Konuralp Journal of Mathematics, 10 (1) (2022) 34-39

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

The Foundations of Homotopic Fuzzy Sets

Mustafa Burç Kandemir[1]

[1] *Department of Mathematics, Faculty of Science, Mugla Sıtkı Koçman University, 48000, Mugla, Turkey*

Abstract

Fuzzy sets are determined by membership functions. Many methods have been developed when determining the membership function of a fuzzy set. However, a fuzzy set can be specified with more than one membership function. Therefore, *the membership function fitting problem* is a well-known problem in fuzzy set theory. In this article, we have introduced the concepts of topologically continuous fuzzy set and homotopic fuzzy set whose membership functions are topologically continuous and homotopic, using the basic concepts of topology to overcome this problem. We have studied its basic structural properties. Finally, we proposed a solution method to the membership function fitting problem in fuzzy set theory using the homotopic fuzzy set concept.

Keywords: *Fuzzy set; Homotopy; Homotopic fuzzy set; Topology.*
2010 Mathematics Subject Classification: *03E72; 54-xx; 55Pxx.*

1. Introduction

In the classic paper [21], Zadeh introduced the notion of fuzzy sets and fuzzy set operations. A fuzzy set A in X is characterized by a membership function μ_A which associates with each point $x \in X$ its "grade of membership" $\mu_A(x) \in I$ where $I = [0,1]$ is a unit interval. Many scientists have worked its theory in this area for modeling uncertainties. However, there has been an existing problem since the establishment of the fuzzy set theory. This problem is that the membership functions that characterize a fuzzy set is not unique. For instance, we can describe a fuzzy set A on the real number set $\mathbb{R}$ as "*real numbers close to* 0". A membership function of it can be defined by $\mu_A(x) = \dfrac{1}{1+x^2}$. But we can also define another membership function for given the fuzzy set A as $\mu_A(x) = \dfrac{1}{1+x^4}$. More generally, $\mu_A(x) = \dfrac{1}{1+x^{2n}}$ for each $n \in \mathbb{N}$. This means that we can express a fuzzy set defined on the set of real numbers $\mathbb{R}$ with infinitely different functions. This, in fact, is one of the biggest problems in fuzzy set theory. For this reason, it is not interested in the membership function of the fuzzy set which model the problem. So in theory, it has been studied with fuzzy sets whose membership functions are determined. Of course, some special membership functions have been devised for fuzzy sets, especially on the set of real numbers such as triangular, trapezoidal, bell curvilinear, Gaussian, sigmoidal etc. Properties of related fuzzy sets have been examined with these functions. However, these types of membership functions still represent the same fuzzy set.

Many scientists have developed various membership function fitting methods to model fuzzy situations in practice. In [1, 2, 5, 7, 8, 20], they demonstrated the experimental construction of membership functions according to the relevant problem. Methods of creating membership degrees with various ways have been given in [6, 10, 12, 13, 14, 17, 18, 19]. Triangular and trapezoidal membership functions which are special membership functions are examined in detail in [16]. Measurement theory is often used to relate problems to the real environment, to model fuzzy situations and to make up a membership function. Examples of this situation can be given as [12, 13, 14]. In [3], the author gave an overview of the different kinds of mathematical forms of the membership functions, it is extracted the different demands and determined the rational class of the membership functions. In [23], emphasizing that there are many speculations about determining the membership function in fuzzy set theory, experimental method of determining membership function is given. When the literature is analyzed in general terms, it is clear that many scientists use an experimental method to adapt the membership function, and of course, the uniqueness of the membership function remains a major problem.

We will try to overcome the problem mentioned above using topology in this paper. We know that *topology* is the major mathematical area which study properties of spaces that are preserved under deformations, twistings, and stretching objects. We do these with *homotopy* in topology. The concept of homotopy is defined as two mathematical objects can be continuously deformed with each other.

In this work, we will first give the definition of a continuous fuzzy set in the topological sense, and study their basic properties. We will then give the concept of homotopicity of the two fuzzy sets defined on the topological universe. Hence, we shall state that the membership

Email addresses: **mbkandemir@mu.edu.tr (Mustafa Burç Kandemir)**

function of a continuous fuzzy set defined on topological space is topologically unique. It is important to note that the given definitions are not to bring a new definition to the known fuzzy set definition, but to define specific fuzzy sets on a topological space.

