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Research Article

Weak A-frames and weak A-semi-frames

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ABSTRACT. After reviewing the interplay between frames and lower semi-frames, we introduce the notion of lower semi-frame controlled by a densely defined operator A or, for short, a *weak lower A-semi-frame* and we study its properties. In particular, we compare it with that of lower atomic systems, introduced by one of us (GB). We discuss duality properties and we suggest several possible definitions for weak A-upper semi-frames. Concrete examples are presented.

Keywords: *A-frames, weak (upper and lower) A-semi-frames, lower atomic systems, G-duality.*

2020 Mathematics Subject Classification: 41A99, 42C15.

Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION AND BASIC FACTS

We consider an infinite dimensional Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$, linear in the first entry, and norm $\|\cdot\|$. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Given a linear operator A, we denote its domain by $\mathcal{D}(A)$, its range by $\mathcal{R}(A)$ and its adjoint by A^*, if A is densely defined. Given a locally compact, σ-compact space (X,μ) with a (Radon) measure μ, a function $\psi : X \mapsto \mathcal{H}, x \mapsto \psi_x$ is said to be *weakly measurable* if for every $f \in \mathcal{H}$ the function $x \mapsto \langle f|\psi_x\rangle$ is measurable. As a particular case, we obtain a discrete situation if $X = \mathbb{N}$ and μ is the counting measure. Given a weakly measurable function ψ, the operator $C_\psi : \mathcal{D}(C_\psi) \subseteq \mathcal{H} \to L^2(X,\, \mathrm{d}\mu)$ with domain

$$\mathcal{D}(C_\psi) := \left\{ f \in \mathcal{H} : \int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) < \infty \right\}$$

and $(C_\psi f)(x) = \langle f|\psi_x\rangle$, $f \in \mathcal{D}(C_\psi)$, C_ψ is called the *analysis* operator of ψ.

Remark 1.1. *In general, the domain of C_ψ is not dense, hence C_ψ^* is not well-defined. An example of function whose analysis operator is densely defined can be found in* [10, Example 2.8], *where $\mathcal{D}(C_\psi)$ coincides with the domain of a densely defined sesquilinear form associated to ψ. Moreover, a sufficient condition for $\mathcal{D}(C_\psi)$ to be dense in $\mathcal{H}$ is that $\psi_x \in \mathcal{D}(C_\psi)$ for every $x \in X$, see* [3, Lemma 2.3].

Proposition 1.1. [3, Lemma 2.1] *Let (X,μ) be a locally compact, σ-compact space, with a Radon measure μ and $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ a weakly measurable function. Then, the analysis operator C_ψ is closed.*

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Consider the set $\mathcal{D}(\Omega_\psi) = \mathcal{D}(C_\psi)$ and the mapping $\Omega_\psi : \mathcal{D}(C_\psi) \times \mathcal{D}(C_\psi) \to \mathbb{C}$ defined by

$$\Omega_\psi(f,g) := \int_X \langle f|\psi_x\rangle\langle\psi_x|g\rangle \,\mathrm{d}\mu(x). \tag{1.1}$$

Ω_ψ is clearly a nonnegative symmetric sesquilinear form which is well defined for every $f,g \in \mathcal{D}(C_\psi)$ because of the Cauchy-Schwarz inequality. It is unbounded in general. Moreover, since $\mathcal{D}(C_\psi)$ is the largest domain such that Ω_ψ is defined on $\mathcal{D}(C_\psi) \times \mathcal{D}(C_\psi)$, it follows that

$$\Omega_\psi(f,g) = \langle C_\psi f|C_\psi g\rangle, \quad \forall f,g \in \mathcal{D}(C_\psi), \tag{1.2}$$

where C_ψ is the analysis operator defined above. Since C_ψ is a closed operator, the form Ω_ψ is closed, see e.g. [16, Example VI.1.13]. If $\mathcal{D}(C_\psi)$ is dense in $\mathcal{H}$, then by Kato's first representation theorem [16, Theorem VI.2.1], there exists a positive self-adjoint operator T_ψ *associated to the sesquilinear form* Ω_ψ on

