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Integral Inequalities for Different Kinds of Convexity via Classical Inequalities

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Abstract. The main purpose of this study is to prove new integral inequalities for product of different classes of convex functions via some classical inequalities such as general Cauchy inequality and reverse Minkowski inequality.

1. INTRODUCTION

The function $f:[a,b]\rightarrow\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. This definition is well-known in the literature and a huge amount of the researchers interested in this definition. We can define starshaped functions on $[0,b]$ which satisfy the condition

$$f(tx)\leq tf(x)$$

for $t\in[0,1]$.

Because of the importance of convex functions in inequality theory, integral inequalities including convex function classes have an important place in the literature of mathematical inequalities. Especially in recent years, many researchers have done many studies in this field. Interested readers can find different aspects of this subjects in references.

The concept of $m-$convexity has been introduced by Toader in [5], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1.1. *The function $f:[0,b]\rightarrow\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

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Several papers have been written on $m-$convex functions and we refer the papers [1], [2], [3], [7], [8] and [9].

In [4], Miheşan gave definition of $(\alpha, m)-$convexity as following;

Definition 1.2. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0, b]$ for which $f(0) \leq 0$. If we choose $(\alpha, m) = (1, m)$, it can be easily seen that $(\alpha, m)-$convexity reduces to $m-$convexity and for $(\alpha, m) = (1, 1)$, we have ordinary convex functions on $[0, b]$. In [6], Set *et al.* proved some inequalities related to $(\alpha, m)-$convex functions.

The following inequality which well known in the literature as Minkowski inequality is given as;

Let $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, and $0 < \int_a^b g(x)^p dx < \infty$. Then

$$\left(\int_a^b (f(x) + g(x))^p \, dx \right)^{\frac{1}{p}} \leq \left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}}. \tag{1}$$

The reverse of this inequality was given by Bougoffa in [16], as the following;

Theorem 1.3. *Let f and g be positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \, [a, b].$$

Then

$$\left(\int_a^b f(x)^p dx \right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx \right)^{\frac{1}{p}} \leq c \left(\int_a^b (f(x) + g(x))^p \, dx \right)^{\frac{1}{p}}. \tag{2}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Definition 1.4. *[See [10]] Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be an $s-$convex function in the second sense if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{3}$$

for all $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$.

In [11], $s-$convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function in [12]. Several properties of $s-$convexity in the first sense are discussed in the paper [10]. Obviously, $s-$convexity means just convexity when $s = 1$.

Theorem 1.5. *[See [14]] Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[0, 1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (4). The above inequalities are sharp.

Some new Hermite-Hadamard type inequalities based on concavity and $s-$convexity established by Kırmacı *et al.* in [15]. For related results see the papers [13], [14] and [15].

This paper organized as follows.

In Section 2, we prove some inequalities for $m-$convex and $s-$convex functions and in Section 3, we give some new inequalities for $(\alpha, m)-$convex functions by using some classical inequalities and fairly elementary analysis.

2. RESULTS FOR $m-$CONVEX AND $s-$CONVEX FUNCTIONS

We will start with the following Theorem which is involving $m-$convex functions.

Theorem 2.1. *Suppose that $f, g : [a,b] \to [0,\infty), 0 \le a < b < \infty$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0,1]$. If $f, g \in L_1[a,b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\,dx \le \frac{1}{3}\left[f(b) + m_2 g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[g(b) + m_1 f\left(\frac{a}{m_1}\right)\right]. \tag{5}$$

Proof. From m_1-convexity and m_2-convexity of f and g, we can write

$$f^t(tb + (1-t)a) \le \left[tf(b) + m_1(1-t) f\left(\frac{a}{m_1}\right)\right]^t$$

and

$$g^{(1-t)}(tb + (1-t)a) \le \left[tg(b) + m_2(1-t) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.$$

Since f, g are non-negative, we have

$$\begin{aligned} & f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a) \\ \le\ & \left[tf(b) + m_1(1-t) f\left(\frac{a}{m_1}\right)\right]^t \left[tg(b) + m_2(1-t) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}. \end{aligned} \tag{6}$$

Recall the General Cauchy Inequality (see [17], Theorem 3.1), let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for every positive real numbers x and y, we always have

$$\alpha x + \beta y \ge x^\alpha y^\beta.$$

By using the General Cauchy Inequality in (6), we get

$$\begin{aligned} & f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a) \\ \le\ & t\left[tf(b) + m_1(1-t) f\left(\frac{a}{m_1}\right)\right] + (1-t)\left[tg(b) + m_2(1-t) g\left(\frac{a}{m_2}\right)\right]. \end{aligned}$$

By integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} & \int_0^1 f^t(tb + (1-t)a)\, g^{(1-t)}(tb + (1-t)a)\,dt \\ \le\ & \frac{1}{3}\left[f(b) + m_2 g\left(\frac{a}{m_2}\right)\right] + \frac{1}{6}\left[g(b) + m_1 f\left(\frac{a}{m_1}\right)\right]. \end{aligned}$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. □

Corollary 2.2. *If we choose $m_1 = m_2 = 1$ in Theorem 3, we have the inequality;*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \le \frac{1}{3}\left[f(b)+g(a)\right]+\frac{1}{6}\left[g(b)+f(a)\right].$$

Another result for $m-$convex functions is emboided in the following Theorem.

Theorem 2.3. *Suppose that $f, g : [0, b] \to \mathbb{R}$, $b > 0$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0, 1]$. If $f \in L_1[a, b]$, then the following inequality holds:*

$$\begin{aligned} & \frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)\,dx + m_2\frac{g\left(\frac{a}{m_2}\right)}{(b-a)^2}\int_a^b (b-x)f(x)\,dx \\ & +\frac{f(b)}{(b-a)^2}\int_a^b (x-a)g(x)\,dx + m_1\frac{f\left(\frac{a}{m_1}\right)}{(b-a)^2}\int_a^b (b-x)g(x)\,dx \\ \le\ & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{3}f(b)\,g(b) + \frac{m_1}{6}f\left(\frac{a}{m_1}\right)g(b) \\ & +\frac{m_2}{6}f(b)\,g\left(\frac{a}{m_2}\right) + \frac{m_1m_2}{3}f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right). \end{aligned} \tag{7}$$

Proof. Since f and g are m_1-convex and m_2-convex functions, respectively, we can write

$$f(tb+(1-t)a) \le tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right)$$

and

$$g(tb+(1-t)a) \le tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right).$$

By using the elementary inequality, $e \le f$ and $p \le r$, then $er + fp \le ep + fr$ for $e, f, p, r \in \mathbb{R}$, then we get

$$\begin{aligned} & f(tb+(1-t)a)\left[tg(b)+m_2(1-t)g\left(\frac{a}{m_2}\right)\right] \\ & +g(tb+(1-t)a)\left[tf(b)+m_1(1-t)f\left(\frac{a}{m_1}\right)\right] \\ \le\ & f(tb+(1-t)a)\,g(tb+(1-t)a) \\ & +\left[tg(b)+m_2(1-t)g\left(\frac{a}{m_2}\right)\right]\left[tf(b)+m_1(1-t)f\left(\frac{a}{m_1}\right)\right]. \end{aligned}$$

So, we obtain

$$\begin{aligned} & tf(tb+(1-t)a)\,g(b) + m_2(1-t)f(tb+(1-t)a)\,g\left(\frac{a}{m_2}\right) \\ & +tf(b)\,g(tb+(1-t)a) + m_1(1-t)f\left(\frac{a}{m_1}\right)g(tb+(1-t)a) \\ \le\ & f(tb+(1-t)a)\,g(tb+(1-t)a) + t^2f(b)\,g(b) + m_1t(1-t)f\left(\frac{a}{m_1}\right)g(b) \\ & +m_2t(1-t)f(b)\,g\left(\frac{a}{m_2}\right) + m_1m_2(1-t)^2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right). \end{aligned}$$

By integrating this inequality with respect to t over $[0, 1]$ and by using the change of the variable $tb+(1-t)a = x$, $(b-a)dt = dx$, the proof is completed. □

Corollary 2.4. *If we choose $m_1 = m_2 = 1$ in Theorem 4, we have the inequality;*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^2}\int_a^b (x-a)f(x)\,dx + \frac{g(a)}{(b-a)^2}\int_a^b (b-x)f(x)\,dx \\
&+\frac{f(b)}{(b-a)^2}\int_a^b (x-a)g(x)\,dx + \frac{f(a)}{(b-a)^2}\int_a^b (b-x)g(x)\,dx \\
\leq\ &\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b).
\end{aligned}$$

Following inequality also holds for $m-$convex functions.

Theorem 2.5. *Suppose that $f,g : [a,b] \to [0,\infty), 0 \leq a < b < \infty$, are m_1-convex and m_2-convex functions, respectively, where $m_1, m_2 \in (0,1]$. If $f,g \in L_1[a,b]$ and f,g satisfy following condition*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a,b]$$

then the following inequality holds:

$$\begin{aligned}
&\frac{1}{c}\left[\left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}}\right] \\
\leq\ &\left(\frac{2^{p-1}(b-a)}{p+1}\right)^{\frac{1}{p}}\left([f(b)+g(b)]^p - \left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right)^{\frac{1}{p}}
\end{aligned}$$

where $c = \frac{M(m+1)+(M+1)}{(m+1)(M+1)}$ and $p \geq 1$.