2. Preliminaries

2.1. Basic Concepts in Fuzzy Sets

Let X be a set and I be the unit interval $[0,1]$. A fuzzy set defined as follows;

Definition 2.1.1. *[21] A fuzzy set A in X is defined by a membership function $\mu_A : X \to I$ whose membership value $\mu_A(x)$ specifies the degree to which $x \in X$ belongs to the fuzzy set A, for $x \in X$.*

The family of all fuzzy sets in X will denote by $\mathscr{F}(X)$. If $A,B \in \mathscr{F}(X)$ then some basic fuzzy set operations are given componentwise proposed by Zadeh [21] as follows:

1) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in X$.
3) $C = A \cup B \Leftrightarrow \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.
4) $D = A \cap B \Leftrightarrow \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.
5) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, for all $x \in X$.

More generally, for a family of fuzzy sets $\mathscr{A} = \{A_j \mid j \in J\}$, the union, $C = \bigcup_{j\in J} A_j$, and the intersection, $D = \bigcap_{j\in J} A_j$, are defined by

$$\mu_C(x) = \max_{j\in J} \mu_{A_j}(x)$$

and

$$\mu_D(x) = \min_{j\in J} \mu_{A_j}(x)$$

for all $x \in X$, respectively [21, 22].
A fuzzy set $A \in \mathscr{F}(X)$ is called *null fuzzy set* if $\mu_A(x) = 0$ for all $x \in X$, and denoted by $\widetilde{\varnothing}$. A fuzzy set B is called *universal fuzzy set* if $\mu_B(x) = 1$ for all $x \in X$, and denoted by $\widetilde{X}$ [21, 22].
Let X and Y be sets, and let $f : X \to Y$ be a function. For a fuzzy set A in Y, the *inverse image* of A under f is the fuzzy set $f^{-1}[A]$ in X by the rule

$$\mu_{f^{-1}[A]}(x) = \mu_A(f(x)) \tag{2.1}$$

for all $x \in X$, i.e. $\mu_{f^{-1}[A]} = \mu_A \circ f$.
For a fuzzy set A in X, the *image* of A under f is the fuzzy set $f[A]$ in Y, and its membership function is defined by

$$\mu_{f[A]}(y) = \begin{cases} \max_{x\in f^{-1}[\{y\}]} \mu_A(x) & , \text{ if } f^{-1}[\{y\}] \neq \varnothing \\ 0 & , \text{ if } f^{-1}[\{y\}] = \varnothing \end{cases} \tag{2.2}$$

in [4, 21, 22].
Let $A \in \mathscr{F}(X)$. For $\alpha \in [0,1]$, we call that αA is an α-*layer* of A which is a fuzzy set on X such that its membership function is $\mu_{\alpha A}(x) = \alpha \wedge \mu_A(x)$ for each $x \in X$ [9].
The *Cartesian product* of two fuzzy sets A and B on any given set X is denoted by $A \otimes B$ and its membership function is defined by $\mu_{A\otimes B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$ for each $(x,y) \in X \times X$ [4, 22].

2.2. Basic Topological Concepts

The concept of topology is considered as a generalization of the structure of Euclidean space and the continuous functions between these spaces. The definition of topology is given as follows:
Let X be a set, $\mathscr{P}(X)$ be the power set of X and $\mathscr{T}$ be a subfamily of $\mathscr{P}(X)$. If $\mathscr{T}$ satisfies the following conditions, $\mathscr{T}$ is called *topology* on X.

(1) $\varnothing$ and X are in $\mathscr{T}$.
(2) If U and V are in $\mathscr{T}$, then their intersection $U \cap V$ is in $\mathscr{T}$.
(3) If $\mathscr{T}'$ is any subfamily of $\mathscr{T}$, then $\bigcup \mathscr{T}'$ is in $\mathscr{T}$.

If $\mathscr{T}$ is a topology on X, then we call that the pair $(X, \mathscr{T})$ is *topological space*. Each element of $\mathscr{T}$ is called an *open set*, and the subset $K \subseteq X$ is called *closed set*, if its complement is open [11, 15].
As is known, in topology, two *continuous* functions from one topological space to another are called *homotopic* if one can be "continuously deformed" into the other, such a deformation being called a *homotopy* between the two functions. Formal definitions of continuity and homotopy is as follows.

