k=1$ and $\alpha=0$ into the above formula, we get

$$\int_0^1 \mathfrak{D}_n(x;\lambda)\,\mathrm{d}x = \sum_{m=0}^{n} m!\binom{n}{m}\lambda^m\mathfrak{D}_{n-m}(\lambda)\,p_m.$$

When we compare the above formula with the equation (36)*, it is shown that the numbers* $m!p_m$ *considered in the formula above actually correspond to the Cauchy numbers* $b_m(0)$ *which is obtained by the techniques of generating functions and their functional equations. Thus, we conclude that Choi* [4] *modified the numbers* $b_m(0)$ *as follows:*

$$m!p_m = b_m(0)$$

in order to obtain an integral formula for the higher-order λ*-Apostol-Daehee polynomials.*

Integrating both-sides of the equation (2), with respect to the variable x, from 0 to z, we get the following integral formula:

$$\int_0^z G_{\mathfrak{D}}(t,x;\lambda)\,\mathrm{d}x = \int_0^z \frac{\log\lambda+\log(1+\lambda t)}{\lambda(1+\lambda t)-1}(1+\lambda t)^x\,\mathrm{d}x \tag{37}$$

$$= \frac{\left((1+\lambda t)^z-1\right)\left(\log\lambda+\log(1+\lambda t)\right)}{\left(\lambda(1+\lambda t)-1\right)\log(1+\lambda t)} \tag{38}$$

which, by (1) and (2), yields the following functional equation:

$$\int_0^z G_{\mathfrak{D}}(t,x;\lambda)\,\mathrm{d}x = \frac{\left(G_{\mathfrak{D}}(t,z;\lambda)-G_{\mathfrak{D}}(t;\lambda)\right)G_C(\lambda t)}{\lambda t}. \tag{39}$$

Combining the above equation with (1), (2) and (17) yields

$$\int_0^z \sum_{n=0}^{\infty}\mathfrak{D}_n(x;\lambda)\frac{t^n}{n!}\mathrm{d}x = \frac{1}{\lambda t}\left(\sum_{n=0}^{\infty}\left(\mathfrak{D}_n(z;\lambda)-\mathfrak{D}_n(\lambda)\right)\frac{t^n}{n!}\right)\sum_{n=0}^{\infty}\lambda^n b_n(0)\frac{t^n}{n!}.$$

By applying the Cauchy product rule to the right-hand side of the above equation, after some elementary calculations, we get

$$\int_0^z \sum_{n=0}^{\infty}\mathfrak{D}_n(x;\lambda)\frac{t^n}{n!}\mathrm{d}x = \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\lambda^{m-1}b_m(0) \times\left(\mathfrak{D}_{n+1-m}(z;\lambda)-\mathfrak{D}_{n+1-m}(\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 12. *Let* $n\in\mathbb{N}_0$ *and* $\lambda\neq 1$*. Then we have*

$$\int_0^z \mathfrak{D}_n(x;\lambda)\,\mathrm{d}x = \frac{1}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\lambda^{m-1}b_m(0)\left(\mathfrak{D}_{n+1-m}(z;\lambda)-\mathfrak{D}_{n+1-m}(\lambda)\right). \tag{40}$$

4. Implementation of computation formulas involving λ-Apostol-Daehee polynomials

In this section, by implementing some of our results with the aid of the Wolfram programming language in Mathematica [35], we compute a few values of the λ-Apostol-Daehee polynomials. In addition, we also give some illustrations involving two dimensional plots of the λ-Apostol-Daehee polynomials.

We first give Mathematica implementation of the equation (25) in Implementation 1 in which we utilized from the Implementation 2 and the Implementation 3 given by Simsek and Kucukoglu [33] in order to compute the rational functions $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$.

