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Fixed Point Theorem for Single valued Mapping in Convex Partial Symmetric Spaces with an Application

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Abstract

In this paper, we proved a fixed point theorem for single-valued mapping in convex partial symmetric spaces. In doing so, we extended and generalised the theorem due to Asim et al. [5] by employing a convex structure for single-valued Kannan type contractions. We also provided an illustrative example to support the results and an application to integral equation.

Keywords: *Fixed point, partial symmetric spaces, convex partial symmetric spaces, single-valued mapping, integral equation.*
2010 Mathematics Subject Classification: *47H10, 54H25.*

1. Introduction and Preliminaries

In 1922, Banach [10] gave a contraction principle (BCP) for self-mappings in metric space. Nadler [25] extended the single-valued map of Banach contraction principle into multi-valued contraction mapping. Ćirić [14] gave a generalised Banach contraction principle and proved fixed point theorems for single-valued and multivalued quasi-contraction. Takahashi [31] introduced the notion of convexity in metric spaces and studied some fixed point theorems for non-expansive mappings in such spaces. Assad and Kirk [7] extended the results of Nadler to the subset of metrically convex metric space with Rothe's boundary condition to obtain the fixed point. A convex metric space is a generalised space. For example, every normed space and cone Banach space is a convex metric space and convex complete metric space. Subsequently, Karapinar [21], Moosaei [24], Nazam *et al.* [26], Asadi [3], Oussaeif [27], Cho and Bae [13], Beg and Abbas [11], and several others studied fixed point theorems in convex metric spaces.
In 1931, Wilson [34] initiated the study of the fixed point theorem in symmetric spaces by defining the metric as follows: Let X be a nonempty set. A symmetric d is a non-negative real function defined on $X \times X$ such that

(s_1) $d(x,y) = 0, \iff x = y$,
(s_2) $d(x,y) = d(y,x)$, (symmetric property).

We denote a nonempty set X equipped with symmetric d on X by (X,d) and call it symmetric space because the triangular inequality not included in this concept. Later, Jachymski *et al.* [20] proved the results using nonlinear contractions on semi-metric spaces. Hicks and Rhoades's [16] proved fixed point theory in symmetric spaces with applications to probabilistic spaces. Imdad and Javid [18] proved the common fixed point theorem in symmetric spaces employing a new implicit function and common property $(E.A)$. Aamri and El Moutawakil [1] proved common fixed points under contractive conditions in symmetric spaces. Finally, Imdad *et al.* [19] proved the coincidence and fixed points in symmetric spaces under strict contractions. In 2007, Turkoglu and Altun [32] proved a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation.
In 1994, as a part of the study of denotational semantics of data flow networks, Matthews [23] introduced a generalisation of the metric space called the partial metric space. He also extended Banach's fixed theorem from metric space to partial metric spaces. Since then, several researchers have worked in this direction; some of them are Bukatin *et al.* [12] gave a review on partial metric spaces. Shatanawi *et al.* [29] gave the results for a generalisation of some coupled fixed point results on partial metric spaces. Karapinar *et al.* [22] introduced the fixed point theorem on partial metric spaces involving rational expressions. Aydi *et al.* [8] gave a Common fixed points for multivalued generalized contractions on partial metric spaces Aydi *et al.* [9] extended the Banach concepts on ϕ-contraction type couplings in partial

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metric spaces. In 2018, Şahin *et al.* [30] proved fixed point theorem for neutrosophic triplet partial metric space. In 2019, Perveeni *et al.* [28] proved the fixed point theorem for the generalised contraction principle under relatively weaker contraction in partial metric spaces. Recently, Asim *et al.* [5] introduced the class of partial symmetric spaces by combining the concept of partial metric space due to Matthew [23] and the symmetric space concept due to Wilson [34] and proved some results. Moreover, they proved some related fixed point results for single-valued and multivalued mappings using this space. Since then, several researchers have been motivated to do their research in this direction. In 2019, Gesmundo *et al.* [15] gave the results on partially symmetric variants of Common's problem via simultaneous rank. In 2021, Wangwe and Kumar [33] proved the fixed point theorem for multivalued non-self mappings in partial symmetric spaces. Furthermore, in 2021, Asim and Imdad [4] proved a common fixed point result in partial symmetric space. Furthermore, Asim *et al.* [6] proved a multivalued result using Suzuki and Wardowski-type contraction mapping in partial symmetric space.
Combining the concepts of convex space and symmetric space, we introduce the class of convex partial symmetric space, wherein we prove the existence and uniqueness of fixed point results for certain types of contractions in convex partial symmetric spaces.
In this paper, we seek to generalise the theorem by Asim *et al.* [5] for single-valued non-self mappings on convex partial symmetric space. In this paper we will denote (X, p_s) as partial symmetric space.