Proof. Since f and g are m_1-convex and m_2-convex functions, respectively, we can write

$$f(tb+(1-t)a) \leq tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \tag{8}$$

and

$$g(tb+(1-t)a) \leq tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right). \tag{9}$$

By adding (8) and (9), we get

$$\begin{aligned}
f(tb+(1-t)a) + g(tb+(1-t)a) \leq\ & tf(b) + m_1(1-t)f\left(\frac{a}{m_1}\right) \\
& + tg(b) + m_2(1-t)g\left(\frac{a}{m_2}\right).
\end{aligned} \tag{10}$$

For $p \geq 1$, taking $p-$th power of both sides of the inequality (10) and by using the elementary inequality, $(c+d)^p \leq 2^{p-1}(c^p + d^p)$, then we get

$$\begin{aligned}
&[f(tb+(1-t)a) + g(tb+(1-t)a)]^p \\
\leq\ & 2^{p-1}\left(t^p[f(b)+g(b)]^p + (1-t)^p\left[m_1 f\left(\frac{a}{m_1}\right) + m_2 g\left(\frac{a}{m_2}\right)\right]^p\right).
\end{aligned}$$

Integrating with respect to t over $[0,1]$ and by using the change of the variable $tb+(1-t)a=x$ and $(b-a)dt=dx$, we obtain

$$\frac{1}{b-a}\int_a^b (f(x)+g(x))^p\,dx \leq \frac{2^{p-1}}{p+1}\left([f(b)+g(b)]^p-\left[m_1 f\left(\frac{a}{m_1}\right)+m_2 g\left(\frac{a}{m_2}\right)\right]^p\right). \tag{11}$$

By taking $\frac{1}{p}$−th power of both sides of the inequality (11) and by using the inequality (2), we get the desired inequality. Which completes the proof. □

We will give an inequality for s−convex functions in the following theorem. In the next theorem we will also make use of the Beta function of Euler type, which is for $x,y>0$ defined as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Theorem 2.6. *Suppose that $f,g:[0,\infty)\to[0,\infty)$ are s_1−convex and s_2−convex functions, respectively, where $s_1,s_2\in[0,1]$. Then the following inequality holds:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\,g^{\frac{b-x}{b-a}}(x)\,dx &\leq \frac{1}{s_1+2}f(b)+\beta(2,s_1+1)f(a)\\ &+\frac{1}{s_2+2}g(b)+\beta(2,s_2+1)g(a).\end{aligned}$$

Proof. Since f and g are s_1−convex and s_2−convex functions, respectively, we can write

$$f^t(tb+(1-t)a)\leq\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]^t$$

and

$$g^{(1-t)}(tb+(1-t)a)\leq\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right]^{(1-t)}.$$

Since f,g are non-negative, we have

$$\begin{aligned}&f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\\ \leq\ &\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]^t\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right]^{(1-t)}.\end{aligned} \tag{12}$$

By using the General Cauchy Inequality in (12), we get

$$\begin{aligned}&f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\\ \leq\ &t\left[t^{s_1}f(b)+(1-t)^{s_1}f(a)\right]+(1-t)\left[t^{s_2}g(b)+(1-t)^{s_2}g(a)\right].\end{aligned}$$

By integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 f^t(tb+(1-t)a)\,g^{(1-t)}(tb+(1-t)a)\,dt\\ \leq\ &\int_0^1\left[t^{s_1+1}f(b)+t(1-t)^{s_1}f(a)+t^{s_2+1}g(b)+t(1-t)^{s_2}g(b)\right]dt.\end{aligned}$$

Hence, by taking into account the change of the variable $tb+(1-t)a=x$, $(b-a)dt=dx$, we obtain the required result. □

Corollary 2.7. *If we choose $s_1 = s_2 = 1$ in Theorem 6, we have the inequality;*

$$\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \le \frac{1}{3}\left[f(b)+g(b)\right]+\frac{1}{6}\left[f(a)+g(a)\right].$$

3. RESULTS FOR (α, m)−CONVEX FUNCTIONS

Similar results to Section 2 are given in this section, but now for (α, m)−convex functions.