Definition 2.2.1. *[15, 11] Let $(X, \mathscr{T})$ and $(Y, \mathscr{T}')$ be topological spaces. A function $f : X \to Y$ is* continuous at a point *a in X provided that for each neighborhood V of $f(a)$ there is neighborhood U of a such that $f(U) \subseteq V$. A function f from X to Y is* continuous *provided it is continuous at each point of X.*

Note that, f is *continuous* if and only if $V \in \mathscr{T}' \Rightarrow f^{-1}(V) \in \mathscr{T}$.

Definition 2.2.2. *[15, 11] Let X be set, Λ be an index set, $(Y_i, \mathscr{T}_i)$ be a indexed family of topological spaces and $f_i : X \to Y_i$ be an indexed family of functions indexed by Λ. We call that the topology $\mathscr{T}$ generated by the family*

$$\mathscr{S} = \{f_i^{-1}(U) \mid i \in \Lambda, U \in \mathscr{T}_i\} \subseteq \mathscr{P}(X)$$

is initial topology *on X with respect to $\{f_i\}_{i\in\Lambda}$.*

Definition 2.2.3. *[15, 11] Let $(X,\mathscr{T})$ and $(Y,\mathscr{T}')$ be topological spaces and let $f, g : X \to Y$ be continuous functions. Then f is* homotopic *to g, denoted by $f \simeq g$, if there is a continuous function $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The function H is called a* homotopy *between f and g.*

Definition 2.2.4. *[15, 11] A continuous function $f : X \to Y$ between topological spaces is said to be* null-homotopic *if it is homotopic to constant function from X to Y.*

We know that, $\simeq$ is an equivalence relation on $C(X,Y)$ which is the collection of continuous functions that map X into Y.

3. Results

In this section, firstly we will define a continuous fuzzy set in the object space $(X,\mathscr{T})$ in topological sense. Then we will try to explain homotopic fuzzy sets. From now on this paper X denotes $(X,\mathscr{T})$ topological space, I denotes the relative topology for $[0,1]$ of usual topology of $\mathbb{R}$.

3.1. Topologically Continuous Fuzzy Sets

Definition 3.1.1. *Let A be a fuzzy set in X. Then A is called* topologically continuous fuzzy set *(or breifly c-fuzzy set) if and only if its membership function μ_A is continuous from X to I, i.e. $\mu_A^{-1}(O) \in \mathscr{T}$ for all open set O in I.*

Example 3.1.2. *Let $X = \mathbb{R}$ and $\mathscr{T}$ be usual topology on $\mathbb{R}$. Let A be a fuzzy set in $\mathbb{R}$ and its membership function $\mu_A : \mathbb{R} \to I$ defined as $\mu_A(x) = \dfrac{|x|}{|x|+1}$ for all $x \in \mathbb{R}$. Since μ_A is continuous, then A is a c-fuzzy set on $\mathbb{R}$.*

Since all type of membership functions on $\mathbb{R}$ such as *triangular*, *trapezoidal*, *bell curvilinear*, *Gaussian*, *sigmoidal* are continuous, related fuzzy sets are c-fuzzy set on $\mathbb{R}$.

Example 3.1.3. *Let $X = \{a,b,c\}$ and $\mathscr{T} = \{\varnothing, X, \{a\}, \{b,c\}\}$ be topology on X. Let A be a fuzzy set on X such that $\mu_A(a) = 0.4$, $\mu_A(b) = 1$ and $\mu_A(c) = 1$. Since*

$$\mu_A^{-1}[(\alpha,\beta)] = \begin{cases} \varnothing & ,0 \le \alpha < \beta \le 0.4 \\ \{a\} & ,\alpha < 0.4 < \beta < 1 \\ \varnothing & ,0.4 < \alpha < \beta < 1 \\ \{b,c\} & ,0.4 < \alpha, \beta = 1 \end{cases} \in \mathscr{T}$$

for arbitrary open basic set (α,β) in I, A is a c-fuzzy set on X.

