On Complex Interval Arithmetic Using Polar Form

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Highlights
• Presentation of sources of difficulties of addition and subtraction of sectors.
• Introduction of effective and low-cost algorithms solving the problem.
• Case studies demonstrating the superiority of the proposed algorithms.

Abstract

In this paper, complex arithmetic in polar form, briefly sectors arithmetic is discussed. The fact that the family of sectors is not closed under addition and subtraction gives rise to the need of an optimized approximation. Precisely, the problem discussed, is to determine the smallest possible sector that contains the sum of two sectors. An efficient algorithm that is solving this problem is introduced and compared with existing algorithms in the literature. The algorithms are implemented in MATLAB R2020a and tested for all cases. It is also shown by examples that the proposed algorithms perform much better than their ancestors and avoid their errors.

1. INTRODUCTION

In complex interval arithmetic three different types of intervals are introduced: rectangular [1-3], circular [4], and sector representation [5]. All these representations suffer from not being closed under one or more of the basic operations addition, subtraction, multiplication and division.

Whenever an operation fails to be closed for a given representation, the issue is to obtain the smallest possible rectangle, circle or sector respectively that just includes the result.

Representations based on sectors fail to be closed under addition and subtraction. Klatte and Ulrich [5] first introduced arithmetic operations on sectors. They proposed sector arithmetic as an alternative to rectangular and circular arithmetic in the complex interval space. They defined six different alternatives; all of them are based on mapping sectors to rectangles or circles, performing the operations on the chosen domain, and then returning to the original representation. Obviously, this forth and back transformations between rectangular and polar forms introduces a good deal of pessimism and is in no way optimal. Then, Flores [6] proposed algorithms for performing basic arithmetic operations on sectors; he called complex fans arithmetic. Even though all algorithms in [6] are proven mathematically and verified empirically, they require a significant amount of computation and moreover, they fail in the case where the origin is a boundary point of the sum of two sectors. Finally, Candau et al. in [7] derived analytical algorithms for the computation of the smallest sector circumscribing the sum of two sectors. Unfortunately, we realized that the algorithm 3. MinAngle proposed in [7] may not work correctly in some scenarios and we demonstrated this by presenting some counter-examples. These facts motivated this work: the research for an effective algorithm to perform addition and subtraction.

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Since the basic operations multiplication and division of sectors are defined flawlessly using real interval arithmetic our focus is addition and subtraction. For polar complex interval addition (subtraction) we will introduce a highly efficient and low cost algorithm compared to the algorithms in [6] and [7]. Sectors, from a different perspective, are two dimensional intervals on the real or complex plane. Therefore all results obtained in this work can be used not only for complex intervals but also for 2-D real intervals.

The rest of the paper is organized as follows: Section 2 presents the definition and arithmetic of polar intervals. The proposed algorithm is introduced in Section 3. In Section 3 we preferred to analyze the problem and summarize the results in an algorithm at the end rather than delivering the material in theorem-proof form. Section 4 contains the implementation of the proposed algorithms and the comparison with the available algorithms in the literature. Finally, Section 5 concludes the work.

2. POLAR COMPLEX INTERVALS

Hereafter $\mathbb{I}(\mathbb{R})$ denotes the set of closed real intervals. Similarly, $\mathbb{I}(\mathbb{R}^+)$ will denote the set of non-negative real intervals.

Definition 1 *Let* $[\rho] = [\rho^-, \rho^+] \in \mathbb{I}(\mathbb{R}^+)$ *and* $[\varphi] = [\varphi^-, \varphi^+] \in \mathbb{I}(\mathbb{R})$*; the set defined by* $S = \{s \in \mathbb{C} : s = \rho e^{i\varphi}, \rho \in [\rho], \varphi \in [\varphi]\}$ *is called a polar complex interval (or sector), with magnitude interval* $[\rho]$ *and angle interval* $[\varphi]$ *denoted by* $[\rho]e^{i[\varphi]}$*.*

For simplicity, and without loss of generality, we will restrict our study to the case in which $[\varphi] = [\varphi^-, \varphi^+] \in \mathbb{I}(\mathbb{R}^+)$, since all other cases can be reduced to the study of this one.

2.1. Classification of Sectors According to Angle Intervals

It is important to distinguish between types of sectors according to their angle intervals $[\varphi]$. It turns out to be sufficient to distinguish according to whether $\varphi^+ \leq 2\pi$ or $\varphi^+ > 2\pi$.

• The set $S(\mathbb{C})$ is defined as the set of sectors with angle interval $[\varphi]$ satisfying $\varphi^+ \leq 2\pi$. Thus, for all $[\rho]e^{i[\varphi]}$ in $S(\mathbb{C})$, we have $\varphi^+ - \varphi^- \leq 2\pi$, $0 \leq \varphi^- < 2\pi$ and $0 < \varphi^+ \leq 2\pi$ (see Figure 1).
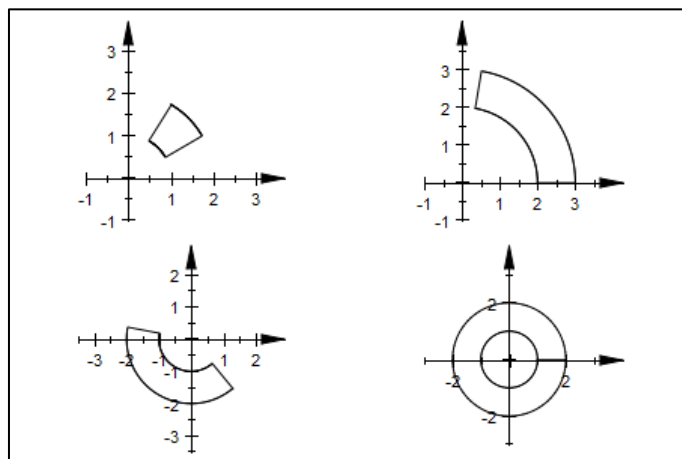

Figure 1. *Some members of the set* $S(\mathbb{C})$

• The set $S^*(\mathbb{C})$ is defined as the set of sectors with angle interval $[\varphi]$ satisfying $\varphi^+ > 2\pi$. Thus, for all $[\rho]e^{i[\varphi]}$ in $S^*(\mathbb{C})$, we have $\varphi^+ - \varphi^- \leq 2\pi$, $0 \leq \varphi^- < 2\pi$ and $2\pi < \varphi^+ < 4\pi$ (see Figure 2).
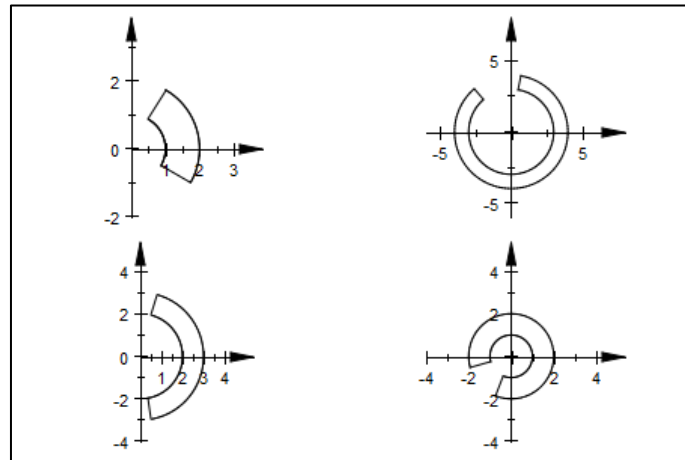
Figure 2. *Some members of the set* $S^*(\mathbb{C})$

2.2. Operations on Sectors

Definition 2 *Let* $S_1 = [\rho_1]e^{i[\theta_1]}$, $S_2 = [\rho_2]e^{i[\theta_2]}$ *be two sectors, and* $* \in \{+, -, \cdot, /\}$. *Then we define the corresponding operations for* S_1 *and* S_2 *by* $S_1 * S_2 := \circledS\{S_1 \circledast S_2\}$, *where*
$S_1 \circledast S_2 := \{s_1 * s_2 : s_1 \in S_1, s_2 \in S_2\}$ *and* $\circledS\{S_1 \circledast S_2\}$ *is the smallest sector that encloses* $S_1 \circledast S_2$.

