Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 2, 229–231.
https://doi.org/10.31197/atnaa.831128
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

Comment on strongly preirresolute topological vector spaces

Madhu Ram[a], Sayed K Elagan[b]

[a]*Department of Mathematics, University of Jammu, Jammu-180006, J& K, India.*

[b]*Department of Mathematics and Statistics, Faculty of Science, P.O. 888, Taif University, Saudi Arabia.*

Abstract

A subset A of a topological space X is said to be pre-open if $A \subseteq Int(Cl(A))$. Let $PO(X)$ denote the family of all pre-open sets in a given topological space X. In general, $PO(X)$ does not form a topology on X. Furthermore, in topological vector spaces, it is not always true that $PO(L)$ forms a topology on L when L is a topological vector space. In this note, we prove that the class of strongly preirresolute topological vector spaces is that subclass of topological vector spaces in which $PO(L)$ forms a topology and thereby we will observe that all results which are proven in [5] concerning strongly preirresolute topological vector spaces are obvious.

Keywords: Pre-open sets strongly preirresolute topological vector spaces.
2010 MSC: 57N17, 57N99.

1. Introduction and the main result

Let $(X, \Im)$ (or simply, X) be a topological space. A subset $A \subseteq X$ is called pre-open if $A \subseteq Int(Cl(A))$. The complement of a pre-open set is called pre-closed set. Let $PO(X)$ denote the collection of all pre-open subsets of X. It is well-known that in general, $PO(X)$ does not form a topology on X. Furthermore, consider a topological vector space $L = \mathbb{R}$, where $\mathbb{R}$ is endowed with the standard topology. Now,

let $A = \{x \in \mathbb{Q} \colon 0 < x < 1\}$ and $B = \{x \in \mathbb{R} \colon x \notin \mathbb{Q},\ 0 < x < 1\} \cup \{\frac{1}{2}\}$ where $\mathbb{Q}$ denotes the set of rational numbers.

Email addresses: `madhuram0502@gmail.com` (Madhu Ram), `sayed_khalil2000@yahoo.com` (Sayed K Elagan)

Received November 25 , 2020; Accepted: March 08, 2021; Online: April 4, 2021.

Obviously, both A and B are pre-open subsets of $\mathbb{R}$ but $A \cap B = \{\frac{1}{2}\}$ is not pre-open. Thus, we have seen that in topological vector spaces, $PO(L)$ need not form a topology on L when L is a topological vector space.

Definition 1.1. *Let X, Y be two topological spaces. A function $f\colon X \to Y$ is called $p-$continuous if the inverse image of any pre-open subset of Y is open in X.*

Definition 1.2. *A topological space X is called pre-T_2 [3] if for each pair of distinct points x and y in X, there exist disjoint pre-open sets U and V of X such that $x \in U$ and $y \in V$.*

Definition 1.3. *A subset A of a topological space X is called strongly compact [4] if every cover of A by pre-open sets in X has a finite subcover.*

In [5], Rajesh and Vijayabharathi (2013) introduced the notion of strongly preirresolute topological vector spaces and established several results in strongly preirresolute topological vector spaces.

A pair $(L, \Im)$ (or simply, L) is called a strongly preirresolute topological vector space if:

- L is a real vector space, and
- $\Im$ is a topology on L such that the vector space operations are $p-$continuous.

In fact, this definition can be extended to all complex vector spaces like topological vector spaces. Evidently, every strongly preirresolute topological vector space is a topological vector space but the converse is not true, in general because $(\mathbb{R}, \Im)$ is not strongly preirresolute topological vector space.

This note concerns the paper [5] by Rajesh and Vijayabharathi. We exhibit that all theorems in [5] are the particular cases of well-known results of topological vector spaces which follow directly from the following fact:

Theorem 1.1. *Let $(L, \Im)$ be a strongly preirresolute topological vector space. Then $PO(L)$ forms a topology on L.*

Proof. To prove this theorem, it is enough to show that every pre-open set of L is open. For, let A be any pre-open set of L and let $x \in A$ be any element.

Since the vector addition mapping of cartesian product $L \times L$ into L is $p-$continuous, there exist open sets U of L containing 0 and V of L containing x such that $U + V \subseteq A$. In particular, $0 + V \subseteq A$. This indicates that x is an interior point of A. Thus, A is open. Hence $PO(L) = \Im$. □

Corollary 1.1. *If $(L, \Im)$ is a strongly preirresolute topological vector space, then we have*

(1) A subset $A \subseteq L$ is strongly compact if and only if it is compact.

(2) $(L, \Im)$ is pre-T_2 space if and only if it is T_2 space.

Remark 1.1. *All results (for example, Theorem 3.9, Theorem 3.11, Theorem 3.13 and Theorem 3.18) in [5] follow directly by Corollary 1.1.1 together with corresponding well-known results in topological vector spaces (for example, see [2, Proposition* 2.2.3*, Corollary* 2.2.4*], [6, Theorem 1.10] and [7]).*

We now formulate an alternative definition of Hahn Banach Separation Theorem in strongly preirresolute topological vector spaces.

Theorem 1.2. *Suppose A, B are disjoint, non-empty convex sets in a strongly preirresolute topological vector space L.*

(a) If A is pre-open, then there is a linear continuous map $\varphi\colon L \to \mathbb{R}$, $\lambda \in \mathbb{R}$ s.t. $Sup\{Re\ \varphi(x)\colon x \in A\} < Re\ \varphi(y)$, for all $y \in B$.

(b) If B is strongly compact, A pre-closed, and L is locally convex, then there is a linear continuous map $\varphi\colon L \to \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\epsilon > 0$ s.t. $\forall x \in B,\ y \in A$, $Re\ \varphi(x) < \lambda < \lambda + \epsilon < Re\ \varphi(y)$.

Proof. Follows from Theorem 1.4 and [1, Theorem 5.7]. □

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