2. Preliminaries

We will require the following preliminaries and results in developing our theorem.

Definition 2.1. [5] *Let X be a non-empty set. A mapping $p_s : X \times X \to \mathbb{R}_+$ is said to be partial symmetric if, for all $x, y, z \in X$, we have the following properties:*

(p_{s1}) : $x = y$ *if and only if* $p_s(x,y) = p_s(x,x) = p_s(y,y)$,
(p_{s2}) : $p_s(x,x) \le p_s(x,y)$,
(p_{s3}) : $p_s(x,y) = p_s(y,x)$.

Then the pair (X, p_s) is said to be partial symmetric space.

From p_{s1} and p_{s2} we have

$$p_s(x,y) = 0 \Rightarrow p_s(x,y) = p_s(x,x) = p_s(y,y) \Rightarrow x = y. \tag{2.1}$$

A partial symmetric space (X, p_s) reduces to a symmetric space if $p_s(x,x) = 0$, for all $x \in X$. Obviously, every symmetric space is partial symmetric space, but not conversely.
Let (X, p_s) be a partial symmetric space. Then, the p_s-open ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_{p_s}(x,\varepsilon) = \{y \in X : p_s(x,y) < p_s(x,x) + \varepsilon\}$,
Similarly, the p_s-closed ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_{p_s}[x,\varepsilon] = \{y \in X : p_s(x,y) \le (x,x) + \varepsilon\}$.
The family of p_s-open balls for all $x \in X$ *and* $\varepsilon > 0$, $U_{p_s} = \{B_{p_s}(x,\varepsilon) : x \in X, \varepsilon > 0\}$, forms basis of some topology τ_{p_s} on X.

Definition 2.2. [5] *Let (X, p_s) be a partial symmetric space. Then,*

(i) a sequence $\{x_n\}$ in (X, p_s) is said to be p_s-convergent to $x \in X$, with respect to τ_{p_s} if $p_s(x,x) = \lim_{n\to\infty} p_s(x_n, x)$.
(ii) a sequence $\{x_n\}$ in (X, p_s) is called a p_s-Cauchy sequence if only if $\lim_{n,m\to\infty} p_s(x_n, x_m)$ exists and is finite.
(iii) a partial symmetric space (X, p_s) is said to be p_s-complete if every p_s- Cauchy sequence $\{x_n\}$ in X is p_s convergent, with respect to τ_{p_s} to a point $x \in X$, such that

$$p_s(x,x) = \lim_{n\to\infty} p_s(x_n, x) = \lim_{n,m\to\infty} p_s(x_n, x_m).$$

Definition 2.3. [5] *Let (X, p_s) be a partial symmetric space. Then*

(A_1) $\lim_{n\to\infty} p_s(x_n,x) = p_s(x,x)$ *and* $\lim_{n\to\infty} p_s(x_n,y) = p_s(x,y)$ *imply that* $x = y$*, for a sequence* $\{x_n\}$, $x, y \in X$.

(A_2) *a partial symmetric p_s is said to be 1-continuous if $\lim_{n\to\infty} p_s(x_n,x) = p_s(x,x)$ implies that $\lim_{n\to\infty} p_s(x_n,y) = p_s(x,y)$, where $\{x_n\}$ is a sequence in X and $x, y \in X$.*

(A_3) *a partial symmetric p_s is said to be continuous if $\lim_{n\to\infty} p_s(x_n,x) = p_s(x,x)$ and $\lim_{n\to\infty} p_s(x_n,y) = p_s(x,y)$ imply that $\lim_{n\to\infty} p_s(x_n,y_n) = p_s(x,y)$ where $\{x_n\}$ and $\{y_n\}$ are sequences in X and $x, y \in X$.*

(A_4) $\lim_{n\to\infty} p_s(x_n,x) = p_s(x,x)$ *and* $\lim_{n\to\infty} p_s(x_n,y_n) = p_s(x,x)$ *imply* $\lim_{n\to\infty} p_s(y_n,x) = p_s(x,x)$*, for sequences* $(x_n), (y_n)$*, and* x *in* X.