Let $S_1 = [\rho_1]e^{i[\theta_1]} = [\rho_1^-, \rho_1^+]e^{i[\theta_1^-, \theta_1^+]}$ and $S_2 = [\rho_2]e^{i[\theta_2]} = [\rho_2^-, \rho_2^+]e^{i[\theta_2^-, \theta_2^+]}$ be given sectors. If the set $S_1 \circledast S_2$ is already a sector, then clearly $\circledS\{S_1 \circledast S_2\} = S_1 \circledast S_2$. Some important properties of $S_1 \circledast S_2$ are summarized below.

1. The results from multiplication $S_1 \odot S_2$ or division $S_1 \oslash S_2$ are sectors. Therefore, $\circledS\{S_1 \odot S_2\} = S_1 \odot S_2$ and $\circledS\{S_1 \oslash S_2\} = S_1 \oslash S_2$.

2. The results from addition $S_1 \oplus S_2$ or subtraction $S_1 \ominus S_2$ are not sectors in general.

For this reason the addition of sectors has been handled with different approaches in the papers [5,6] and [7]. The method in [5] is to transform the sectors into rectangular regions, perform the addition exactly and map the result back into a sector. Obviously this method introduces a serious amount of pessimism and is far from being optimal. The method in [6] is optimal except for some certain cases but with high computational cost. Finally the algorithm introduced in [7] contains some errors as we will show in the sequel.
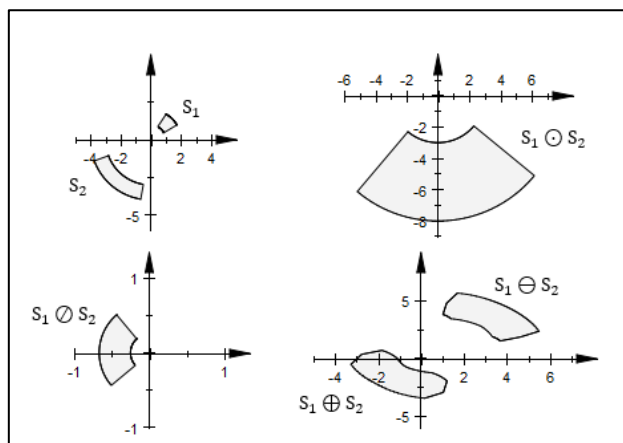

Figure 3. *Sectors are not closed under addition and subtraction*

Consider the example, $S_1 = [1,2]e^{i[30°,60°]}, S_2 = [3,4]e^{i[200°,260°]}$. Then, the resulting set from $S_1 \odot S_2$ or $S_1 \oslash S_2$ is a sector, while the results from $S_1 \oplus S_2$ or $S_1 \ominus S_2$ are not sectors. This can be seen in Figure 3.

By the rules for arithmetic operations in $\mathbb{I}(\mathbb{R})$ and the continuity of the exponential function the multiplication and division operations can easily performed as follows:

• Multiplication

The multiplication of S_1 and S_2 is given by,

$$S_1 \cdot S_2 := \circledS\{S_1 \odot S_2\}$$
$$= \circledS\{\rho_1\rho_2 e^{i(\theta_1+\theta_2)}: \rho_1 \in [\rho_1], \rho_2 \in [\rho_2], \theta_1 \in [\theta_1], \theta_2 \in [\theta_2]\}$$
$$= [\rho_1^-\rho_2^-, \rho_1^+\rho_2^+]e^{i[\theta_1^-+\theta_2^-, \theta_1^++\theta_2^+]}.$$

Note that, from the definition of a sector, when $(\theta_1^+ + \theta_2^+) - (\theta_1^- + \theta_2^-) \geq 2\pi$ we have to set $[\theta_1^- + \theta_2^-, \theta_1^+ + \theta_2^+] = [0,2\pi]$.

• Division

The division of S_1 and S_2 is given by,

$$S_1 / S_2 := \circledS\{S_1 \oslash S_2\}$$
$$= \circledS\left\{\frac{\rho_1}{\rho_2}e^{i(\theta_1-\theta_2)}: \rho_1 \in \rho_1], \rho_2 \in \rho_2], \theta_1 \in \theta_1], \theta_2 \in \theta_2]\right\}, 0 \notin \rho_2]$$
$$= \begin{cases} \left[\frac{\rho_1^-}{\rho_2^+}, \frac{\rho_1^+}{\rho_2^-}\right]e^{i[\theta_1^--\theta_2^++2n\pi, \theta_1^+-\theta_2^-+2n\pi]}, & \text{if}\,\theta_1^- - \theta_2^+ < 0 \\ \left[\frac{\rho_1^-}{\rho_2^+}, \frac{\rho_1^+}{\rho_2^-}\right]e^{i[\theta_1^--\theta_2^+, \theta_1^+-\theta_2^-]}, & \text{otherwise} \end{cases},$$

where n is the smallest positive integer such that $\theta_1^- - \theta_2^+ + 2n\pi \geq 0$. Also we set $[\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-] = [0,2\pi]$ when $(\theta_1^+ - \theta_2^-) - (\theta_1^- - \theta_2^+) \geq 2\pi$.

3. ADDITION AND SUBTRACTION OF TWO SECTORS

In general there is no simple, direct way to add or subtract two sectors S_1 and S_2, but the minimal sector, $\circledS\{S_1 \circledast S_2\}$ can be obtained from $S_1 \circledast S_2$. In order to evaluate this hull, we have to compute the best possible lower and upper bounds for both $[\rho]$ and $[\varphi]$ such that the sector they represent is the smallest possible sector that contains the set $S_1 \circledast S_2$.

The problem of subtraction can be transformed into addition. That is subtraction of S_1 and S_2 can be calculated as: $S_1 - S_2 = S_1 + (-S_2)$, where $(-S_2) := \{w: -w \in S_2\}$. Therefore, we can use addition to solve subtraction problems as well.