$$\mathcal{D}(\mathsf{T}_\psi) = \left\{ f \in \mathcal{D}(\Omega_\psi) : h \mapsto \int_X \langle f|\psi_x\rangle\langle\psi_x|h\rangle \,\mathrm{d}\mu(x) \text{ is bounded in } \mathcal{D}(C_\psi) \right\} \tag{1.3}$$

defined by

$$\mathsf{T}_\psi f := h \tag{1.4}$$

with h as in (1.3). The density of $\mathcal{D}(\Omega_\psi)$ ensures the uniqueness of the vector h. The operator T_ψ is the greatest one whose domain is contained in $\mathcal{D}(\Omega_\psi)$ and such that

$$\Omega_\psi(f,g) = \langle \mathsf{T}_\psi f|g\rangle, \quad f \in \mathcal{D}(\mathsf{T}_\psi),\ g \in \mathcal{D}(\Omega_\psi).$$

The set $\mathcal{D}(\mathsf{T}_\psi)$ is dense in $\mathcal{D}(\Omega_\psi)$, see [16, p. 279]. In addition, by Kato's second representation theorem [16, Theorem VI.2.23], we have $\mathcal{D}(\Omega_\psi) = \mathcal{D}(\mathsf{T}_\psi^{1/2})$ and

$$\Omega_\psi(f,g) = \langle \mathsf{T}_\psi^{1/2} f|\mathsf{T}_\psi^{1/2} g\rangle, \quad \forall f,g \in \mathcal{D}(\Omega_\psi),$$

hence, comparing with (1.2), we deduce $\mathsf{T}_\psi = C_\psi^* C_\psi = |C_\psi|^2$ on $\mathcal{D}(\mathsf{T}_\psi)$.

Definition 1.1. *The operator* $\mathsf{T}_\psi : \mathcal{D}(\mathsf{T}_\psi) \subset \mathcal{H} \to \mathcal{H}$ *defined by* (1.4) *will be called the* generalized frame operator *of the function* $\psi : x \in X \to \psi_x \in \mathcal{H}$.

Now, we recall a series of notions well-known in the literature, see e.g. [1, 3, 15]. A weakly measurable function ψ is said to be

- μ-total if $\langle f|\psi_x\rangle = 0$ for a.e. $x \in X$ implies that $f = 0$;
- a *continuous frame* of $\mathcal{H}$ if there exist constants $0 < \mathsf{m} \le \mathsf{M} < \infty$ (the frame bounds) such that
$$\mathsf{m}\|f\|^2 \le \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \mathsf{M}\,\|f\|^2, \qquad \forall\, f \in \mathcal{H};$$
- a *Bessel mapping* of $\mathcal{H}$ if there exists $\mathsf{M} > 0$ such that
$$\int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \mathsf{M}\,\|f\|^2, \qquad \forall\, f \in \mathcal{H};$$
- an *upper semi-frame* of $\mathcal{H}$ if there exists $\mathsf{M} < \infty$ such that
$$0 < \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \mathsf{M}\,\|f\|^2, \qquad \forall\, f \in \mathcal{H},\ f \neq 0,$$
i.e., if it is a μ-total Bessel mapping;
- a *lower semi-frame* of $\mathcal{H}$ if there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\,\|f\|^2 \le \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x), \qquad \forall\, f \in \mathcal{H}. \tag{1.5}$$

Note that the integral on the right hand side in (1.5) may diverge for some $f \in \mathcal{H}$, namely, for $f \notin \mathcal{D}(C_\psi)$. Moreover, if ψ satisfies (1.5), then it is automatically μ-total.

2. From semi-frames to frames and back

Starting from a lower semi-frame, one can easily obtain a genuine frame, albeit in a smaller space. Indeed, we have proved a theorem [8, Prop.3.5], which implies the following:

Proposition 2.2. *A weakly measurable function ϕ on $\mathcal{H}$ is a lower semi-frame of $\mathcal{H}$ whenever $\mathcal{D}(C_\phi)$ is complete for the norm $\|f\|^2_{C_\phi} = \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) = \|C_\phi f\|^2$, continuously embedded into $\mathcal{H}$ and for some $\alpha, \mathsf{m}, \mathsf{M} > 0$, one has*

$$\alpha \|f\| \leq \|f\|_{C_\phi} \text{ and } \tag{2.1}$$

$$\mathsf{m} \|f\|^2_{C_\phi} \leq \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M} \|f\|^2_{C_\phi}, \ \forall f \in \mathcal{D}(C_\phi). \tag{2.2}$$