Theorem 3.1.4. *Null and universal fuzzy sets on an topological space are c-fuzzy sets.*

Proof. Since the memberships functions of null and universal fuzzy sets are constant function, and constant functions are continuous, then null and universal fuzzy sets are c-fuzzy sets. □

Theorem 3.1.5. *Let A and B be c-fuzzy sets and $\{A_i\}_{i\in I}$ be family of c-fuzzy sets in X. Then $A \cup B$, $A \cap B$ and A^c are also c-fuzzy sets in X.*

Proof. If A and B c-fuzzy sets, then $\mu_A, \mu_B : X \to I$ are continuous functions. At that case, for any open set O in I, we obtain that

$$\begin{aligned} \mu_{A\cup B}^{-1}[O] &= \{x \in X \mid \mu_{A\cup B}(x) \in O\} \\ &= \{x \in X \mid \max\{\mu_A(x), \mu_B(x)\} \in O\}. \end{aligned}$$

Since I is a total ordered set, either $\mu_A(x) \le \mu_B(x)$ or $\mu_B(x) \le \mu_A(x)$. Therefore, if we assume that $\mu_A(x) \le \mu_B(x)$, then $\mu_{A\cup B}^{-1}[O] = \mu_B^{-1}[O]$ is open set in X. It is also valid for $\mu_B(x) \le \mu_A(x)$, i.e. we have $\mu_{A\cup B}^{-1}[O] = \mu_A^{-1}[O]$. Thus $\mu_{A\cup B}$ is continuous, hence $A \cup B$ is a c-fuzzy set on X.
In addition to this, assume that A is a c-fuzzy set in X. Let (α,β) be an open basic set in I. We know that $\alpha, \beta \in I$ and $\alpha < \beta$ implies $1-\beta \le 1-\alpha$. Then, we have

$$\begin{aligned} \mu_{A^c}^{-1}[(\alpha,\beta)] &= \{x \in X \mid \mu_{A^c}(x) \in (\alpha,\beta)\} \\ &= \{x \in X \mid 1-\mu_A(x) \in (\alpha,\beta)\} \\ &= \{x \in X \mid \mu_A(x) \in (1-\beta, 1-\alpha)\} \\ &= \mu_A^{-1}[(1-\beta,1-\alpha)] \in \mathscr{T}. \end{aligned}$$

So A^c is a c-fuzzy set.
Proof of other expressions is made in a similar way. □

Theorem 3.1.6. *If A be a c-fuzzy set, then its α-layer αA is also c-fuzzy set for $\alpha \in [0,1]$.*

Proof. It is straightforward. □

Theorem 3.1.7. *Let B be a c-fuzzy set in Y and f be a continuous function from X to Y. $f^{-1}[B]$ is a c-fuzzy set on X.*

Proof. Let B be a c-fuzzy set in Y. For all $x \in X$, we have that $\mu_{f^{-1}[B]}(x) = (\mu_B \circ f)(x)$ from Equation (2.1). Assume that O be an open set in I. Since μ_B and f are continuous function, then we obtain that

$$\mu^{-1}_{f^{-1}[B]}[O] = (\mu_B \circ f)^{-1}[O] \in \mathscr{T}.$$

Hence $f^{-1}[B]$ is a c-fuzzy set in X. □

Theorem 3.1.8. *If A and B are c-fuzzy set on X, then $A \times B$ is a c-fuzzy set on $X \times X$.*

Proof. Similar to proof of Theorem 3.1.5. □

Theorem 3.1.9. *If A is a c-fuzzy set in the topological space X, then $\mathfrak{s}(A)$ is an open set in the space X.*

Proof. Assume that A is a c-fuzzy set. Then the membership function $\mu_A : X \to I$ is continuous. We know that $(0,1]$ is an open set in the space I. So, the inverse image of $(0,1]$ under μ_A is

$$\mu_A^{-1}[(0,1]] = \{x \in X \mid \mu_A(x) \in (0,1]\} = \{x \in X \mid \mu_A(x) > 0\} = \mathfrak{s}(A).$$

Since μ_A is continuous, then $\mathfrak{s}(A)$ is an open set in X. □

Similar to Theorem 3.1.9, for each $\alpha \in [0,1)$, if A is a c-fuzzy set then its strong level sets

$$A^{>\alpha} = \{x \in X \mid \mu_A(x) > \alpha\}$$

and

$$A^{<\alpha} = \{x \in X \mid \mu_A(x) < \alpha\}$$

are open sets in the topological space X.
Let X be a non-empty set and A be a fuzzy set on X. We can construct a topology on X by using the membership function of A and the relative topology of $[0,1]$. Using the concept of initial topology from Definition 2.2.2, we obtain a topology on X that the membership function is continuous from X to $[0,1]$. Thus, each fuzzy set on X can be made continuous by this method.