In this section, the basis of a method for determining $\circledS\{S_1 \oplus S_2\}$ will be presented.
Let $s_1 \in S_1, s_2 \in S_2$ and formualte $s_1 + s_2 = s$ as:

$$\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2} = \rho e^{i\varphi}. \tag{1}$$

To determine ρ, we simply take the complex absolute value of both sides of Equation (1). Hence,

$$\begin{aligned}
\rho^2 &= \left|\rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}\right| \\
&= \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos(\theta_1 - \theta_2).
\end{aligned}$$

Taking the positive square roots yields,

$$\rho = \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos(\theta_1 - \theta_2)}. \tag{2}$$

To determine φ, from Equation (1), we have,

$$Re(s) = \rho\cos(\varphi) = \rho_1\cos(\theta_1) + \rho_2\cos(\theta_2),$$
$$Im(s) = \rho\sin(\varphi) = \rho_1\sin(\theta_1) + \rho_2\sin(\theta_2).$$

From the last two equations we get,

$$\tan(\varphi) = \frac{\rho_1\sin(\theta_1) + \rho_2\sin(\theta_2)}{\rho_1\cos(\theta_1) + \rho_2\cos(\theta_2)}. \tag{3}$$

3.1. Computing The Bounds of Angle Interval $[\varphi]$

Recall that $\tan\varphi$ was given in Equation (3) and thus:

$$\varphi = \arctan\left(\frac{\rho_1\sin(\theta_1) + \rho_2\sin(\theta_2)}{\rho_1\cos(\theta_1) + \rho_2\cos(\theta_2)}\right). \tag{4}$$

where $\varphi \in]\frac{-\pi}{2}, \frac{\pi}{2}[$ and the quadrant in which it falls is determined by the signs of $\rho_1\sin(\theta_1) + \rho_2\sin(\theta_2)$ and $\rho_1\cos(\theta_1) + \rho_2\cos(\theta_2)$. If $\rho_1\cos(\theta_1) + \rho_2\cos(\theta_2) = 0$, then

$$\varphi = \begin{cases} \frac{\pi}{2}, & \text{if } \rho_1\sin(\theta_1) + \rho_2\sin(\theta_2) > 0, \\ \frac{3\pi}{2}, & \text{if } \rho_1\sin(\theta_1) + \rho_2\sin(\theta_2) < 0. \end{cases}$$

Hence for each $s \in S_1 \oplus S_2$ we can compute the corresponding value of φ, except when $s = 0$.
Our goal is to compute the bounds φ^- and φ^+ such that $[\varphi^-, \varphi^+]$ is the least interval that encloses all possible angles of $S_1 \oplus S_2$. This means that we have to compute the minimum and the maximum of the function φ with respect to all points $(\rho_1, \rho_2, \theta_1, \theta_2) \in \Omega = [\rho_1] \times [\rho_2] \times [\theta_1] \times [\theta_2]$.
Differentiating φ we obtain:

$$\begin{aligned}
\frac{\partial\varphi}{\partial\rho_1} &= \frac{\rho_2\sin(\theta_1-\theta_2)}{\rho^2}, \\
\frac{\partial\varphi}{\partial\rho_2} &= \frac{-\rho_1\sin(\theta_1-\theta_2)}{\rho^2}, \\
\frac{\partial\varphi}{\partial\theta_1} &= \frac{\rho_1\rho_2\cos(\theta_1-\theta_2)+\rho_1^2}{\rho^2}, \\
\frac{\partial^2\varphi}{\partial\theta_1^2} &= \frac{\rho_1\rho_2\sin(\theta_1-\theta_2)(\rho_1^2-\rho_2^2)}{\rho^4}, \\
\frac{\partial\varphi}{\partial\theta_2} &= \frac{\rho_1\rho_2\cos(\theta_1-\theta_2)+\rho_2^2}{\rho^2}, \\
\frac{\partial^2\varphi}{\partial\theta_2^2} &= \frac{\rho_1\rho_2\sin(\theta_1-\theta_2)(\rho_1^2-\rho_2^2)}{\rho^4}.
\end{aligned}$$

Note that $\nabla\varphi(\rho_1, \rho_2, \theta_1, \theta_2)$ is defined whenever φ is. Thus, it is easy to check that the function φ has no critical points on its domain of definition.

We will find two points $x_{\min}, x_{\max} \in \Omega$ such that,
$$\min\varphi = \varphi^- = \varphi(x_{\min}) \leq \varphi(x) \leq \varphi(x_{\max}) = \varphi^+ = \max\varphi \text{ for all } x \in \Omega.$$

From $\frac{\partial \varphi}{\partial \rho_1}$ and $\frac{\partial \varphi}{\partial \rho_2}$ we note that the possible values of ρ_1 and ρ_2 are $\{\rho_1^+, \rho_1^-\}$ and $\{\rho_2^+, \rho_2^-\}$, respectively. Hence the optimal point ($x_{\min}$ or $x_{\max}$) must fall into one of the following types of points:

• $X_1 = \{(\rho_1, \rho_2, \theta_1, \theta_2): \rho_1 \in \{\rho_1^+, \rho_1^-\}, \rho_2 \in \{\rho_2^+, \rho_2^-\}, \theta_1 \in]\theta_1[, \theta_2 \in \{\theta_2^+, \theta_2^-\}\}$, where θ_1 is given by solving the equation $\frac{\partial \varphi}{\partial \theta_1} = 0$.

• $X_2 = \{(\rho_1, \rho_2, \theta_1, \theta_2): \rho_1 \in \{\rho_1^+, \rho_1^-\}, \rho_2 \in \{\rho_2^+, \rho_2^-\}, \theta_1 \in \{\theta_1^+, \theta_1^-\}, \theta_2 \in]\theta_2[\}$, where θ_2 is given by solving the equation $\frac{\partial \varphi}{\partial \theta_2} = 0$.

• $X_3 = \{(\rho_1, \rho_2, \theta_1, \theta_2): \rho_1 \in \{\rho_1^+, \rho_1^-\}, \rho_2 \in \{\rho_2^+, \rho_2^-\}, \theta_1 \in \{\theta_1^+, \theta_1^-\}, \theta_2 \in \{\theta_2^+, \theta_2^-\}\}$. Notice that we only need to select θ_1 and θ_2 because the signs of $\frac{\partial \varphi}{\partial \rho_1}$ and $\frac{\partial \varphi}{\partial \rho_2}$ are determined by the sign of $\sin(\theta_1 - \theta_2)$. In what follows we identify the optimality conditions of θ_1 and θ_2.

Minimality condition:

$$\theta_1 = \begin{cases} \theta_1^-, & \text{if } \rho_1\rho_2\cos(\theta_1^- - \theta_2) + \rho_1^2 \geq 0, \\ \theta_1^+, & \text{if } \rho_1\rho_2\cos(\theta_1^+ - \theta_2) + \rho_1^2 < 0 \end{cases}, \theta_2 = \begin{cases} \theta_2^-, & \text{if } \rho_1\rho_2\cos(\theta_1 - \theta_2^-) + \rho_2^2 \geq 0, \\ \theta_2^+, & \text{if } \rho_1\rho_2\cos(\theta_1 - \theta_2^+) + \rho_2^2 < 0. \end{cases}$$

Maximality condition:

$$\theta_1 = \begin{cases} \theta_1^+, & \text{if } \rho_1\rho_2\cos(\theta_1^+ - \theta_2) + \rho_1^2 \geq 0, \\ \theta_1^-, & \text{if } \rho_1\rho_2\cos(\theta_1^- - \theta_2) + \rho_1^2 < 0 \end{cases}, \theta_2 = \begin{cases} \theta_2^+, & \text{if } \rho_1\rho_2\cos(\theta_1 - \theta_2^+) + \rho_2^2 \geq 0, \\ \theta_2^-, & \text{if } \rho_1\rho_2\cos(\theta_1 - \theta_2^-) + \rho_2^2 < 0. \end{cases}$$

Without loss of generality, we will assume that $\rho_1^- \leq \rho_2^-$. We write $int(S_1 \oplus S_2)$ and $\partial(S_1 \oplus S_2)$ to denote the interior of $S_1 \oplus S_2$ and the boundary of $S_1 \oplus S_2$, respectively and split the problem into two cases.