Note that (2.2) is trivial here. Following the notation of our previous papers, denote by $\mathcal{H}(\mathsf{T}_\phi^{1/2})$ the Hilbert space $\mathcal{D}(\mathsf{T}_\phi^{1/2})$ with the norm $\|f\|^2_{1/2} = \left\|\mathsf{T}_\phi^{1/2} f\right\|^2$, where T_ϕ is the generalized frame operator defined in (1.4). In the same way, denote by $\mathcal{H}(C_\phi)$ the Hilbert space $\mathcal{D}(C_\phi)$ with the inner product $\langle\cdot|\cdot\rangle_{C_\phi} = \langle C_\phi\cdot|C_\phi\cdot\rangle$, and the corresponding norm $\|f\|^2_{C_\phi} = \|C_\phi f\|^2$. Then, clearly $\mathcal{H}(\mathsf{T}_\phi^{1/2}) = \mathcal{H}(C_\phi)$. What we have obtained in Proposition 2.2 is a frame in $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$. Indeed, assume that $\mathcal{D}(C_\phi)$ is dense. Then, for every $x \in X$, the map $f \mapsto \langle f|\phi_x\rangle$ is a bounded linear functional on the Hilbert space $\mathcal{H}(C_\phi)$. By the Riesz Lemma, there exists an element $\chi^\phi_x \in \mathcal{D}(C_\phi)$ such that

$$\langle f|\phi_x\rangle = \langle f|\chi^\phi_x\rangle_{C_\phi} \quad \forall f \in \mathcal{D}(C_\phi).$$

By Proposition 2.2, χ^ϕ is a frame. Actually, one can say more [8]. The norm $\|f\|^2_{1/2} = \left\|\mathsf{T}_\phi^{1/2} f\right\|^2$ is equivalent to the the graph norm of $\mathsf{T}_\phi^{1/2}$. Hence, $\langle f|\phi_x\rangle = \langle f|\chi^\phi_x\rangle_{C_\phi} = \langle f|\mathsf{T}_\phi \chi^\phi_x\rangle$ for all $f \in \mathcal{D}(C_\phi)$. Thus, $\chi^\phi_x = \mathsf{T}_\phi^{-1}\phi_x$ for all $x \in X$, i.e., χ^ϕ is the canonical dual Bessel mapping of ϕ (we recall that ϕ may have several duals).

Proposition 2.3. *Let ϕ be a lower semi-frame of $\mathcal{H}$ with $\mathcal{D}(C_\phi)$ dense. Then, the canonical dual Bessel mapping of ϕ is a tight frame for the Hilbert space $\mathcal{H}(C_\phi)$.*

Conversely, starting with a frame $\chi \in \mathcal{D}(C_\phi)$, does there exists a lower semi-frame η of $\mathcal{H}$ such that χ is the frame χ^η constructed from η in the way described above. The answer is formulated in the following [13, Prop. 6].

Proposition 2.4. *Let χ be a frame of $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$. Then,*

(i) there exists a lower semi-frame η of $\mathcal{H}$ such that $\chi = \chi^\eta$ if and only if $\chi \in \mathcal{D}(\mathsf{T}_\phi)$;
(ii) if $\chi = \chi^\eta$ for some lower semi-frame η of $\mathcal{H}$, then $\eta = \mathsf{T}_\phi \chi$.