Example 3.1.10. *Let $X = \{a,b,c\}$ be a set, A be a fuzzy set on X such that $\mu_A(a) = 0.2$, $\mu_A(b) = 0.7$ and $\mu_A(c) = 0.7$. $\mathscr{B} = \{[0,\alpha),(\alpha,\beta),(\alpha,1] \mid 0 < \alpha < \beta < 1\}$ is a basis for relative topology on $[0,1]$ with respect to usual topology on $\mathbb{R}$. For arbitrary basis element $(\alpha,\beta) \in \mathscr{B}$,*

$$\mu_A^{-1}[(\alpha,\beta)] = \begin{cases} \varnothing & ,0 \leq \alpha < \beta \leq 0.2 \\ \{a\} & ,\alpha < 0.2 < \beta < 0.7 \\ \varnothing & ,0.2 < \alpha < \beta < 0.7 \\ \{b,c\} & ,0.2 < \alpha < 0.7 < \beta \leq 1 \end{cases}$$

We obtain the family $\mathscr{S} = \{\varnothing,\{a\},\{b,c\}\}$, and the topology

$$\mathscr{T} = \{\varnothing, X, \{a\}, \{b,c\}\}$$

is generated by $\mathscr{S}$ which μ_A is made continuous.

The coarsest topology which all fuzzy sets are c-fuzzy sets can be found in this way from Definition 2.2.2. Thus, we can give following result.

Corollary 3.1.11. *All fuzzy sets on X can be made c-fuzzy sets.*

3.2. Homotopic Fuzzy Sets

Definition 3.2.1. *Let A and B be continuous fuzzy set over X. It is called that A and B are* homotopic fuzzy sets *if there exist a homotopy function $H : X \times I \to I$ such that $H(x,0) = \mu_A(x)$ and $H(x,1) = \mu_B(x)$ for all $x \in X$, and denoted by $A \cong B$.*

Also, we call that the fuzzy sets A and B are *topologically same* to each other if $A \cong B$. Since $\cong$ is an equivalence relation on $\mathscr{C}(X)$ which is the collection of c-fuzzy sets, the equivalence class of homotopic fuzzy sets to A is denoted by $[A]$. Family of all classes of homotopic fuzzy sets on the space X is denoted by $\mathscr{H}(X)$. Obviously, $\mathscr{H}(X) = \mathscr{C}(X)/\cong$ quotient set.

Example 3.2.2. *Let $\mathbb{R}$ be a usual topological space, define the fuzzy set A as $A =$"real numbers near to 0". Its membership function can be defined as $\mu_A(x) = \dfrac{1}{1+x^2}$. Besides, the membership function $\mu'_A(x) = \dfrac{1}{1+x^4}$ is also modelled the fuzzy set A. μ_A and μ'_A are continuous functions from $\mathbb{R}$ to $[0,1]$. If we define the function $H : \mathbb{R} \times I \to I$ such that $H(x,t) = t\mu_A(x) + (1-t)\mu'_A(x)$ which is called* linear homotopy, *then we obtain that μ_A is homotopic to μ'_A. More generally, $\mu_A(x) = \dfrac{1}{1+x^n}$ for each $n \in \mathbb{N}$ defines the fuzzy set A. For each $n,m \in \mathbb{N}$ and $n \neq m$, we can define same linear homotopy between $\mu_A(x) = \dfrac{1}{1+x^n}$ and $\mu_A(x) = \dfrac{1}{1+x^m}$, then we can say that membership function of the fuzzy set A is unique under homotopy.*

Theorem 3.2.3. *The membership function of c-fuzzy set A defined on $\mathbb{R}$ is unique in topological sense.*

Proof of this theorem is straightforward. We can use linear homotopy between all membership functions for the fuzzy set A.
As a result of this theorem, all type of membership function on $\mathbb{R}$ such as triangular, trapezoidal, bell curvilinear, Gaussian, sigmoidal are homotopic each other.
Obviously, we obtain following theorem.