IMPLEMENTATION 1. The following Mathematica code (i.e. the procedure DPoly) returns the values of the polynomials $\mathfrak{D}_n(x;\lambda)$

```
DPoly[x_, \[Lambda]_, n_] := (Log[\[Lambda]]/2) * YPoly[x, \[Lambda], n]+
(Factorial[n]/2) * Sum[((-1)^(n - j - 1)) * ((((\[Lambda])^(n - j))
*YPoly[x, \[Lambda], j])/(Factorial[j] * (n - j))), {j, 0, n - 1}]
```

IMPLEMENTATION 2. The following Mathematica code (i.e. the procedure YPoly) returns the values of the polynomials $Y_n(x;\lambda)$ (*cf.* [33])

```
YPoly[x_, \[Lambda]_, n_] := Sum[Binomial[n, j] * (\[Lambda]^(n - j))*
FactorialPower[x, n - j, 1] * YNum[\[Lambda], j], {j, 0, n}]
```

IMPLEMENTATION 3. The following Mathematica code (i.e. the procedure YNum) returns the values of the rational functions $Y_n(\lambda)$ (*cf.* [33])

```
YNum[\[Lambda]_, n_] := 2 * ((-1)^n) * (Factorial[n]/(\[Lambda] - 1))
*(((\[Lambda]^2)/(\[Lambda] - 1))^n)
```

By using the Implementation 1 in Mathematica, the values of the λ-Apostol-Daehee polynomials are computed as follows:

$$\mathfrak{D}_0(x;\lambda) = \frac{\log\lambda}{\lambda-1},$$

$$\mathfrak{D}_1(x;\lambda) = \frac{\lambda}{\lambda-1} + \left(\frac{\lambda x}{\lambda-1} - \frac{\lambda^2}{(\lambda-1)^2}\right)\log\lambda,$$

$$\mathfrak{D}_2(x;\lambda) = -\frac{2\lambda^3}{(\lambda-1)^2} + \frac{\lambda^2(2x-1)}{\lambda-1} + \left(\frac{2\lambda^4}{(\lambda-1)^3} - \frac{2\lambda^3 x}{(\lambda-1)^2} + \frac{\lambda^2 x(x-1)}{\lambda-1}\right)\log\lambda,$$

and so on.

By using the Implementation 1 and the `Plot` command in Mathematica, we also give some two dimensional plots of the polynomials $\mathfrak{D}_n(x;\lambda)$ in Figure 1-2. The curves, provided in Figure 1-2, illustrate the behaviour of the polynomials $\mathfrak{D}_n(x;\lambda)$ in some randomly selected special cases.

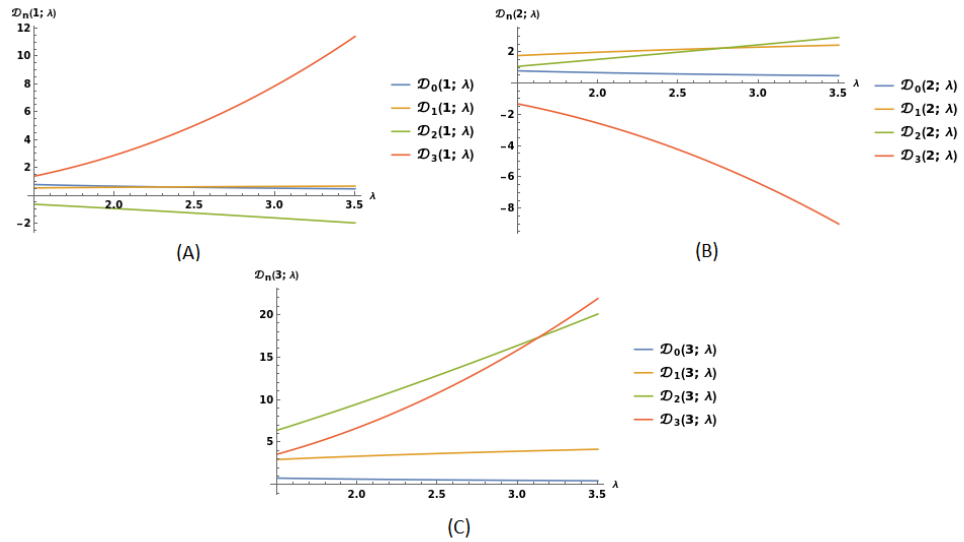

FIGURE 1. Plots of the polynomials $\mathfrak{D}_n(x;\lambda)$ for the randomly selected special cases when $\lambda \in \left[\frac{3}{2}, \frac{7}{2}\right]$ and $n \in \{0,1,2,3\}$ with **(A)** $x = 1$; **(B)** $x = 2$; **(C)** $x = 3$.

Firstly, in order to illustrate the effects of the parameter x on the graphs of the polynomials $\mathfrak{D}_n(x;\lambda)$ in the case when the other parameter λ belongs to a fixed randomly selected interval, we present Figure 1 which contains some plots of the polynomials $\mathfrak{D}_n(x;\lambda)$ for some randomly selected special cases when $\lambda \in \left[\frac{3}{2}, \frac{7}{2}\right]$ and $n \in \{0,1,2,3\}$ with $x \in \{1,2,3\}$.