Case 1. $[\rho_1] \cap \rho_2] = \emptyset$.

In this case neither $0 \in int(S_1 \oplus S_2)$ nor $0 \in \partial(S_1 \oplus S_2)$, i.e., $0 \notin S_1 \oplus S_2$. Also, since $\rho_1 < \rho_2$ for all $\rho_1 \in [\rho_1]$, $\rho_2 \in [\rho_2]$, we have $X_2 = \emptyset$ because $\frac{\partial \varphi}{\partial \theta_2} \neq 0$ for any $(\rho_1, \rho_2, \theta_1, \theta_2) \in \Omega$.

In fact we always have $\frac{\partial \varphi}{\partial \theta_2} > 0$, and accordingly φ reaches its minimum (maximum) value when θ_2 is minimum (maximum). Therefore we have

$$\varphi^- = \min\varphi(\rho_1, \rho_2, \theta_1, \theta_2^-),$$
$$\varphi^+ = \max\varphi(\rho_1, \rho_2, \theta_1, \theta_2^+).$$

Hence the optimal point falls in X_1 or X_3 and rules out X_2.

Points of type X_1

Suppose that $[\theta_1]$ is not a degenerate interval (i.e., $\theta_1^+ > \theta_1^-$). We have $\frac{\partial \varphi}{\partial \theta_1} = 0$ when, $\rho_1\rho_2\cos(\theta_1 - \theta_2) + \rho_1^2 = 0$, from which it follows that $\cos(\theta_1 - \theta_2) = \frac{-\rho_1}{\rho_2}$. Using the fact that, $\sin^2 + \cos^2 = 1$, we get $\sin(\theta_1 - \theta_2) = \pm\sqrt{1 - (\frac{\rho_1}{\rho_2})^2}$.

Since $\frac{\partial^2 \varphi}{\partial \theta_1^2} > 0$ when $\sin(\theta_1 - \theta_2) < 0$ and $\frac{\partial^2 \varphi}{\partial \theta_1^2} < 0$ when $\sin(\theta_1 - \theta_2) > 0$, it follows that φ is

minimum (maximum) when $\sin(\theta_1 - \theta_2) < 0$ ($\sin(\theta_1 - \theta_2) > 0$).
Thus, since $\frac{\partial \varphi}{\partial \rho_1} < 0$ and $\frac{\partial \varphi}{\partial \rho_2} > 0$ when $\sin(\theta_1 - \theta_2) < 0$, we conclude that the minimum of φ, among all points of X_1, is attainable at the point $x_{1\min} = (\rho_1^+, \rho_2^-, \theta_1, \theta_2^-)$ if $\sin(\theta_1 - \theta_2^-) < 0$, where θ_1 can be determined as follows

$$\theta_1 - \theta_2^- = \arctan\left(\frac{\sin(\theta_1 - \theta_2^-)}{\cos(\theta_1 - \theta_2^-)}\right) + \pi$$

$$= \arctan\left(\frac{-\sqrt{1 - (\frac{\rho_1^+}{\rho_2^-})^2}}{-\frac{\rho_1^+}{\rho_2^-}}\right) + \pi$$

i.e,

$$\theta_1 = \theta_2^- + \arctan\left(\frac{\sqrt{(\rho_2^-)^2 - (\rho_1^+)^2}}{\rho_1^+}\right) + \pi \pm 2\pi k,$$

where $k \in \{0,1\}$ (Throughout the rest of the paper k will be in $\{0,1\}$).
Similarly, assuming $\theta_2 = \theta_2^+$, the maximum of φ, among all points of X_1, is attainable at the point $x_{1\max} = (\rho_1^+, \rho_2^-, \theta_1, \theta_2^+)$ if $\sin(\theta_1 - \theta_2^+) > 0$, where θ_1 is given by

$$\theta_1 = \theta_2^+ - \arctan\left(\frac{\sqrt{(\rho_2^-)^2 - (\rho_1^+)^2}}{\rho_1^+}\right) + \pi \pm 2\pi k.$$

The above analysis can be summarized as follows.

1. If there exists $\theta_1 \in]\theta_1[$ such that $\cos(\theta_1 - \theta_2^-) = \frac{-\rho_1^+}{\rho_2^-}$ and $\sin(\theta_1 - \theta_2^-) < 0$, then,
$\varphi(x_{1\min}) \leq \varphi(x_1)$ for all $x_1 \in X_1$.

2. If there exists $\theta_1 \in]\theta_1[$ such that $\cos(\theta_1 - \theta_2^+) = \frac{-\rho_1^+}{\rho_2^-}$ and $\sin(\theta_1 - \theta_2^+) > 0$, then,
$\varphi(x_{1\max}) \geq \varphi(x_1)$ for all $x_1 \in X_1$.

Points of type X_3

In general, X_3 contains 16 points. Among these 16 points, we will determine the pair $x_{3\min}$ and $x_{3\max}$ such that $\varphi(x_{3\min}) \leq \varphi(x_3) \leq \varphi(x_{3\max})$ for all $x_3 \in X_3$.
To find the point $x_{3\min}$, recall that $\varphi^- = \min \varphi(\rho_1, \rho_2, \theta_1, \theta_2^-)$. Since θ_1 has two possible values (i.e., $\theta_1 \in \{\theta_1^+, \theta_1^-\}$), we have

$$x_{3\min} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^-), (\rho_1, \rho_2, \theta_1^+, \theta_2^-)\},$$

where at least one of θ_1^+ and θ_1^- must achieve the minimality condition. If both θ_1^+ and θ_1^- achieve the minimality condition, then,

$$x_{3\min} = \begin{cases} (\rho_1, \rho_2, \theta_1^-, \theta_2^-), & \text{if } \sin(\theta_1^- - \theta_2^-) < 0 \text{ and } \sin(\theta_1^+ - \theta_2^-) \geq 0, \\ (\rho_1, \rho_2, \theta_1^+, \theta_2^-), & \text{if } \sin(\theta_1^+ - \theta_2^-) < 0 \text{ and } \sin(\theta_1^- - \theta_2^-) \geq 0, \\ \min\{(\rho_1, \rho_2, \theta_1^-, \theta_2^-), (\rho_1, \rho_2, \theta_1^+, \theta_2^-)\}, & \text{otherwise} \end{cases}$$

where

$$\begin{cases} \rho_1 = \rho_1^+, \rho_2 = \rho_2^-, & \text{if } \sin(\theta_1 - \theta_2^-) \leq 0, \\ \rho_1 = \rho_1^-, \rho_2 = \rho_2^+ & \text{otherwise.} \end{cases}$$

Similarly, by fixing θ_2 at θ_2^+, we have, $x_{3\max} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^+), (\rho_1, \rho_2, \theta_1^+, \theta_2^+)\}$, where $\theta_1 \in \{\theta_1^+, \theta_1^-\}$ is determined according to the maximality condition. If both θ_1^+ and θ_1^- achieve the maximality condition, then,

$$x_{3\max} = \begin{cases} (\rho_1, \rho_2, \theta_1^-, \theta_2^+), & \text{if } \sin(\theta_1^- - \theta_2^+) > 0 \text{ and } \sin(\theta_1^+ - \theta_2^+) \leq 0, \\ (\rho_1, \rho_2, \theta_1^+, \theta_2^+), & \text{if } \sin(\theta_1^+ - \theta_2^+) > 0 \text{ and } \sin(\theta_1^- - \theta_2^+) \leq 0, \\ \min\{(\rho_1, \rho_2, \theta_1^-, \theta_2^+), (\rho_1, \rho_2, \theta_1^+, \theta_2^+)\} & \text{otherwise} \end{cases}$$

where

$$\begin{cases} \rho_1 = \rho_1^+, \rho_2 = \rho_2^-, & \text{if } \sin(\theta_1 - \theta_2^+) \geq 0, \\ \rho_1 = \rho_1^-, \rho_2 = \rho_2^+, & \text{otherwise.} \end{cases}$$