Theorem 3.2.4. *Null and universal fuzzy sets are homotopic.*

Theorem 3.2.5. $A \cong A^c$ *and* $A \cong \alpha A$.

Proof. It is obvious. □

A space is said to be *contractible* if the identity map $i_X : X \to X$ is a null-homotopic which defines in Definition 2.2.4. Let I be a space and $i_I : I \to I$ be an identity map and $\mathbf{0} : I \to I$ be constant map such that $\mathbf{0}(x) = 0$ for all $x \in I$. Define the linear homotopy function $H : I \times I \to I$ such that $H(x,t) = (1-t)i_X(x) + t\mathbf{0}(x)$. Hence I is contractible [11].
Since $[0,1]$ is contractible space, then $[X,I]$ has a single element. Then we obtain following theorem.

Theorem 3.2.6. *Membership function of all c-fuzzy sets on any given topological space X is unique.*

As a result of the above theorem, if A and B are c-fuzzy set on X, then $A \cong B$, $A \cap B \cong A \cup B$, etc.

Theorem 3.2.7. *Let X and Y topological spaces,* $f : X \to Y$ *be a homeomorphism and A and B be c-fuzzy set on Y. If* $A \cong B$*, then* $f^{-1}[A] \cong f^{-1}[B]$.

Proof. If A is a c-fuzzy set on Y, then $f^{-1}[A]$ is a c-fuzzy set on X from Theorem 3.1.7. Assume that $A \cong B$. Then there exist a homotopy $H : Y \times I \to I$ such that $H(y,0) = \mu_A(y)$ and $H(y,1) = \mu_B(y)$. Since f is a bijection, then $y = f(x) \Leftrightarrow x = f^{-1}(y)$ for each $x \in X$, $y \in Y$. Define the function $H' : X \times I \to I$ such that $H'(x,0) = H(f(x),0)$ and $H'(x,1) = H(f(x),1)$ then H' is continuous and we have

$$H'(x,0) = H(f(x),0) = \mu_A(f(x)) = \mu_{f^{-1}[A]}(x)$$

and

$$H'(x,1) = H(f(x),1) = \mu_B(f(x)) = \mu_{f^{-1}[B]}(x).$$

Hence, $f^{-1}[A] \cong f^{-1}[B]$. □

Theorem 3.2.8. *Let X be a topological space, A,B,C and D be c-fuzzy sets on X. If* $A \cong B$ *and* $C \cong D$*, then* $A \otimes C \cong B \otimes D$.

Proof. From Theorem 3.1.8, $A \otimes C$ and $B \otimes D$ are c-fuzzy sets on $X \times X$ with respect to product topology. Since $A \cong B$ and $C \cong D$, there exist homotopies $H_1 : X \times I \to I$ such that $H_1(x,0) = \mu_A(x)$ and $H_1(x,1) = \mu_B(x)$, and $H_2 : X \times I \to I$ such that $H_2(y,0) = \mu_C(y)$ and $H_2(y,1) = \mu_D(y)$ for all $x, y \in X$. We define the function $H : (X \times X) \times I \to I$ such that

$$H((x,y),0) = \min\{H_1(x,0), H_2(y,0)\}$$

and

$$H(x,y),1) = \min\{H_1(x,1), H_2(y,1)\}.$$

Since H_1 and H_2 is continuous then H is continuous. From definition of H, we obtain that

$$H((x,y),0) = \min\{H_1(x,0), H_2(y,0)\} = \min\{\mu_A(x), \mu_C(y)\} = \mu_{A \otimes C}(x,y)$$

and

$$H(x,y),1) = \min\{H_1(x,1), H_2(y,1)\} = \min\{\mu_B(x), \mu_D(y)\} = \mu_{B \otimes D}(x,y).$$

Thus $A \otimes C \cong B \otimes D$. □

4. Conclusion

Fitting membership function problem is one of the most problem in fuzzy set theory. As we mentioned in the introduction, the membership function that defines a fuzzy set may not be unique. In this article, we have shown that the membership function of the fuzzy set defined on a topological universe is topologically unique up to the concept of homotopy. We also say that each fuzzy set can be continuous fuzzy set, since we can establish a topology that is called initial topology on the given universe using the membership function of the fuzzy set defined on it. With this method, we can uniquely identify fuzzy sets on the given universe in topological sense.
In future, it can be studied that how homotopic fuzzy sets affect decision-making.