So far, we have discussed the interplay between frames and lower semi-frames. But, one question remains: how does one obtain semi-frames? A standard construction is to start from an unbounded operator A and build a lattice of Hilbert spaces out of it, as described in [4] and in [8]. As we will see in Section 6 (1) and (2) below, this approach indeed generates a weak lower A-semi-frame. Before that, we need a new ingredient, namely the notion of metric operator. Given a closed unbounded operator S with dense domain $\mathcal{D}(S)$, define the operator $G = I + S^*S$, which is unbounded, with $G > 1$ and bounded inverse. This is a *metric operator*, that is, a strictly positive self-adjoint operator G, that is, $G > 0$ or $\langle Gf|f\rangle \geq 0$ for every $f \in \mathcal{D}(G)$ and $\langle Gf|f\rangle = 0$ if and only if $f = 0$. Then, the norm $\|f\|_{G^{1/2}} = \|G^{1/2}f\|$ is equivalent to the graph norm of $G^{1/2}$ on $\mathcal{D}(G^{1/2}) = \mathcal{D}(S)$ and makes the latter into a Hilbert space continuously embedded into $\mathcal{H}$, denoted by $\mathcal{H}(G)$. Then $\mathcal{H}(G^{-1})$, built in the same way from G^{-1}, coincides, as a vector space, with the conjugate dual of $\mathcal{H}(G)$. On the other hand, G^{-1} is bounded. Hence, we get the triplet

$$\mathcal{H}(G) \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}(G^{-1}) = \mathcal{H}(G)^{\times}. \tag{2.3}$$

Two developments arise from these relations. First, the triplet (2.3) is the central part of the discrete scale of Hilbert spaces $V_{\mathcal{G}}$ built on the powers of $G^{1/2}$. This means that $V_{\mathcal{G}} := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = \mathcal{D}(G^{n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^{\times}$:

$$\ldots \subset \ \mathcal{H}_2 \ \subset \ \mathcal{H}_1 \ \subset \ \mathcal{H} \ \subset \ \mathcal{H}_{-1} \subset \ \mathcal{H}_{-2} \subset \ \ldots .$$

Thus $\mathcal{H}_1 = \mathcal{H}(G^{1/2}) = \mathcal{D}(S)$, $\mathcal{H}_2 = \mathcal{H}(G) = \mathcal{D}(S^*S)$, and $\mathcal{H}_{-2} = \mathcal{H}(G^{-1})$, and so on. What we have obtained in this way is a Lattice of Hilbert Spaces (LHS), the simplest example of a Partial Inner Product Spaces (PIP-space). See our monograph [2] about this structure.

One may also add the end spaces of the scale, namely,

$$\mathcal{H}_\infty(G) := \cap_{n\in\mathbb{Z}}\mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{2.4}$$

In this way, we get a genuine Rigged Hilbert Space:

$$\mathcal{H}_\infty(G) \subset \mathcal{H} \subset \mathcal{H}_{-\infty}(G).$$

In fact, one can go one more step farther. Namely, following [2, Sec. 5.1.2], we can use quadratic interpolation theory [12] and build a continuous scale of Hilbert spaces $\mathcal{H}_\alpha, \alpha \geq 0$, where $\mathcal{H}_\alpha = \mathcal{D}(G^{\alpha/2})$, with the graph norm $\|\xi\|_\alpha^2 = \|\xi\|^2 + \|G^{\alpha/2}\xi\|^2$ or, equivalently, the norm $\left\|(I+G)^{\alpha/2}\xi\right\|^2$. Indeed, every $G^\alpha, \alpha \geq 0$, is an unbounded metric operator. Next, we define $\mathcal{H}_{-\alpha} = \mathcal{H}_\alpha^{\times}$ and thus obtain the full continuous scale $V_{\widetilde{\mathcal{G}}} := \{\mathcal{H}_\alpha, \alpha \in \mathbb{R}\}$. Of course, one can replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition (2.4) of the end spaces of the scale. A second development of the previous analysis is that we have made a link to the formalism based on metric operators that we have developed for the theory of pseudo-Hermitian operators, in particular non-self-adjoint Hamiltonians, as encountered in the so-called pseudo-Hermitian or $\mathcal{PT}$-symmetric quantum mechanics. This is not the place, however, to go into details, instead we refer the reader to [4, 5] for a complete mathematical treatment.

3. Weak lower A-semi-frames

The following concept was introduced and studied in [10].