Theorem 3.1. *Suppose that $f, g : [a,b] \to [0,\infty)$, $0 \le a < b < \infty$, are (α_1, m_1)−convex and (α_2, m_2)−convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$. If $f, g \in L_1[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f^{\frac{x-a}{b-a}}(x)\, g^{\frac{b-x}{b-a}}(x)\, dx \\
\le\ & \frac{1}{\alpha_1+2} f(b) + \frac{m_1}{2(\alpha_1+2)} f\left(\frac{a}{m_1}\right) \\
&+\frac{1}{(\alpha_2+1)(\alpha_2+2)} g(b) + \frac{m_2\left(\alpha_2^2+3\alpha\right)}{2(\alpha_2+1)(\alpha_2+2)} g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. Since f and g are (α_1, m_1)−convex and (α_2, m_2)−convex functions, respectively, we can write

$$f^t(tb+(1-t)a) \le \left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right]^t$$

and

$$g^{(1-t)}(tb+(1-t)a) \le \left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.$$

Since f, g are non-negative, we have

$$\begin{aligned}
& f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a) \\
\le\ & \left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right]^t \left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right]^{(1-t)}.
\end{aligned} \tag{13}$$

By using the General Cauchy Inequality in (13), we get

$$\begin{aligned}
& f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a) \\
\le\ & t\left[t^{\alpha_1} f(b) + m_1(1-t^{\alpha_1}) f\left(\frac{a}{m_1}\right)\right] + (1-t)\left[t^{\alpha_2} g(b) + m_2(1-t^{\alpha_2}) g\left(\frac{a}{m_2}\right)\right].
\end{aligned}$$

By integrating with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f^t(tb+(1-t)a)\, g^{(1-t)}(tb+(1-t)a)\, dt \\
\le\ & \frac{1}{\alpha_1+2} f(b) + \frac{m_1}{2(\alpha_1+2)} f\left(\frac{a}{m_1}\right) \\
&+\frac{1}{(\alpha_2+1)(\alpha_2+2)} g(b) + \frac{m_2\left(\alpha_2^2+3\alpha\right)}{2(\alpha_2+1)(\alpha_2+2)} g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Hence, by taking into account the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we obtain the required result. □

Corollary 3.2. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 7, we have the inequality (5).*

Theorem 3.3. *Suppose that $f, g : [a, b] \to [0, \infty)$, $0 \le a < b < \infty$, are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, where $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$. If $f, g \in L_1[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{g(b)}{(b-a)^{\alpha_2+1}} \int_a^b (x-a)^{\alpha_2} f(x)\,dx + m_2 \frac{g\left(\frac{a}{m_2}\right)}{(b-a)^{\alpha_2+1}} \int_a^b \left[(b-a)^{\alpha_2} - (x-a)^{\alpha_2}\right] f(x)\,dx \\
&+ \frac{f(b)}{(b-a)^{\alpha_1+1}} \int_a^b (x-a)^{\alpha_1} g(x)\,dx + m_1 \frac{f\left(\frac{a}{m_1}\right)}{(b-a)^{\alpha_1+1}} \int_a^b \left[(b-a)^{\alpha_1} - (x-a)^{\alpha_1}\right] g(x)\,dx \\
\le\ & \frac{1}{b-a} \int_a^b f(x) g(x)\,dx + \frac{1}{\alpha_1+\alpha_2+1} f(b) g(b) + \frac{m_2\alpha_2}{(\alpha_1+1)(\alpha_1+\alpha_2+1)} g\left(\frac{a}{m_2}\right) f(b) \\
&+ \frac{m_1\alpha_1}{(\alpha_1+1)(\alpha_1+\alpha_2+1)} f\left(\frac{a}{m_1}\right) g(b) + \frac{m_1 m_2 \alpha_1 \alpha_2 (\alpha_1+\alpha_2+2)}{(\alpha_1+1)(\alpha_2+1)(\alpha_1+\alpha_2+1)} f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).
\end{aligned}$$

Proof. Since f and g are $(\alpha_1, m_1)-$convex and $(\alpha_2, m_2)-$convex functions, respectively, we can write

$$f(tb + (1-t)a) \le t^{\alpha_1} f(b) + m_1 (1 - t^{\alpha_1}) f\left(\frac{a}{m_1}\right)$$

and

$$g(tb + (1-t)a) \le t^{\alpha_2} g(b) + m_2 (1 - t^{\alpha_2}) g\left(\frac{a}{m_2}\right).$$

By using the elementary inequality, $e \le f$ and $p \le r$, then $er + fp \le ep + fr$ for $e, f, p, r \in \mathbb{R}$ and by a similar argument to the proof of Theorem 4, we get the required result. □

Corollary 3.4. *If we choose $\alpha_1 = \alpha_2 = 1$ in Theorem 8, we have the inequality (7).*

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