References

[1] Chen, J.E., Otto, K.N., Constructing membership functions using interpolation and measurement theory. Fuzzy Set Syst. 1995;73:313–327.
[2] Civanlar, M. R., Trussel, H. J. Constructing membership functions using statistical data. Fuzzy Set Syst. 1986;18:1–14.
[3] Dombi, J. Membership function as an evaluation. Fuzzy Set Syst. 1990;35:1–22.
[4] Dubois, D., Prade, H. Fuzzy sets and systems : theory and applications. New York: Academic Press; 1980.
[5] Furukawa, M., Yamakawa, T. The design algorithms of membership functions for a fuzzy neuron. Fuzzy Set Syst. 1995;71(3):329–343.
[6] Giles, R. The concept of grade of membership. Fuzzy Set Syst. 1988;25:297–323.
[7] Hersh, H., Carmazza, A., Brownell, H. H. Effects of context on fuzzy membership functions. In M. M. Gupta, R. M. Ragade, R. R. Yager, editors, Advances in Fuzzy Set Theory, Amsterdam: North Holland; 1979. 389–408.
[8] Lambert, J. The fuzzy set membership problem using the hierarchy decision method, Fuzzy Set Syst. 1992;48(3):323–330.
[9] Ying-Ming, L., Mao-Kang, L. Fuzzy Topology. Singapore: World Scientific Publishing Co. Pte. Ltd.; 1997.
[10] Mabuchi, S. An interpretation of membership functions and the properties of general probabilistic operators as fuzzy set operators .1. case of type1 fuzzy sets. Fuzzy Set Syst. 1992;49(3):271–283.
[11] Munkres, J. R. Topology 2nd Ed. Upper Saddle River, NJ: Prentice Hall, Inc.; 2000.
[12] Norwich, A. M., Türkşen, I. B. The construction of membership functions. In R. R. Yager editor, Fuzzy Sets and Possibility Theory: Recent Developments, New York: Pergamon Press; 1982. 61–67.
[13] Norwich, A. M., Türkşen, I. B. The fundamental measurement of fuzziness. In R. R. Yager editor, Fuzzy Sets and Possibility Theory: Recent Developments, New York: Pergamon Press; 1982. 49–60.
[14] Norwich, A. M., Türkşen, I. B. A model for the measurement of membership and the consequences of its empirical implementation, Fuzzy Set Syst. 1984;12:1–25.
[15] Patty, C. W. Foundations of topology. Boston: PWS-KENT Pub.; 1992.
[16] Pedrycz, W. Why triangular membership functions?. Fuzzy Set Syst. 1994;64(1):21–30.
[17] Saaty, T. L. Scaling the membership function. Eur. J. Oper. Res. 1986;25:320–329.
[18] Triantaphyllou, E., Mann, S. H. An evaluation of the eigenvalue approach for determining the membership values in fuzzy sets. Fuzzy Set Syst. 1990;35:295–301.
[19] Türkşen, I. B. Measurement of membership functions and their acquisition. Fuzzy Set Syst. 1991;40:5–38.
[20] Yamakawa, T., Furukawa, M. A design algorithm of membership functions for a fuzzy neuron using example based learning. Proceedings of the First IEEE Conference on Fuzzy Systems; San Diego: 1992. p. 75–82.
[21] Zadeh, L. A. Fuzzy sets, Inform. Contr. 1965;8:338–353.
[22] Zimmermann, H. J. Fuzzy set theory and its applications 2nd Ed. Boston: Kluwer Academic Publishers; 1991.
[23] Zysno, P. Modeling membership functions. In B. B. Rieger, editor, Empirical Semantics I, Vol. 1 of Quantitative Semantics, Vol. 12, Bochum: StudienverlagBrockmeyer; 1982. 350–375.