Definition 3.2. *Let A be a densely defined operator on $\mathcal{H}$. A* (continuous) weak A-frame *is a function $\phi : x \in X \mapsto \phi_x$ such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is a measurable function on X and, for some $\alpha > 0$,*

$$\alpha \left\|A^*u\right\|^2 \leq \int_X |\langle u|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) < \infty, \qquad \forall\, u \in \mathcal{D}(A^*). \tag{3.1}$$

If $X = \mathbb{N}$ and μ is the counting measure, we recover the discrete situation (so that the word "continuous" is superfluous in the definition above). We get a simpler situation when A is bounded and ϕ is Bessel. This is in fact the construction of Găvruţa [14]. Now, we introduce a structure that generalizes both concepts of lower semi-frame and weak A-frame. We follow mostly the terminology of [10] and keep the term "weak" because the notion leads to a weak decomposition of the range of the operator A (see Theorem 4.2). We begin with giving the following definitions.

Definition 3.3. *Let A be a densely defined operator on $\mathcal{H}$, $\phi : x \in X \mapsto \phi_x$ a function such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. We say that a closed operator B is a ϕ-*extension *of A if*

$$A \subset B \ \text{ and } \ \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi).$$

We denote by $\mathcal{E}_\phi(A)$ the set of ϕ-extensions of A.

Remark 3.2. *It is worth noting that, if A has a ϕ-extension, then A is automatically closable.*

Definition 3.4. *Let A and ϕ be as in Definition 3.3. Then ϕ is called a* weak lower A-semi-frame *if A admits a ϕ-extension B such that ϕ is a weak B-frame.*

Let us put $\mathcal{D}(A,\phi) := \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi)$. If ϕ and A are as in Definition 3.3, and $B := (A^* \upharpoonright \mathcal{D}(A,\phi))^*$ is a ϕ-extension of A, it would be the smallest possible extension for which ϕ is a weak B-frame, but in general, we could have a larger extension enjoying the same property. Indeed, if B is a closed extension of A such that ϕ is a weak B-frame, we have

$$A \subset A^{**} \subset (A^* \upharpoonright \mathcal{D}(A,\phi))^* \subset B.$$

Remark 3.3.

(1) *If A is bounded, $\mathcal{D}(A^*) = \mathcal{H}$ and we recover the notion of lower semi-frame, under some minor restrictions on A, hence the name (see Proposition 5.5).*

(2) *If A is a densely defined operator on $\mathcal{H}$ such that the integral on the right hand side of (3.1) is finite for every $f \in \mathcal{D}(A^*)$, then $\mathcal{D}(A^*) \subset \mathcal{D}(C_\phi)$ and the weak lower A-semi-frame ϕ is, in fact, a weak A-frame, in the sense of Definition 3.2.*

(3) *Let us assume that ϕ is both a lower semi-frame and a weak A-frame, then we have simultaneously*

$$\mathsf{m}\,\|f\|^2 \le \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x), \quad \forall\, f \in \mathcal{H},$$

$$\alpha\,\|A^* f\|^2 \le \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty, \quad \forall\, f \in \mathcal{D}(A,\phi) = \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi).$$

It follows that

$$\alpha'(\|f\|^2 + \|A^* f\|^2) \le \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty, \quad \forall\, f \in \mathcal{D}(A^*) \cap \mathcal{D}(C_\phi) \tag{3.2}$$

with $\alpha' \le \frac{1}{2}\min\{\mathsf{m}, \alpha\}$. If we consider the domain $\mathcal{D}(A^*)$ with its graph norm ($\|f\|_{A^*} = (\|f\|^2 + \|A^* f\|^2)^{1/2}$, $f \in \mathcal{D}(A^*)$), we are led to the triplet of Hilbert spaces

$$\mathcal{H}(A^*) \subset \mathcal{H} \subset \mathcal{H}(A^*)^\times,$$

as discussed in Section 2. Let us consider the sesquilinear form Ω_ϕ defined in (1.1) and suppose in particular that $\mathcal{D}(A^*) = \mathcal{D}(\Omega_\phi) = \mathcal{D}(C_\phi)$. Then, using Proposition 1.1, it is not difficult to prove that Ω_ϕ is closed in $\mathcal{H}(A^*)$ and then bounded. Thus, there exists $\gamma > 0$ such that, for every $f \in \mathcal{D}(A^*)$,

$$\alpha'(\|f\|^2 + \|A^* f\|^2) \le \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \gamma(\|f\|^2 + \|A^* f\|^2).$$