Special subcase of Case1

There remains a case in which the optimal bounds of φ can not be determined in the usual ways. Figure 4 demonstrates two examples.
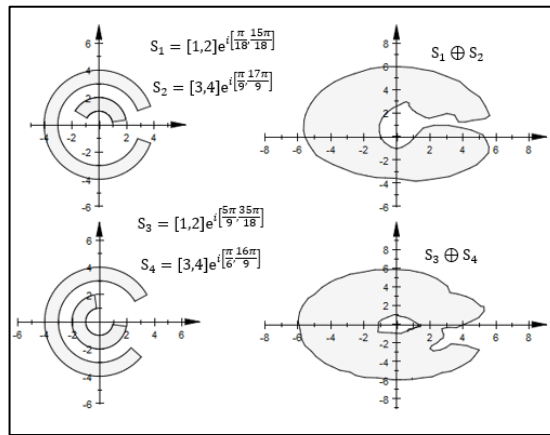

Figure 4. *Case of annulus*

It is clear that any sector, which encloses the set $S_1 \oplus S_2$ or $S_3 \oplus S_4$ is an annulus and has an angle interval $[\varphi^-, \varphi^+] = [0, 2\pi]$. In this special case, one can check that one of the following holds:

- $(\rho_1, \rho_2, \theta_1^-, \theta_2^-)$ and $(\rho_1, \rho_2, \theta_1^+, \theta_2^-)$ are both candidates for φ^- with, $|\varphi(\rho_1, \rho_2, \theta_1^-, \theta_2^-) - \varphi(\rho_1, \rho_2, \theta_1^+, \theta_2^-)| > \pi$,

- $(\rho_1, \rho_2, \theta_1^-, \theta_2^+)$ and $(\rho_1, \rho_2, \theta_1^+, \theta_2^+)$ are both candidates for φ^+ with, $|\varphi(\rho_1, \rho_2, \theta_1^-, \theta_2^+) - \varphi(\rho_1, \rho_2, \theta_1^+, \theta_2^+)| > \pi$,

- $\varphi^+ - \varphi^- < \dfrac{\theta_2^+ - \theta_2^-}{2}$.

Case 2. $[\rho_1] \cap [\rho_2] \neq \emptyset$.

In this case it may happen that $\rho_1 \cos(\theta_1) + \rho_2 \cos(\theta_2) = 0$ and $\rho_1 \sin(\theta_1) + \rho_2 \sin(\theta_2) = 0$. This clearly happens when either $\rho_1 = \rho_2 = 0$; or $\rho_1 = \rho_2$ and $\theta_1 - \theta_2 = \pm(2k+1)\pi$. When this happens, we have either $0 \in \partial(S_1 \oplus S_2)$ or $0 \in int(S_1 \oplus S_2)$. The case of $0 \in int(S_1 \oplus S_2)$ needs special attention because φ^- and φ^+ cannot be determined in the usual way. That $0 \in int(S_1 \oplus S_2)$ occurs in the following two situations.

1. $[\rho_1] \cap \rho_2] = \rho_1^+ = \rho_2^-$ and both $-(2k+1)\pi \in]\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-[$ and

$(2k + 1)\pi \in]\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-[$ provided that $[\rho_1]$ and $[\rho_2]$ are not both degenerate (thin) intervals. Figure 5 shows two examples.
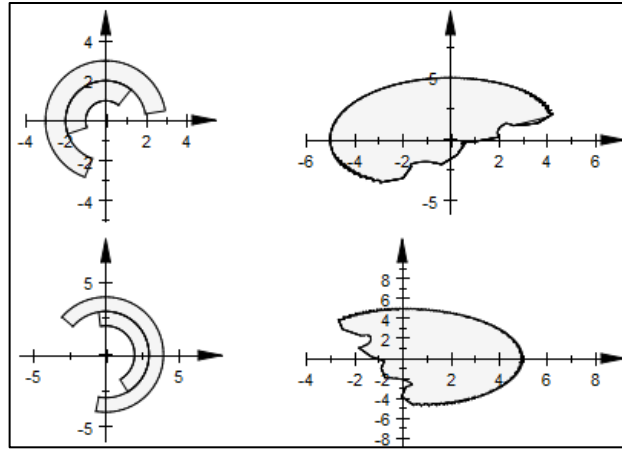

Figure 5. *The origin as an interior point*

2. $]\rho_1[\cap]\rho_2[\neq \emptyset$ and either $-(2k + 1)\pi \in]\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-[$ or $(2k + 1)\pi \in]\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-[$. It is obvious that under these conditions $\varphi^- = 0, \varphi^+ = 2\pi$.

Now, we proceed to determine φ^- and φ^+ in cases where $0 \notin int(S_1 \oplus S_2)$. We consider the following two subcases.

Subcase 2.1. $\rho_1^+ = \rho_2^-$.

Similar to Case 1, we have $\frac{\partial \varphi}{\partial \theta_2} > 0$ for any $(\rho_1, \rho_2, \theta_1, \theta_2)$ in the domain of φ. Therefore,

$$\varphi^- = \min\varphi(\rho_1, \rho_2, \theta_1, \theta_2^-),$$
$$\varphi^+ = \max\varphi(\rho_1, \rho_2, \theta_1, \theta_2^+).$$

We distinguish two cases: $0 \in \partial(S_1 \oplus S_2)$ and $0 \notin \partial(S_1 \oplus S_2)$.

• $0 \in \partial(S_1 \oplus S_2)$.

This case occurs when either $-(2k + 1)\pi \in \theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-]$ or $(2k + 1)\pi \in \theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-]$. Figure 6 shows two examples.
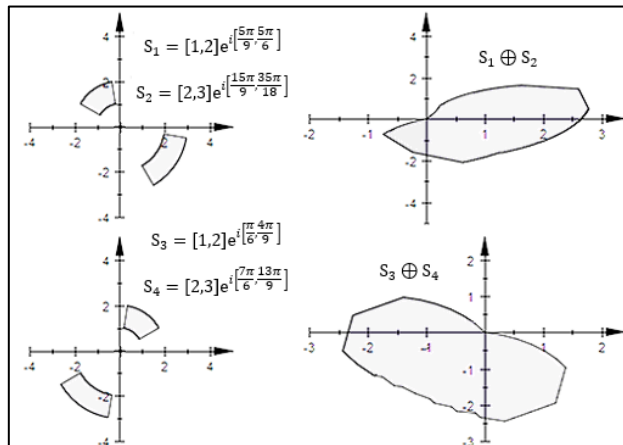

Figure 6. *The origin as a boundary point*

Notice that φ is undefined at $x = (\rho_1^+, \rho_2^-, \theta_1, \theta_2)$ when $\theta_1 - \theta_2 = \pm(2k+1)\pi$. Let us investigate how φ behaves near the point x. Consider the function

$$h_1(\theta_1, \theta_2) = \varphi(\rho_1^+, \rho_2^-, \theta_1, \theta_2) = \arctan\left(\frac{\sin(\theta_1)+\sin(\theta_2)}{\cos(\theta_1)+\cos(\theta_2)}\right),$$

where $\theta_1 \in [\theta_1]$ and $\theta_2 \in [\theta_2]$.

