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De Moivre-Type Identities for the Pell Numbers

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Abstract. This paper aims to present a method for constructing the second order Pell and Pell-Lucas numbers and the third order Pell and Pell-Lucas numbers. Moreover, we obtain the De Moivre-type identities for these numbers. In addition, we define a Pell sequence with new initial conditions and give some identities for these third order Pell numbers such as Binet's formulas, generating functions, sums.

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1. Introduction

In the literature, the roots of the equation $x^2 - x - 1 = 0$ are given as $x_1 = (1 + \sqrt{5})/2$, $x_2 = (1 - \sqrt{5})/2$, and the following relation is satisfied

$$(\frac{1 \pm \sqrt{5}}{2})^n = \frac{L_n \pm \sqrt{5}F_n}{2} \tag{1.1}$$

where L_n denotes the n^{th} Lucas numbers and F_n denotes the n^{th} Fibonacci numbers for the integers n. The relation (1.1) is called *De Moivre-type identity*, [2]. In [9] and [10], Lin gave the De Moivre-type identities for Tribonacci and Tetrabonacci numbers by using the recurrence relation $x^3 - x^2 - x - 1$ and the recurrence relation $x^4 - x^3 - x^2 - x - 1$, respectively.

In this paper, we give a way for constructing the second order Pell and Pell-Lucas numbers and the third order Pell and Pell-Lucas numbers by using the roots of characteristic equations $x^2 - 2x - 1 = 0$ and $x^3 - 2x^2 - x - 1 = 0$, respectively. Moreover, we obtain the De Moivre-type identity for the second order Pell numbers and the third order Pell numbers. Furthermore, we define a third order Pell sequence with new initial conditions and find some identities between all of these sequences.

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2. De Moivre-Type Identities for Pell numbers

Horadam in [5], defines the Pell sequence with the recurrence relation

$$P_{n+2} = 2P_{n+1} + P_n$$

and with the initial conditions $P_0 = 0, P_1 = 1$, where P_n denotes the n^{th} Pell number. Some terms of $\{P_n\}$ are as follows

$$\{P_n\} = 0, 1, 2, 5, 12, 29, ...$$

Also, some identities for the Pell numbers are given by Horadam. In the literature, there are a lot of studies about Pell number such as [4, 6–8] and [1]. The characteristic equation of the Pell sequence is $x^2 - 2x - 1 = 0$ and the roots of it are $r_1 = 1 + \sqrt{2}$, $r_2 = 1 - \sqrt{2}$, [5]. By calculating the powers of the roots, the De Moivre-type identity for the Pell numbers can be found as follows;

$$(1 \pm \sqrt{2})^n = \frac{A_n \pm \sqrt{2}P_n}{2},$$

analog to the De Moivre-type identity for Fibonacci numbers. Note that the sequence $\{A_n\}$ takes place in The On-Line Encyclopedia of Integer Sequences (OEIS, A002203) called as the Companion Pell numbers or Pell- Lucas numbers in [11] with the initial conditions $A_0 = 2, A_1 = 2$.

Table 1. Some terms of the sequences $\{A_n\}$ and $\{P_n\}$

n	0	1	2	3	4	5	6	7	8	9	10
A_n	2	2	6	14	34	82	198	1154	2786	6726	16238
P_n	0	1	2	5	12	29	70	169	408	985	2378

3. De Moivre-Type Identities for the Third Order Pell Numbers

In [12], the third order Pell numbers is defined and some identities for this numbers are found. We know from [12] that the recurrence relation for the third order Pell numbers is

$$x^3 - 2x^2 - x - 1 = 0. \tag{3.1}$$

Addition to this, the three roots of the equation (3.1) are

$$r_1 = \frac{2}{3} + (\frac{61}{54} + \sqrt{\frac{29}{36}})^{1/3} + (\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3},$$

$$r_2 = \frac{2}{3} + \omega(\frac{61}{54} + \sqrt{\frac{29}{36}})^{1/3} + \omega^2(\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3},$$

$$r_3 = \frac{2}{3} + \omega^2(\frac{61}{54} + \omega\sqrt{\frac{29}{36}})^{1/3} + \omega(\frac{61}{54} - \sqrt{\frac{29}{36}})^{1/3},$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$, [12].