One could notice that (3.2) is similar to a frame condition. The inequality (3.2) says that the sesquilinear form Ω_ϕ defined in (1.1) is coercive on $\mathcal{H}(A^*)$ and thus the Lax-Milgram theorem *applies* [16, VI §2, 2] *or* [18, Lemma 11.2]. *This means that for every* $F \in \mathcal{H}(A^*)^\times$ *there exists* $w \in \mathcal{H}(A^*)$ *such that*

$$\langle F|f\rangle = \Omega_\phi(w,f) = \int_X \langle w|\phi_x\rangle\langle\phi_x|f\rangle \,\mathrm{d}\mu(x), \quad \forall f \in \mathcal{H}(A^*).$$

Therefore, in the case under consideration, we get expansions in terms of ϕ *of elements that do not belong to the domain of* A^**; in particular, those of* $\mathcal{H}$*. The price to pay is that the form of this expansion is necessarily weak since vectors of* $\mathcal{H}$ *do not belong to the domain of the analysis operator* C_ϕ*.*

In the sequel, we will need the following:

Lemma 3.1. [11, Lemma 3.8] *Let* $(\mathcal{H}, \|\cdot\|), (\mathcal{H}_1, \|\cdot\|_1)$ *and* $(\mathcal{H}_2, \|\cdot\|_2)$ *be Hilbert spaces and* $T_1 : \mathcal{D}(T_1) \subseteq \mathcal{H}_1 \to \mathcal{H}$, $T_2 : \mathcal{D}(T_2) \subseteq \mathcal{H} \to \mathcal{H}_2$ *densely defined operators. Assume that* T_1 *is closed and* $\mathcal{D}(T_1^*) = \mathcal{D}(T_2)$*. If* $\|T_1^* f\|_1 \le \lambda \|T_2 f\|_2$ *for all* $f \in \mathcal{D}(T_1^*)$ *and some* $\lambda > 0$*, then there exists a bounded operator* $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $T_1 = T_2^* U$*.*

Remark 3.4. *Lemma 3.1 is still valid if we replace closedness of* T_1 *by its closability, and in this hypothesis,* $\overline{T_1} = T_2^* U$*.*

In the literature [19], two measurable functions ψ and ϕ are said to be *dual* to each other if one has

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle \,\mathrm{d}\mu(x), \qquad \forall f, g \in \mathcal{H}. \tag{3.3}$$

If ϕ is a lower semi-frame of $\mathcal{H}$, then its dual ψ is a Bessel mapping of $\mathcal{H}$ [8]. In addition, if $\mathcal{D}(C_\phi)$ is dense, its dual ψ is an upper semi-frame. However, this definition is too general, in the sense that the right hand side may diverge for arbitrary $f, g \in \mathcal{H}$. A more useful definition will be given below, namely (5.3). A notion of duality related to a given operator G can be formulated as follows.

Definition 3.5. *Let* G *be a densely defined operator and* $\phi : x \in X \mapsto \phi_x$ *a function such that, for all* $u \in \mathcal{D}(G^*)$ *the map* $x \mapsto \langle u|\phi_x\rangle$ *is a measurable function on* X*. Then a function* $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ *such that, for all* $f \in \mathcal{D}(G)$ *the map* $x \mapsto \langle f|\psi_x\rangle$ *is a measurable function on* X *is called a weak G-dual of* ϕ *if*

$$\langle Gf|u\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|u\rangle \,\mathrm{d}\mu(x), \quad \forall f \in \mathcal{D}(G)\cap\mathcal{D}(C_\psi), \forall u \in \mathcal{D}(G^*)\cap\mathcal{D}(C_\phi). \tag{3.4}$$

This is a generalization of the notion of weak G-dual in [10].

Remark 3.5.