We observe that the function h_1 reaches its minimum value when $|\theta_1 - \theta_2|$ approaches π from the right-hand side, and it reaches its maximum value when $|\theta_1 - \theta_2|$ approaches π from the left-hand side. Assuming that $\theta_2 = \theta_2^-$ we compute φ^- as follows. If there exists $\theta_1 \in]\theta_1[$ such that, $\theta_1^- = \theta_2^- \pm (2k+1)\pi$, then $\sin(\theta_1 - \theta_2^-) < 0$, which makes $\frac{\partial\varphi}{\partial\rho_1} < 0$ and $\frac{\partial\varphi}{\partial\rho_2} > 0$, we compute the minimum of φ using $\rho_1 = \rho_1^+$ and $\rho_2 = \rho_2^-$. That is φ achieves its minimum at the point, $x_{1\min} = (\rho_1^+, \rho_2^-, \theta_1, \theta_2^-) \in X_1$, i.e.,

$$\varphi^- = \varphi(\rho_1^+, \rho_2^-, \theta_1, \theta_2^-) = \arctan\left(\frac{\sin(\theta_1)+\sin(\theta_2^-)}{\cos(\theta_1)+\cos(\theta_2^-)}\right).$$

If $\theta_2^- + \epsilon \pm (2k+1)\pi \notin]\theta_1[$ then clearly φ achieves its minimum at $x_{3\min} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^-), (\rho_1, \rho_2, \theta_1^+, \theta_2^-)\} \subset X_3$, where $x_{3\min}$ is selected as in Case 1 provided that $\rho_1 = \rho_1^-$ and $\rho_2 = \rho_2^+$ when $\theta_1^+ - \theta_2^- = \pm(2k+1)\pi$.
Let us now compute φ^+. If there exists $\theta_1 \in]\theta_1[$ such that $\theta_1 = \theta_2^+ - \epsilon \pm (2k+1)\pi$, then, $\sin(\theta_1 - \theta_2^+) > 0$, which makes $\frac{\partial\varphi}{\partial\rho_1} > 0$ and $\frac{\partial\varphi}{\partial\rho_2} < 0$, we compute the maximum of φ using $\rho_1 = \rho_1^+$ and $\rho_2 = \rho_2^-$. That is φ achieves its maximum at the point $x_{1\max} = (\rho_1^+, \rho_2^-, \theta_1, \theta_2^+) \in X_1$, i.e.,

$$\varphi^+ = \varphi(\rho_1^+, \rho_2^-, \theta_1, \theta_2^+) = \arctan\left(\frac{\sin(\theta_1)+\sin(\theta_2^+)}{\cos(\theta_1)+\cos(\theta_2^+)}\right).$$

If $\theta_2^+ - \epsilon \pm (2k+1)\pi \notin]\theta_1[$, then φ achieves its maximum at, $x_{3\max} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^+), (\rho_1, \rho_2, \theta_1^+, \theta_2^+)\} \subset X_3$, where $x_{3\max}$ is selected as in Case 1 provided that $\rho_1 = \rho_1^-$ and $\rho_2 = \rho_2^+$ when $\theta_1^- - \theta_2^+ = \pm(2k+1)\pi$.

We end this case with the following remark.

Remark 3 *After determining φ^- and φ^+, we have to check whether $\varphi^+ - \varphi^- < \frac{\theta_2^+ - \theta_2^-}{2}$. If that is the case, we set $\varphi^- = 0, \varphi^+ = 2\pi$.*

• $0 \notin \partial(S_1 \oplus S_2)$.

This case occurs when $\theta_1 - \theta_2 \neq \pm(2k+1)\pi$. Consequently,

$$\varphi^- = \varphi(x_{3\min}),$$
$$\varphi^+ = \varphi(x_{3\max}).$$

In other words, φ achieves its maximum and minimum at points of type X_3. In order to prove this, suppose that $X_1 \neq \emptyset$ with $x_{1\min}, x_{1\max} \in X_1$ such that $\varphi(x_{1\min}) \leq \varphi(x_1) \leq \varphi(x_{1\max})$ for all $x_1 \in X_1$. We want to show that

$$\varphi(x_{3\min}) < \varphi(x_{1\min}),$$
$$\varphi(x_{3\max}) > \varphi(x_{1\max}).$$

To prove $\varphi(x_{3\min}) < \varphi(x_{1\min})$, we have $x_{1\min} = (\rho_1, \rho_2, \alpha, \theta_2^-)$, where $\alpha \in]\theta_1[$ such that, $\cos(\alpha - \theta_2^-) = \frac{-\rho_1}{\rho_2}$. If $\sin(\alpha - \theta_2^-) < 0$, then $\frac{\partial\varphi}{\partial\rho_1} < 0$ and $\frac{\partial\varphi}{\partial\rho_2} > 0$, i.e., $x_{1\min} = (\rho_1^+, \rho_2^-, \alpha, \theta_2^-)$. But such a point does not exist in the domain of φ. If $\sin(\alpha - \theta_2^-) > 0$ we have $\frac{\partial\varphi}{\partial\rho_1} > 0$ and $\frac{\partial\varphi}{\partial\rho_2} < 0$, i.e., $x_{1\min} = (\rho_1^-, \rho_2^+, \alpha, \theta_2^-)$. Consider the function,

$$h_2(\theta_1) = \varphi(\rho_1^-, \rho_2^+, \theta_1, \theta_2^-) = \arctan\left(\frac{\rho_1^- \sin(\theta_1) + \rho_2^+ \sin(\theta_2^-)}{\rho_1^- \cos(\theta_1) + \rho_2^+ \cos(\theta_2^-)}\right).$$

Considering the second derivative, we see that for $\sin(\alpha - \theta_2^-) > 0$ we have

$$\frac{\partial^2 h_2}{\partial\theta_1^2} = \frac{\partial^2\varphi(x_{1\min})}{\partial\theta_1^2} < 0.$$

So the graph of h_2 is concave downward about $\theta_1 = \alpha$. Therefore, based on the above discussion we conclude that the minimum of h_2 achieves at one of the end points of $[\theta_1]$. Consequently we have $\varphi(x_{3\min}) < \varphi(x_{1\min})$. In the same manner, we can show that $\varphi(x_{3\max}) > \varphi(x_{1\max})$.

Subcase 2.2. $]\rho_1[\cap]\rho_2[\neq \emptyset$.

In this case we have $\rho_1^+ > \rho_2^-$ and the only places where φ can have extreme values are points of type X_3, or in other words

$$\varphi(x_{3\min}) < \begin{cases} \varphi(x_1), & \text{for all } x_1 \in X_1, \\ \varphi(x_2), & \text{for all } x_2 \in X_2, \end{cases} \text{ and } \varphi(x_{3\max}) > \begin{cases} \varphi(x_1), & \text{for all } x_1 \in X_1, \\ \varphi(x_2), & \text{for all } x_2 \in X_2. \end{cases}$$

To see that $\varphi(x_{3\min}) < \varphi(x_1)$, suppose that there exists $\alpha \in]\theta_1[$ such that $\cos(\alpha - \theta_2) = \frac{-\rho_1}{\rho_2}$. If $\sin(\alpha - \theta_2) > 0$, then $x_1 = (\rho_1^-, \rho_2^+, \alpha, \theta_2)$. So $\frac{\partial^2\varphi(x_1)}{\partial\theta_1^2} < 0$, showing that $\varphi(x_{3\min}) < \varphi(x_1)$. Now suppose that $\sin(\alpha - \theta_2) < 0$, then $\rho_1 = \rho_1^+$ and $\rho_2 = \rho_2^-$. It follows that $\cos(\alpha - \theta_2) = \frac{-\rho_1^+}{\rho_2^-} < -1$, which is impossible.