Let us rewrite the roots of equation (3.1) as follows:

$$r_1 = \frac{2}{3} + X + Y,$$

$$r_2 = \frac{2}{3} - \frac{1}{2}(X + Y) + \frac{\sqrt{3}}{2}i(X - Y),$$

$$r_2 = \frac{2}{3} - \frac{1}{2}(X + Y) - \frac{\sqrt{3}}{2}i(X - Y),$$

where $X = \sqrt[3]{\frac{61}{54} + \sqrt{\frac{29}{36}}}$ and $Y = \sqrt[3]{\frac{61}{54} - \sqrt{\frac{29}{36}}}$. Also, we know that $X.Y = \frac{7}{9}$ and $X^3 + Y^3 = \frac{61}{27}$ are satisfied. Thus,

$$r_1^2 = 2 + \frac{4}{3}(X + Y) + (X^2 + Y^2),$$

$$r_1^3 = \frac{17}{3} + \frac{11}{3}(X+Y) + 2(X^2+Y^2),$$
$$r_1^4 = 14 + \frac{29}{3}(X+Y) + 5(X^2+Y^2),$$
$$r_1^5 = \frac{107}{3} + \frac{73}{3}(X+Y) + 13(X^2+Y^2),$$
$$r_1^6 = 91 + 62(X+Y) + 33(X^2+Y^2).$$

The coefficients of the above equations construct the third order Pell sequences, which we denote by $\{R_n\}, \{S_n\}$ and $\{T_n\}$, respectively. Especially,

(1) the sequence $\{R_n\}$ is the third order Pell-Lucas in [12],

(2) the sequence $\{S_n\}$ is the third order Pell numbers sequence with the initial conditions $S_0 = 3, S_1 = 4$ and $S_3 = 11$,

(3) the sequence $\{T_n\}$ is the usual third order Pell sequence which is also called as Tripell sequence or the sequence A077939 in OEIS, [11].

Table 2. Some terms of the sequences $\{R_n\}, \{S_n\}, \{T_n\}$ and $\{U_n\}$

n	0	1	2	3	4	5	6	7	8	9	10
R_n	3	2	6	17	42	107	273	695	1770	4508	11481
S_n	3	4	11	29	73	186	474	1207	3074	7829	19939
T_n	1	2	5	13	33	84	214	545	1388	3535	9003
U_n	0	2	5	12	31	79	201	512	1304	3321	8458

By using the sequences $\{R_n\}, \{S_n\}$ and $\{T_n\}$ applying induction over n, we obtain the followings:

$$r_1^n = \frac{1}{3}R_n + \frac{1}{3}S_{n-1}(X+Y) + T_{n-2}(X^2+Y^2).$$

Similarly, we get

$$r_2^n = \frac{1}{3}R_n - \frac{1}{6}S_{n-1}(X+Y) - \frac{1}{2}T_{n-2}(X^2+Y^2) + \sqrt{3}i[\frac{1}{6}S_{n-1}(X-Y) + \frac{1}{2}T_{n-2}(X^2+Y^2)],$$

and

$$r_3^n = \frac{1}{3}R_n - \frac{1}{6}S_{n-1}(X+Y) - \frac{1}{2}T_{n-2}(X^2+Y^2) - \sqrt{3}i[\frac{1}{6}S_{n-1}(X-Y) + \frac{1}{2}T_{n-2}(X^2+Y^2)],$$

where R_n denotes the n^{th} term of $\{R_n\}$, S_n denotes the n^{th} term of $\{S_n\}$ and T_n denotes the n^{th} term of $\{T_n\}$. So, we calculate the powers of the roots r_1^n, r_2^n and r_3^n in terms of R_n, S_n and T_n. Thus, we get the De Moivre-type identity for the third order Pell numbers.