(A_5) $\lim_{n\to\infty} p_s(x_n,y_n) = p_s(x,x)$ *and* $\lim_{n\to\infty} p_s(y_n,z_n) = p_s(x,x)$ *imply* $\lim_{n\to\infty} p_s(x_n,z_n) = p_s(x,x)$*, for sequences* $(x_n), (y_n), (z_n)$*, and* x *in* X.

Definition 2.4. [5] *Let (X, p_s) be a partial symmetric space. A mapping $f : X \to X$ is said to be a Kannan-Ćirić type k-contraction if, for all $x, y \in X$,*

$$p_s(fx, fy) \le k \max\Big\{p_s(x, fx), p_s(y, fy)\Big\} \tag{2.2}$$

where $k \in [0,1)$.

In 2019 Asim et al. [5], proved a Kannan-Ćirić type k-contractions in the setting of partial symmetric space.

Theorem 2.5. [5] *Let (X, p_s) be a complete partial symmetric space and $f : X \longrightarrow X$. Assume that the following conditions are satisfied:*

(i) f *is a Kannan-Ćirić type k-contraction,*
(ii) f *is continuous.*

Then T *has a unique fixed point* $x \in X$ *such that* $p_s(x,x) = 0$.

Now, we introduce some definitions on convex structure:

Definition 2.6. ([2], [31]) *Let* (X,d) *be a metric space and* $I = [0,1]$. *A mapping* $W : X \times X \times I \to X$ *is said to be a convex structure on* X *if for each* $(x,y,\lambda) \in X \times X \times I$ *and* $u \in X$, *such that*

$$d(u, W(x,y,\lambda)) \leq \lambda\, d(u,x) + (1-\lambda) d(u,y). \tag{2.3}$$

A metric space (X,d) *together with a convex structure* W *is called a convex metric space, which is denoted by* (X,d,W). *Obviously,* $W(x,x,\lambda) = x$, *and* $W(x,y,\lambda) = \lambda x + (1-\lambda) y$.

Now, we extend the concept of partial symmetric space to convex partial symmetric space structure:

Definition 2.7. *Let* (X, p_s) *be a partial symmetric space and* $I = [0,1]$. *A mapping* $W : X \times X \times I \to X$ *is said to be a convex structure on* X *if for each* $(x,y,\lambda) \in X \times X \times I$ *and* $u \in X$, *such that*

$$p_s(u, W(x,y,\lambda)) \leq \lambda\ p_s(u,x) + (1-\lambda)\ p_s(u,y). \tag{2.4}$$

A partial symmetric space (X, p_s) *together with a convex structure* W *is called a convex partial symmetric space, which is denoted by* (X, p_s, W). *Obviously,* $W(x,x,\lambda) = x$, *and* $W(x,y,\lambda) = \lambda x + (1-\lambda) y$.

Definition 2.8. *Let* (X, p_s, W) *be a convex partial symmetric space. A non empty subset* C *of* X *is said to be convex if* $W(x,y,\lambda) \in C$ *whenever* $(x,y,\lambda) \in C \times C \times I$.

Definition 2.9. *Let* (X, p_s, W) *be a convex partial symmetric space and* C *be a convex subset of* X. *A self-mapping* T *on* C *has a property* $(I = [0,1])$ *if* $T(W(x,y,\lambda)) = W(T(x), T(y), \lambda)$ *for each* $x, y \in C$ *and* $\lambda \in I$.