Next, in order to illustrate the effects of the parameter λ on the graphs of the polynomials $\mathfrak{D}_n(x;\lambda)$ in the case when the other parameter x belongs to a fixed randomly selected interval, we present Figure 2 which contains some plots of the polynomials $\mathfrak{D}_n(x;\lambda)$ for the randomly selected special cases when $x \in [-2,2]$ and $n \in \{0,1,2,3\}$ with $\lambda \in \{\frac{3}{2}, e, \frac{7}{2}, e^2\}$

5. Conclusion

In this paper, we present various identities and computation formulas containing not only the λ-Apostol-Daehee numbers and polynomials, but also Simsek numbers and polynomials, the Stirling numbers of the first kind, the Daehee numbers, and also the Chu-Vandermonde identity. Furthermore, by using functional equations containing the generating functions for the Cauchy numbers and the integrals of

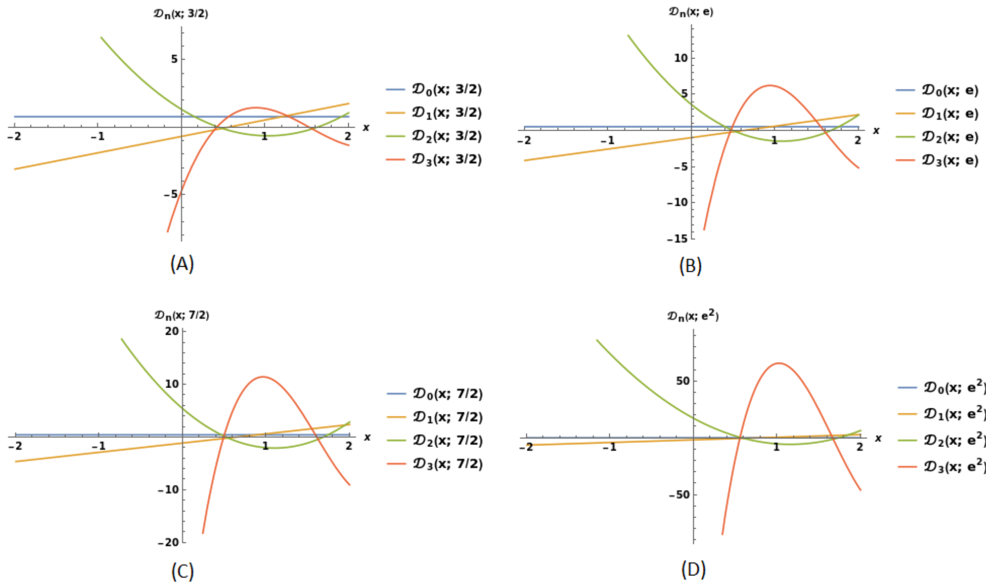

FIGURE 2. Plots of the polynomials $\mathfrak{D}_n(x;\lambda)$ for the randomly selected special cases when $x \in [-2,2]$ and $n \in \{0,1,2,3\}$ with **(A)** $\lambda = \frac{3}{2}$; **(B)** $\lambda = e$; **(C)** $\lambda = \frac{7}{2}$; **(D)** $\lambda = e^2$.

the generating functions for the λ-Apostol-Daehee numbers and polynomials, we also derive some identities and formulas for these numbers and polynomials. In addition, we give Mathematica implementation of a computation formula which computes the λ-Apostol-Daehee polynomials in terms of the polynomials $Y_n(x;\lambda)$. By the aid of the Mathematica implementation, we also give some plots which help the readers to analyze the behaviour of the λ-Apostol-Daehee polynomials in some randomly selected special cases of their parameters. As a conclusion, the results of this paper have the potential to affect many researchers conducting a research not only in computational mathematics, discrete mathematics and combinatorics, but also in other related fields.

For future studies, it is planned to investigate connections of the λ-Apostol-Daehee numbers with some special functions such as the Bernstein basis functions which possess many applications not only in approximation theory, but also in the construction of the Bezier curves widely used in computer-aided geometric design (*cf.* [1, 10, 9, 16, 21, 23, 24, 25, 31] and also cited references therein).

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