To see that $\varphi(x_{3\max}) > \varphi(x_1)$, suppose that there exists $\beta \in]\theta_1[$ such that $\cos(\beta - \theta_2) = \frac{-\rho_1}{\rho_2}$. If $\sin(\beta - \theta_2) < 0$, then $x_1 = (\rho_1^-, \rho_2^+, \beta, \theta_2)$. So $\frac{\partial^2\varphi(x_1)}{\partial\theta_1^2} > 0$, showing that $\varphi(x_{3\max}) > \varphi(x_1)$.

If $\sin(\beta - \theta_2) > 0$, we have $\cos(\beta - \theta_2) = \frac{-\rho_1^+}{\rho_2^-} < -1$, which is impossible.

In the same way we can show that $\varphi(x_{3\min}) < \varphi(x_2)$ and $\varphi(x_{3\max}) > \varphi(x_2)$ for any $x_2 \in X_2$. Thus to compute φ^- and φ^+, it is sufficient to find $x_{3\min}$ and $x_{3\max}$. We have,

$$x_{3\min} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^+), (\rho_1, \rho_2, \theta_1^-, \theta_2^-), (\rho_1, \rho_2, \theta_1^+, \theta_2^-)\},$$
$$x_{3\max} \in \{(\rho_1, \rho_2, \theta_1^-, \theta_2^+), (\rho_1, \rho_2, \theta_1^+, \theta_2^+), (\rho_1, \rho_2, \theta_1^+, \theta_2^-)\}.$$

We will show that $x_{3\min} \neq (\rho_1, \rho_2, \theta_1^+, \theta_2^+)$. If $\sin(\theta_1^+ - \theta_2^+) < 0$, then $\rho_1 = \rho_1^+$ and $\rho_2 = \rho_2^-$. Since $\rho_1^+ > \rho_2^-$, we have $\frac{\partial\varphi}{\partial\theta_1} > 0$, which violates the minimality condition for $\theta_1 = \theta_1^+$. If $\sin(\theta_1^+ - \theta_2^+) > 0$, then $\rho_1 = \rho_1^-$ and $\rho_2 = \rho_2^+$. Since $\rho_1^- < \rho_2^+$, we have $\frac{\partial\varphi}{\partial\theta_2} > 0$, which violates the minimality condition with respect to $\theta_2 = \theta_2^+$.

In the same way we can show that $x_{3\max} \neq (\rho_1, \rho_2, \theta_1^-, \theta_2^-)$.
Before introducing algorithms for computing φ^- and φ^+ we remind that if $\varphi^+ < \varphi^-$, we just add 2π to φ^+.

We summarize all the results of the discussion in the following two algorithms:

Algorithm 1 (φ^-):	**Algorithm 2** (φ^+):
if $0 \in int(S_1 \oplus S_2)$	if $0 \in int(S_1 \oplus S_2)$
$$\varphi^- = 0$$	$$\varphi^+ = 2\pi$$
else if $]\rho_1[\cap]\rho_2[\neq \emptyset$	else if $]\rho_1[\cap]\rho_2[\neq \emptyset$
$$\varphi^- = \varphi(x_{3\min})$$	$$\varphi^+ = \varphi(x_{3\max})$$
else if $\rho_1^+ = \rho_2^-$	else if $\rho_1^+ = \rho_2^-$
if $\theta_2^- + \epsilon \pm (2k+1)\pi \in]\theta_1[$	if $\theta_2^+ - \epsilon \pm (2k+1)\pi \in]\theta_1[$
$\varphi^- = \varphi(x_{1\min})$	$\varphi^+ = \varphi(x_{1\max})$
else	else
$\varphi^- = \varphi(x_{3\min})$	$\varphi^+ = \varphi(x_{3\max})$
end	end
else	else
if $\|\varphi(\rho_1,\rho_2,\theta_1^-,\theta_2^-) - \varphi(\rho_1,\rho_2,\theta_1^+,\theta_2^-)\| > \pi$	if $\|\varphi(\rho_1,\rho_2,\theta_1^-,\theta_2^+) - \varphi(\rho_1,\rho_2,\theta_1^+,\theta_2^+)\| > \pi$
$\varphi^- = 0$	$\varphi^+ = 2\pi$
elseif $\theta_2^- + \arctan\left(\frac{\sqrt{(\rho_2^-)^2-(\rho_1^+)^2}}{\rho_1^+}\right) + \pi \pm 2\pi k \in]\theta_1[$ and $\sin(\theta_1 - \theta_2^-) < 0$	elseif $\theta_2^+ - \arctan\left(\frac{\sqrt{(\rho_2^-)^2-(\rho_1^+)^2}}{\rho_1^+}\right) + \pi \pm 2\pi k \in]\theta_1[$ and $\sin(\theta_1 - \theta_2^+) > 0$
$\varphi^- = \varphi(x_{1\min})$	$\varphi^+ = \varphi(x_{1\max})$
else	else
$\varphi^- = \varphi(x_{3\min})$	$\varphi^+ = \varphi(x_{3\max})$
end	end
end	end

3.2. Computing The Bounds of Magnitude Interval $[\rho]$

Recall that,

$$\rho = \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos(\theta)}$$

where $\theta = \theta_1 - \theta_2 \in [\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-]$.

Since the square root function is an increasing function, we can minimize (maximize) ρ by minimizing (maximizing) $h = \rho^2$, so the function that we are going to determine its extremes is,

$$h: D \to (0, \infty), \text{ where } D = [\rho_1] \times [\rho_2] \times [\theta].$$

We aim to find two points $v_{\min}, v_{\max} \in D$ such that $\min h = h(v_{\min}), \max h = h(v_{\max})$. The extremes of h are located where $\nabla h = 0$, that means,

$$\frac{\partial h}{\partial \theta} = -2\rho_1\rho_2\sin(\theta) = 0, \frac{\partial h}{\partial \rho_1} = 2\rho_1 + 2\rho_2\cos(\theta) = 0, \frac{\partial h}{\partial \rho_2} = 2\rho_2 + 2\rho_1\cos(\theta) = 0.$$

The analysis after here is routine compared with the angle optimization. We ommit the proof and propose the following two improved algorithms for computing $\max h$ and $\min h$ respectively.

Algorithm 3 *(max ($h = \rho^2$))*: if $b \geq 0$ $maxh = h(\rho_1^+, \rho_2^+, \theta)$ else $maxh$ $= \max(h(\rho_1^-, \rho_2^+, \theta), h(\rho_1^+, \rho_2^-, \theta), h(\rho_1^+, \rho_2^+, \theta))$ end	**Algorithm 4** *(min ($h = \rho^2$))*: if $a \geq 0$ $minh = h(\rho_1^-, \rho_2^-, \theta)$ else if $\rho_1 = \rho_2 = 0$ or $[\rho_1] \cap \rho_2] \neq \emptyset$ and $\pm(2k+1)\pi \in [\theta_1^- - \theta_2^+, \theta_1^+ - \theta_2^-]$ $minh = 0$ else if $\rho_1^* = -\rho_2^- a \in]\rho_1[$ $minh = h(\rho_1^*, \rho_2^-, \theta)$ else $minh$ $= \min(h(\rho_1^-, \rho_2^-, \theta), h(\rho_1^-, \rho_2^+, \theta), h(\rho_1^+, \rho_2^-, \theta))$ end

4. NUMERICAL EXAMPLES

In this section, we provide numerical examples to show the efficiency and robustness of the proposed algorithms. We also compare our algorithms with the existing algorithms in [6].