4. Generating Functions and Binet's Formula for the Sequences $\{R_n\}, \{S_n\}$ and $\{T_n\}$

The generating function for $\{R_n\}$ can be found as

$$R(x) = \frac{3-4x-x^2}{1-2x-x^2-x^3}$$

where $R(x) = \sum_{n=0}^{\infty} R_n x^n$, as in [12]. Similarly, the generating function for $\{S_n\}$ and $\{T_n\}$, can be calculated as follows;

$$S(x) = \frac{3-6x}{1-2x-x^2-x^3},$$

and

$$T(x) = \frac{1}{1-2x-x^2-x^3}, \tag{4.1}$$

where $S(x) = \sum_{n=0}^{\infty} S_n x^n$ and $T(x) = \sum_{n=0}^{\infty} T_n x^n$, respectively. Note that the expansion of the equation (4.1) was given as the the sequence A077939 in [11].
The Binet's formulas for $\{R_n\}$, $\{S_n\}$ and $\{T_n\}$, by the help of the generating functions, can be found as follows:

$$\begin{aligned} R_n &= r_1^n + r_2^n + r_3^n, \\ S_n &= \frac{(3r_1-6)r_1^{n+1}}{(r_1-r_2)(r_1-r_3)} + \frac{(3r_2-6)r_2^{n+1}}{(r_2-r_1)(r_2-r_3)} + \frac{(3r_3-6)r_3^{n+1}}{(r_3-r_1)(r_3-r_2)}, \\ T_n &= \frac{r_1^{n+2}}{(r_1-r_2)(r_1-r_3)} + \frac{r_2^{n+2}}{(r_2-r_1)(r_2-r_3)} + \frac{r_3^{n+2}}{(r_3-r_1)(r_3-r_2)}. \end{aligned} \tag{4.2}$$

The equation (4) was obtained by Soykan [12].

5. Some Properties of $\{R_n\}, \{S_n\}$ and $\{T_n\}$

In this section some interesting identities are derived as analogous to the idea in Ian Bruces' article [3], and by the help of the article [12], using the definition of the third order Pell numbers. The recurrence relations of the sequences $\{R_n\}, \{S_n\}, \{T_n\}$ and $\{U_n\}$ with initial conditions are follows:

$$R_{n+3} = 2R_{n+2} + R_{n+1} + R_n, n \geq 0, R_0 = 3, R_1 = 2, R_2 = 6,$$

$$S_{n+3} = 2S_{n+2} + S_{n+1} + S_n, n \geq 0, S_0 = 3, S_1 = 4, S_2 = 11,$$

$$T_{n+3} = 2T_{n+2} + T_{n+1} + T_n, n \geq 0, T_0 = 1, T_1 = 2, T_2 = 5.$$

$$U_{n+3} = 2U_{n+2} + U_{n+1} + U_n, n \geq 0, U_0 = 0, U_1 = 2, U_2 = 5.$$

By using these equations, the following identities are hold:

$$U_n = 2T_{n-1} + T_{n-2}, n \geq 2.$$

$$S_n = 3T_n - 2T_{n-1}, n \geq 1.$$

$$R_n = 2T_{n-1} + 2T_{n-2} + 3T_{n-3}, n \geq 3.$$

$$\sum_{k=0}^{n} U_k = T_{n+1} - T_n - 1.$$

$$\sum_{k=0}^{n} R_k = \frac{7T_n - T_{n-1} - 2T_{n-2} + 2}{3}.$$

$$\sum_{k=0}^{n} S_k = \frac{10T_n + 2T_{n-1} + T_{n-2} + 2}{3}.$$

$$\sum_{k=0}^{n} T_k = \frac{4T_n + 2T_{n-1} + T_{n-2} - 1}{3}.$$

$$U_{n+1}U_{n-1} = T_{n-1}^2 + 2T_n(T_{n-1}T_{n-4}) - T_{n-1}T_{n-4}, n \geq 5.$$

$$U_{n+1}^2 + U_{n-1}^2 = 4T_n^2 + 2T_{n-1}^2 + T_{n-4}^2 + 4T_nT_{n-1} - 2T_{n-1}T_{n-4}, n \geq 5.$$

6. Conclusion

In this study, we give a way for constructing the Pell and Pell-Lucas numbers. Also, we obtain the De Moivre-type identities, occupy an important place in the analysis, for the second and the third order Pell sequences. In future studies, this identity can be derived for the fourth order Pell numbers as similar to the De Moivre-type identity for the tetrabocacci numbers in the literature.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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