We now introduce the following lemma in context of convex partial symmetric space which will be useful in our main results:

Lemma 2.10. *Let* (X, p_s, W) *be a convex partial symmetric space, then the following are true:*

(i)

$$p_s(x,y) = \frac{1}{2}\Big(p_s(x, W(x,y,\lambda)) + p_s(y, W(x,y,\lambda))\Big) \tag{2.5}$$

for all $(x,y,\lambda) \in X \times X \times I$.

(ii)

$$p_s(x, W(x,y,\frac{1}{2})) = p_s(y, W(x,y,\frac{1}{2})) = p_s(x,y) \tag{2.6}$$

for all $x, y \in X$.

Proof. (i) From Equation 2.5 and Definition 2.6 for all $(x,y,\lambda) \in X \times X \times I$, we have

$$\begin{aligned} p_s(x, W(x,y,\lambda)) &\leq \lambda p_s(x,x) + (1-\lambda) p_s(x,y) \\ &\leq \lambda p_s(x,x) + p_s(x,y) - \lambda p_s(x,y) \\ &\leq \lambda p_s(x,y) + p_s(x,y) - \lambda p_s(x,y) \\ &= p_s(x,y),\ by\ (p_{s1}). \end{aligned}$$

Also,

$$\begin{aligned} p_s(y, W(x,y,\lambda)) &\leq \lambda p_s(y,x) + (1-\lambda) p_s(y,y) \\ &\leq \lambda p_s(y,x) + p_s(y,y) - \lambda p_s(y,y) \\ &\leq \lambda p_s(x,y) + p_s(x,y) - \lambda p_s(x,y) \\ &= p_s(x,y),\ by\ (p_{s1}). \end{aligned}$$

By considering the right hand side of (2.5) we get

$$\begin{aligned} &\leq \frac{1}{2}\Big(p_s(x, W(x,y,\lambda)) + p_s(y, W(x,y,\lambda))\Big) \\ &\leq \frac{1}{2}\Big(p_s(x,y) + p_s(x,y)\Big) \\ &= p_s(x,y). \end{aligned}$$

Therefore, $p_s(x,y) = \frac{1}{2}\Big(p_s(x, W(x,y,\lambda)) + p_s(y, W(x,y,\lambda))\Big)$ exists.

(ii) From Equation 2.6 for $(x,y) \in X$ and by using Definition 2.5, we have

$$\begin{aligned} p_s(x,W(x,y,\frac{1}{2})) &\leq \frac{1}{2}p_s(x,x)+(1-\frac{1}{2})p_s(x,y) \\ &\leq \frac{1}{2}p_s(x,x)+p_s(x,y)-\frac{1}{2}p_s(x,y) \\ &\leq \frac{1}{2}p_s(x,y)+p_s(x,y)-\frac{1}{2}p_s(x,y) \\ &= p_s(x,y),\ by\ (p_{s2}). \end{aligned}$$

Also,

$$\begin{aligned} p_s(y,W(x,y,\frac{1}{2})) &\leq \frac{1}{2}p_s(y,x)+(1-\frac{1}{2})p_s(y,y) \\ &\leq \frac{1}{2}p_s(y,x)+p_s(y,y)-\frac{1}{2}p_s(y,y) \\ &\leq \frac{1}{2}p_s(x,y)+p_s(x,y)-\frac{1}{2}p_s(x,y) \\ &= p_s(x,y),\ by\ (p_{s2}). \end{aligned}$$

Hence,

$$p_s(x,W(x,y,\frac{1}{2})) = p_s(y,W(x,y,\frac{1}{2})) = p_s(x,y),$$

for all $x,y \in C$. □

We give some examples on convex partial symmetric space as follows:

Example 2.11. *Let $X=\mathbb{R}$. Define a mapping $p_s : X\times X \to \mathbb{R}_+$ for all $x,y\in X$ and $p,q>1$, as follows:*

$$p_s(x,y) = \left(\frac{1}{2}|x-y|\right)^{\frac{1}{p}} + \left(\frac{1}{2}|x-y|\right)^{\frac{1}{q}}.$$

Then the pair (X,p_s) is a convex partial symmetric space.

Example 2.12. *Let $X=\mathbb{R}$. Define a mapping $p_s : X\times X \to \mathbb{R}_+$ for all $x,y\in X$ and $p,q>1$, as below:*

$$p_s(x,y) = (\max\{x,y\})^{\frac{1}{p}} + (\max\{x,y\})^{\frac{1}{q}}.$$

Then the pair (X,p_s) is a convex partial symmetric space.