Example 1. Consider the sectors,

$$S_1 = [2,3]e^{[\frac{5\pi}{9}, \frac{4\pi}{3}]}, \ S_2 = [3,5]e^{[0, \frac{5\pi}{6}]}$$

Using our algorithms we obtain, $[\rho] = [0,8], \ [\varphi] = [4.71674, 9.68658]$ (Figure 7).
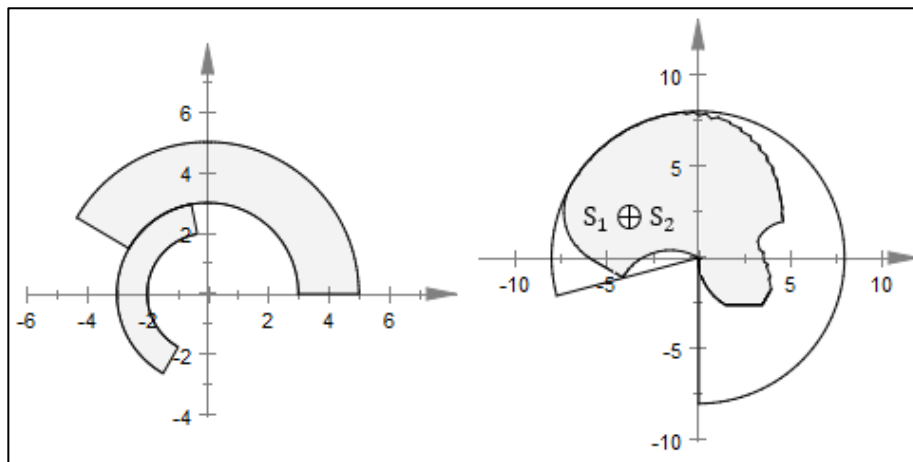

Figure 7. *Graphs of Example 1*

If we use the introduced algorithms in [7] we obtain $[\rho] = [0,8], \ [\varphi] = [0.4, 3.4034]$, a result that is not correct.

Example 2. Consider the sectors,

$$S_1 = [2,3.5]e^{[\frac{5\pi}{9}, \frac{11\pi}{9}]}, \ S_2 = [4,5]e^{[\frac{\pi}{18}, \frac{5\pi}{6}]}$$

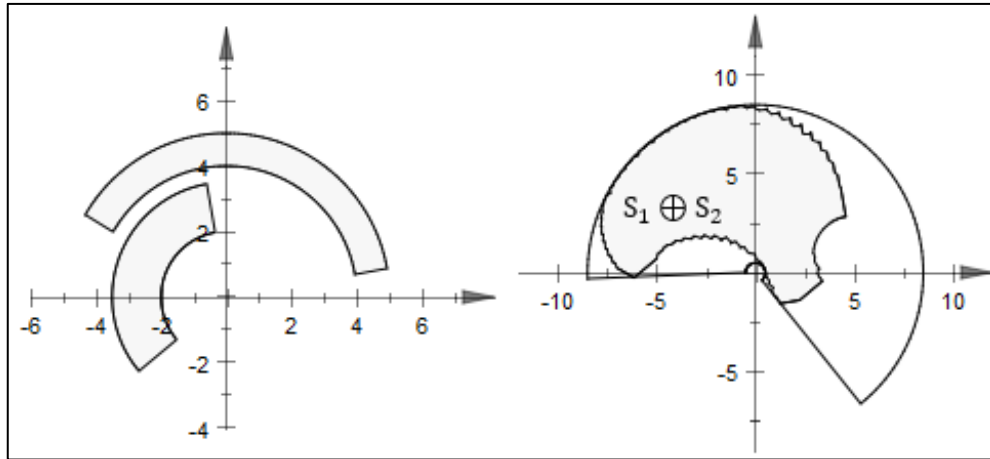Using our algorithms we obtain, $[\rho] = [0.5, 8.5], [\varphi] = [5.39228, 9.4654]$ (Figure 8).

Figure 8. *Graphs of Example 2*

Using the proposed algorithms in [7] we obtain $[\rho] = [0.5, 8.5]$ and $[\varphi] = [0.555, 3.1822]$, which is also not optimal.

5. CONCLUSION

From the results observed in the numerical examples, we conclude that the algorithms in [7] may fail to find the the optimal bounds for φ. As an alternative we developed simple and efficient algorithms to perform the addition (subtraction) operation of sectors. Numerical results demonstrate the efficiency, robustness and accuracy of the proposed algorithms. The algorithms are implemented in MATLAB R2020a environment and tested for all possible cases. For each case the result is computed for 12,960,000 sample points and compared with the outputs of the proposed algorithms. Figure 9 shows the user interface of the computer implementation.
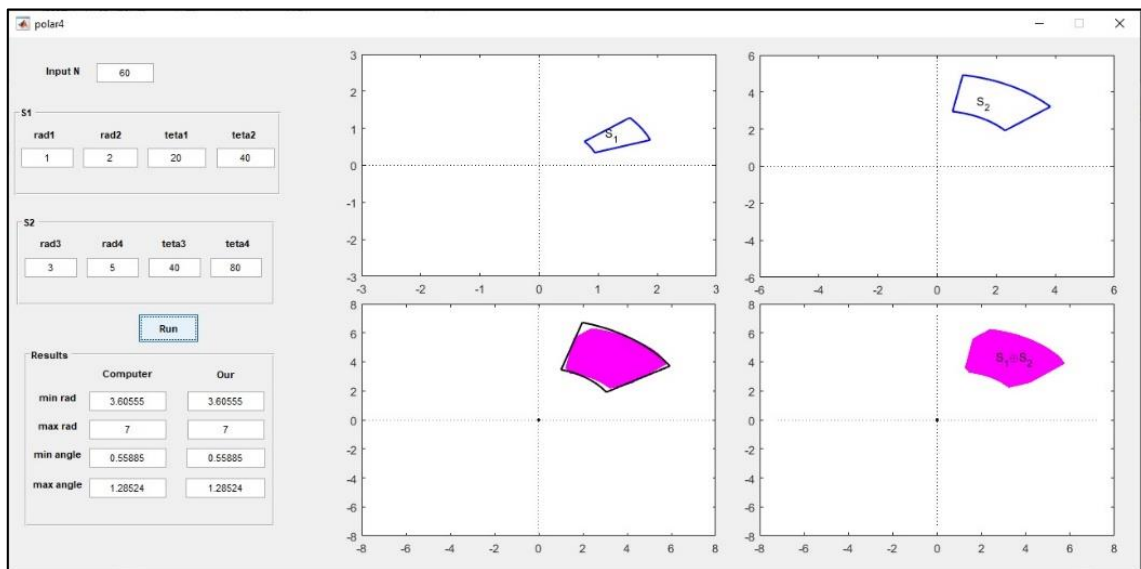

Figure 9. *User Interface of the implemented Algorithms*

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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