3. Main Results

In this section, we proved a fixed point theorem for single-valued mappings using Kannan type Contraction in convex partial symmetric spaces.
To develop our main result, we start by extension of Definition 2.4 in convex partial symmetric spaces.

Definition 3.1. *Let (X,p_s,W) be a convex partial symmetric space and $T : C \to C$ be a mapping. Let T satisfy the following condition:*

$$p_s(Tx,Ty) \leq \lambda\{p_s(x,Tx)+p_s(y,Ty)\}, \tag{3.1}$$

for all $x,y\in C, 0<\lambda<1$, then T is called Kannan-contraction p_s-mapping of C into C.

Now, we give an extension of Theorem 2.5 for a self-mapping satisfying Kannan-type contraction under the setting of convex partial symmetric spaces.

Theorem 3.2. *Let (X,p_s,W) be a complete convex partial symmetric space and $T : C \longrightarrow C$ be a self mapping. Assume that T satisfies the following conditions:*

(i) T is a Kannan- type contraction mapping,
(ii) T is continuous.

Then, T has a fixed point provided $p_s(x,x)=0$.

Proof. Consider $x_0\in C$ and construct an iterative sequence $x_n \in C$ such that;

$$x_{n+1} = W(x_n,Tx_n,\lambda) \tag{3.2}$$

for $n=1,2,3,\ldots$
Applying Definition 2.6, Lemma 2.10 and let x_0 be arbitrary. Define $\{x_n\}$ in the following way

$$x_{n+1} = \frac{x_n+Tx_n}{2},$$

where $n=1,2,3,\ldots$
We know that,

$W(x,y,\lambda)=\lambda x+(1-\lambda)y.$

Taking $\lambda=\frac{1}{2}$ in (3.2), we get

$$\begin{aligned}
W(x,x,\frac{1}{2}) &= \frac{1}{2}x+(1-\frac{1}{2})x, \\
&= \frac{x+x}{2}, \\
&= x. \\
W(x,y,\frac{1}{2}) &= \frac{1}{2}x+(1-\frac{1}{2})y, \\
&= \frac{x+y}{2}.
\end{aligned}$$

Since C is convex, let $x=x_n, y=Tx_n=x_{n+1}$ in the above equation, we get

$$\begin{aligned}
x_n-Tx_n &= x_n-\frac{x_n+Tx_n}{2}, \\
&= \frac{1}{2}(2x_n-x_n-Tx_n), \\
&= \frac{1}{2}(x_n-x_{n+1}), \\
p_s(x_n,Tx_n) &= \frac{1}{2}p_s(x_n,x_{n+1}).
\end{aligned}$$

This can easily be seen by the use of Definition 2.1 and is equivalent to say

$p_s(x_n,x_{n+1})=0.$

For $x=x_{n-1}, y=x_n$ in Equation (3.1), we obtain

$$\begin{aligned}
p_s(Tx_{n-1},Tx_n) &\leq \lambda\ p_s(x_{n-1},Tx_{n-1})+\lambda\ p_s(x_n,Tx_n), \\
p_s(x_n,x_{n+1}) &\leq \lambda\ p_s(x_{n-1},x_n)+\lambda\ p_s(x_n,x_{n+1}), \\
(1-\lambda)\ p_s(x_{n+},x_{n+1}) &\leq \lambda\ p_s(x_{n-1},x_n), \\
p_s(x_n,x_{n+1}) &\leq \frac{\lambda}{1-\lambda}\ p_s(x_{n-1},x_n).
\end{aligned} \tag{3.3}$$

Next, we apply $x=x_n, y=Tx_n=x_{n+1}$ in Equation (3.1), we get

$$\begin{aligned}
p_s(Tx_n,T(Tx_n)) &\leq \lambda\ p_s(x_n,Tx_n)+\lambda\ p_s(Tx_n,T(Tx_n)), \\
p_s(x_{n+1},x_{n+2}) &\leq \lambda\ p_s(x_n,x_{n+1})+\lambda\ p_s(x_{n+1},x_{n+2}), \\
(1-\lambda)\ p_s(x_{n+1},x_{n+2}) &\leq \lambda\ p_s(x_n,x_{n+1}), \\
p_s(x_{n+1},x_{n+2}) &\leq \frac{\lambda}{1-\lambda}\ p_s(x_n,x_{n+1}).
\end{aligned} \tag{3.4}$$

Using (3.3) in (3.4) we have

$$p_s(x_{n+1},x_{n+2}) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 p_s(x_{n-1},x_n). \tag{3.5}$$

For all $n\in\mathbb{N}$, $\frac{\lambda}{1-\lambda}\in[0,\frac{1}{2}]$, it is a Cauchy sequence in C.
Let $\omega=\frac{\lambda}{1-\lambda}$, using (3.3) and (3.5) we obtain,

$$\begin{aligned}
p_s(x_n,x_{n+1}) &\leq \omega p_s(x_{n-1},x_n) \\
p_s(x_{n+1},x_{n+2}) &\leq \omega^2 p_s(x_{n-1},x_n)
\end{aligned}$$

Continuing in the similar way, we obtain

$p_s(x_n,x_{n+1})=\omega^n p_s(x_0,x_1), for\ all\ n\in\mathbb{N}.$

On taking limit of $n\to\infty$, we get

$$\lim_{n\to\infty} p_s(x_n,x_{n+1})=0. \tag{3.6}$$

Since $\{x_n\}$ is a p_s-Cauchy sequence. Using $x_n, x_{n+1} = x_m$ in (3.1), we have, for $n,m \in \mathbb{N}$,

$$p_s(x_n,x_m) = p_s(Tx_{n-1},Tx_{m-1}),$$

$$p_s(x_n,x_m) \leq \lambda\ p_s(x_{n-1},Tx_{n-1}) + \lambda\ p_s(x_{m-1},Tx_{m-1}), \tag{3.7}$$

For all $n,m \in \mathbb{N}$ and $n,m \to \infty$, we get

$$p_s(x_n,x_m) \quad \leq \quad \lambda\ p_s(x,x) + \lambda p_s(y,y).$$

From (3.6) and p_{s_1}, we obtain

$$p_s(x_n,x_m) = 0. \tag{3.8}$$

This shows that x_n is a p_s-Cauchy sequence. Since (X, p_s) is a complete space in C, then x is a fixed point of T in C. Therefore,

$$\begin{aligned} p_s(x_n,x) &= 0, \\ \lim_{n\to\infty} x_n &= x. \end{aligned}$$

If we have x as a fixed point of $T \in C$, then by continuity of T, $x_n \to x \Rightarrow Tx_n \to Tx$. Using Equation (3.1) and let $Tx = x$, $Ty = y$, we have

$$\begin{aligned} p_s(x,y) &\leq \lambda\ p_s(x,x) + \lambda\ p_s(y,y), \\ &\leq \lambda\ p_s(x,x), \\ &\leq \lambda\ p_s(y,y), \\ p_s(x,y) &\leq 0, \end{aligned}$$

which is a contradiction. Hence x is a unique fixed point of T.
Finally, we will show that $p_s(x,x) = 0$, from Equation (3.1) and p_{s_2}, we have

$$\begin{aligned} p_s(x,x) &\leq \lambda\ \{p_s(x,x) + p_s(y,y)\}, \\ p_s(x,x) &\leq \lambda p_s(x,x) + \lambda p_s(y,y), \\ &\leq \frac{\lambda}{1-\lambda} p_s(y,y), \\ p_s(x,x) &= 0. \end{aligned}$$

□

Furthermore, we will provide an illustrative example for Theorem 3.2.

Example 3.3. *Let $X = [0,4]$ be with usual convex partial symmetric endowed with the usual metric $p_s(x,y) = \|x-y\|$. Defined $T : X \longrightarrow [0,4]$ by*

$$Tx = \begin{cases} \frac{x}{3}, & 0 \leq x \leq 3, \\ \frac{x}{4}, & 3 < x \leq 4. \end{cases}$$

For $x,y \in [0,3]$, we obtain

$$\begin{aligned} p_s(Tx,Ty) = \frac{1}{3}|x-y| &= \frac{1}{3}|0-3| = 1, \\ p_s(x,Tx) = \lambda|x-\frac{x}{3}| &= \lambda|0-0| = 0, \\ p_s(y,Ty) = \lambda|y-\frac{y}{3}| &= \lambda|3-1| = 2\lambda, \end{aligned}$$

The above equality deduces to

$$\begin{aligned} p_s(Tx,Ty) &\leq \lambda[p_s(x,Tx) + p_s(y,Ty)], \\ 1 &\leq 2\lambda. \end{aligned} \tag{3.9}$$

For $x,y \in (3,4]$, we have

$$\begin{aligned} p_s(Tx,Ty) = \frac{1}{4}|x-y| &= \frac{1}{4}|4-4| = 0, \\ p_s(x,Tx) = \lambda|x-\frac{x}{4}| &= \lambda|4-1| = 3\lambda, \\ p_s(y,Ty) = \lambda|y-\frac{y}{4}| &= \lambda|4-4| = 3\lambda, \end{aligned}$$

By applying (3.9), the above equalities deduces to

$$0 \leq 6\lambda.$$

For $x \in [0,3]$ and $y \in (3,4]$, we have

$$\begin{aligned}
p_s(Tx,Ty) &\leq |\frac{x}{3} - \frac{y}{4}|, \\
&\leq |1-1|, \\
&= 0. \\
\lambda[p_s(x,Tx) + p_s(y,Ty)] &\leq \lambda|x-Tx| + \lambda|y-Ty|, \\
&\leq \lambda|x - \frac{x}{3}| + \lambda|y - \frac{x}{4}|, \\
&\leq \lambda|3-1| + \lambda|4-1|, \\
&= 5\lambda.
\end{aligned}$$

Using (3.9) together with the equalities above we get

$$0 \leq 5\lambda.$$

for all $x,y \in [0,4]$ Theorem 3.2 satisfy. Hence, 0 is a fixed point of T.

4. An application

In this section, we give an application of Theorem 3.2 to prove existence and uniqueness of the solution of an integral equation

$$x(t) = x(a) + \int_a^b K(t,s,x(s))ds, \; t \in [a,b], \tag{4.1}$$

where $K > 0$. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous functions and define a convex partial symmetric space p_s on X endowed with metric.

$$p_s(x,y) = \Big(\frac{1}{2}\sup_{t\in[a,b]} |x(t)-y(t)|\Big)^{\frac{1}{p}} + \Big(\frac{1}{2}\sup_{t\in[a,b]} |x(t)-y(t)|\Big)^{\frac{1}{q}}, \tag{4.2}$$

for all $x,y \in X$ and $p,q > 1$. Then the pair (X,p), is a convex partial symmetric space. Consider the Banach space $C[a,b]$ of continuous real valued functions with supremum norm by Hunter and Nachtergaele [17] defined by

$$\|x\| = \sup_{t\in[a,b]} |x|, \; t \in [a,b],$$

where u is a unique solution of (4.1). From Equation (4.1), u is the solution of $x^{'}(t) = K(t,x(s))$ satisfying initial condition $x(a) = x$ if and only if

$$x(b) = x(a) + \int_a^b K(t,s,x(s))ds, \; t \in [a,b]. \tag{4.3}$$

Definition 4.1. *Suppose that $K : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $[a,b]$ is an interval in $\mathbb{R}$. We say that $K(t,x(t))$ is a globally Lipschitz continuous function of u uniformly in t if there is a constant $\lambda > 0$ such that*

$$\|K(t,x) - K(t,y)\|_\infty \leq \lambda\|x-y\|.$$

Equation (4.3) can be formulated as a fixed point equation $x = Tx$.
For a map $T : X \longrightarrow X$ defined by

$$Tx(t) = x(a) + \int_a^b K(t,s,x(s))ds, \; t \in [a,b]. \tag{4.4}$$

Theorem 4.2. *Suppose that, for all $x,y \in C([a,b],\mathbb{R})$,*

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq \mu|x(s)-y(s)|, \tag{4.5}$$

where $\mu = \frac{1}{2(b-a)} < 1$, for all $t,s \in [a,b]$ and $x,y \in \mathbb{R}$. Then Equation (4.5) has a fixed point $x \in X$.

Proof: We prove that a mapping T defined in (4.4) is a contraction for two continuous functions x and y on $C([a,b],\mathbb{R})$.

$$
\begin{aligned}
\|Tx(t)-Ty(t)\|^{\frac{1}{p}}+\|Tx(t)-Ty(t)\|^{\frac{1}{q}} &\leq \Big(\int_a^b \|K(t,s,x(s)-K(t,s,y(s)\|ds\Big)^{\frac{1}{p}} \\
&\quad +\Big(\int_a^b \|K(t,s,x(s)-K(t,s,y(s)\|ds\Big)^{\frac{1}{q}} \\
&\leq \Big(\int_a^b \mu \ \|x(s)-y(s)\|ds\Big)^{\frac{1}{p}} \\
&\quad +\int_a^b \mu \ \|x(s)-y(s)\|ds\Big)^{\frac{1}{q}} \\
&\leq \Big(\mu \ \|x(s)-y(s)\|\Big)^{\frac{1}{p}}\Big(\int_a^b ds\Big)^{\frac{1}{p}} \\
&\quad +\Big(\mu \ \|x(s)-y(s)\|\Big)^{\frac{1}{q}}\Big(\int_a^b ds\Big)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
\|Tx(t)-Ty(t)\|^{\frac{1}{p}}+\|Tx(t)-Ty(t)\|^{\frac{1}{q}} &\leq \Big((\mu(b-a)) \ \|x(s)-y(s)\|\Big)^{\frac{1}{p}} \\
&\quad +\Big((\mu(b-a))\|x(s)-y(s)\|\Big)^{\frac{1}{q}} \\
&\leq \mu \ (\frac{1}{2}\sup_{t\in[a,b]}\|x(t)-y(t)\|)^{\frac{1}{p}} \\
&\quad +\mu \ (\frac{1}{2}\sup_{t\in[a,b]}\|x(t)-y(t)\|)^{\frac{1}{q}}).
\end{aligned}
$$

Hence, Equation 4.5 is satisfied for

$$
\lambda = \max\{(\mu(b-a))^{\frac{1}{p}},(\mu(b-a))^{\frac{1}{q}}\}.
$$

Since $x,y\in X$, from $\|x\|_\tau,\|y\|_\tau\leq 1$, we have

$$
|Tx(t)-Ty(t)| = \lambda\|x-y\|_\tau,
$$

or equivalently,

$$
p_\tau(Tx,Ty) = \lambda p_\tau(x,y). \tag{4.6}
$$

From Equation (3.2) and (4.6), assume that u and v are two distinct points on X. Letting $x=x_{n-1}$ and $y=x_n$, we have

$$
\begin{aligned}
p_s(Tx,Ty) &\leq \lambda\{\ p_s(x,y)+\ p_s(Tx,Ty)\}, \\
p_s(Tx_{n-1},Tx_n) &\leq \lambda\{\ p_s(x_{n-1},x_n)+\ p_s(x_n,Tx_n)\}, \\
p_s(x_n,x_{n+1}) &\leq \lambda\{\ p_s(x_{n-1},x_n)+\ p_s(x_n,x_{n+1})\}, \\
p_s(x_n,x_{n+1})-\lambda\ p_s(x_n,x_{n+1}) &\leq \lambda\ p_s(x_{n-1},x_n), \\
(1-\lambda)\ p_s(x_n,x_{n+1}) &\leq \lambda\ p_s(x_{n-1},x_n), \\
p_s(x_n,x_{n+1}) &\leq \frac{\lambda}{(1-\lambda)}\ p_s(x_{n-1},x_n),
\end{aligned}
$$

Letting $\lambda=\dfrac{\lambda}{(1-\lambda)}<1$, we obtain

$$
p_s(Tx,Ty) \leq \lambda\ p_s(x,y). \tag{4.7}
$$

Operator T satisfies condition of Equation (3.2). Hence by Theorem 4.2 we obtained the operator T has a fixed point $u\in X$, which is a solution of (4.1).

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