(i) *The weak G-dual* ψ *of* ϕ *is not unique, in general. On the other hand, Definition 3.5 could be meaningless. For instance, if either* $\mathcal{D}(G)\cap\mathcal{D}(C_\psi) = \{0\}$ *or* $\mathcal{D}(G^*)\cap\mathcal{D}(C_\phi) = \{0\}$*, then everything is "dual".*

(ii) *Note that, if* ϕ *is a weak G-frame, then there exists a weak G-dual* ψ *of* ϕ *such that relation (3.4) must hold only for* $\forall f \in \mathcal{D}(G), \forall u \in \mathcal{D}(G^*)$ *indeed* $\mathcal{D}(G^*) \subset \mathcal{D}(C_\phi)$ *and by Theorem 3.20 in* [10], *there exists a Bessel weak G-dual* ψ *of* ϕ*, hence* $\mathcal{D}(G) \subset \mathcal{D}(C_\psi) = \mathcal{H}$*.*

Example 3.1. *Given a densely defined operator* G *on a separable Hilbert space* $\mathcal{H}$*, we show two examples of G-duality (see* [10, Ex. 3.10]*).*

(i) Let (X, μ) *be a locally compact,* σ*-compact measure space and let* $\{X_n\}_{n\in\mathbb{N}}$ *be a covering of* X *made up of countably many measurable disjoint sets of finite measure. Without loss of generality, we suppose that* $\mu(X_n) > 0$ *for every* $n \in \mathbb{N}$*. Let* $\{e_n\} \subset \mathcal{D}(G)$ *be an orthonormal basis*

of $\mathcal{H}$ and consider ϕ, with $\phi_x = \frac{Ge_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$, then ϕ is a weak G-frame, see [10, Example 3.10]. *One can take ψ with $\psi_x = \frac{e_n}{\sqrt{\mu(X_n)}}$, $x \in X_n, \forall n \in \mathbb{N}$.*

(ii) If $\phi := G\zeta$, where $\zeta : x \in X \mapsto \zeta_x \in \mathcal{D}(G) \subset \mathcal{H}$ is a continuous frame for $\mathcal{H}$, then one can take as ψ any dual frame of ζ.

4. Lower atomic systems

Theorem 4.1. *Let (X, μ) be a locally compact, σ-compact measure space, A a densely defined operator and $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ a map such that, for every $u \in \mathcal{D}(A^*)$, the function $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) ϕ is a weak lower A-semi-frame for $\mathcal{H}$;

(ii) $\mathcal{E}_\phi(A) \neq \emptyset$ and for every $B \in \mathcal{E}_\phi(A)$, there exists a closed densely defined extension R of C_ϕ^*, with $\mathcal{D}(R^*) = \mathcal{D}(B^*)$, such that B can be decomposed as $B = RM$ for some $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$.

Proof. We proceed as in [10, Theor. 3.16].
(i)⇒(ii): If ϕ is a weak lower A-semi-frame for $\mathcal{H}$, by definition, there exists $B \in \mathcal{E}_\phi(A)$. Consider $E : \mathcal{D}(B^*) \to L^2(X, \mu)$ given by $(Eu)(x) = \langle u|\phi_x\rangle$, $\forall u \in \mathcal{D}(B^*)$, $x \in X$ which is a restriction of the analysis operator C_ϕ. E is closable and densely defined.

Apply Lemma 3.1 to $T_1 := B$, and $T_2 := E$, noting that $\|Eu\|_2^2 = \int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)$, $u \in \mathcal{D}(B^*)$. Thus, there exists $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$ such that $B = E^*M$. Then the statement is proved by taking $R = E^*$, indeed $R = E^* \supseteq C_\phi^*$ and $\mathcal{D}(R) \supset \mathcal{D}(C_\phi^*)$ is dense because C_ϕ is closed and densely defined. Note that, we have $\mathcal{D}(B^*) = \mathcal{D}(R^*)$; indeed $\mathcal{D}(R^*) = \mathcal{D}(\overline{E})$,

$$\mathcal{D}(B^*) \subset \mathcal{D}(\overline{E}) = \mathcal{D}(M^*\overline{E}) \subset \mathcal{D}((E^*M)^*) = \mathcal{D}(B^*),$$

hence, in particular, E is closed, recalling that $\mathcal{D}(E) = \mathcal{D}(B^*)$.
(ii)⇒(i): Let $B \in \mathcal{E}_\phi(A)$. For every $u \in \mathcal{D}(B^*) = \mathcal{D}(R^*)$,

$$\|B^*u\|^2 = \|M^*R^*u\|^2 \leq \|M^*\|^2\|R^*u\|^2 = \|M^*\|^2 \int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty$$

since $R^* \subset C_\phi$. This proves that ϕ is a weak lower A-semi-frame. □

Generalizing the notion of continuous weak atomic system for A [10], we consider the following:

Definition 4.6. *Let A be a densely defined operator on $\mathcal{H}$. A* lower atomic system *for A is a function $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ such that*

(i) *for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is a measurable function on X;*

(ii) *the operator A has a closed extension B such that $\mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$; i.e., $\mathcal{E}_\phi(A) \neq \emptyset$;*

(iii) *there exists $\gamma > 0$ such that, for every $f \in \mathcal{D}(A)$, there exists $a_f \in L^2(X, \mu)$, with $\|a_f\|_2 = \left(\int_X |a_f(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \leq \gamma\|f\|$ and*

$$\langle Af|u\rangle = \int_X a_f(x)\langle\phi_x|u\rangle \,\mathrm{d}\mu(x), \qquad \forall u \in \mathcal{D}(B^*).$$

We have chosen not to call ϕ a *weak* lower atomic system for A for brevity, even if it leads to a weak decomposition of the range of the operator A. Theorem 3.20 of [10], gives a characterization of weak atomic systems for A and weak A-frames. The next theorem yields the corresponding result for weak lower A-semi-frames.

Theorem 4.2. *Let (X, μ) be a locally compact, σ-compact measure space, A a densely defined operator in $\mathcal{H}$ and $\phi : x \in X \mapsto \phi_x \in \mathcal{H}$ a function such that, for all $u \in \mathcal{D}(A^*)$, the map $x \mapsto \langle u|\phi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) *ϕ is a lower atomic system for A;*
(ii) *ϕ is a weak lower A-semi-frame for $\mathcal{H}$;*
(iii) *$\mathcal{E}_\phi(A) \neq \emptyset$ and for every $B \in \mathcal{E}_\phi(A)$, ϕ has a Bessel weak B-dual ψ.*

Proof.
(i)⇒(ii): Consider a ϕ-extension B of A. By the density of $\mathcal{D}(A)$, we have, for every $u \in \mathcal{D}(B^*)$

$$\begin{aligned}\|B^*u\| &= \sup_{f\in\mathcal{H},\|f\|=1} |\langle B^*u|f\rangle| = \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle B^*u|f\rangle| \\ &= \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle u|Bf\rangle| = \sup_{f\in\mathcal{D}(A),\|f\|=1} |\langle u|Af\rangle| \\ &= \sup_{f\in\mathcal{D}(A),\|f\|=1} \left|\int_X \overline{a_f(x)}\langle u|\phi_x\rangle \,\mathrm{d}\mu(x)\right| \\ &\leq \sup_{f\in\mathcal{D}(A),\|f\|=1} \left(\int_X |a_f(x)|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \left(\int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\right)^{1/2} \\ &\leq \gamma \left(\int_X |\langle u|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\right)^{1/2} < \infty\end{aligned}$$

for some $\gamma > 0$, the last but one inequality is due to the fact that ϕ is a lower atomic system for A and the last one to the inclusion $\mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$. Then, ϕ is a weak lower A-semi-frame.
(ii)⇒(iii): Following the proof of Theorem 4.1, for every ϕ-extension B of A, there exists a closed densely defined extension R of C_ϕ^*, with $\mathcal{D}(R^*) = \mathcal{D}(B^*)$, such that $B = RM$ for some $M \in \mathcal{B}(\mathcal{H}, L^2(X, \mu))$. By the Riesz representation theorem, for every $x \in X$ there exists a unique vector $\psi_x \in \mathcal{H}$ such that $(Mh)(x) = \langle h|\psi_x\rangle$, for every $h \in \mathcal{H}$. The function $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ is Bessel. Indeed,

$$\begin{aligned}\int_X |\langle h|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) &= \int_X |(Mh)(x)|^2 \,\mathrm{d}\mu(x) \\ &= \|Mh\|_2^2 \leq \|M\|^2\|h\|^2, \qquad \forall h \in \mathcal{H}.\end{aligned}$$

Hence $\mathcal{D}(C_\psi) = \mathcal{H}$. Moreover, for $f \in \mathcal{D}(B)\cap\mathcal{D}(C_\psi) = \mathcal{D}(B)$, $u \in \mathcal{D}(B^*) = \mathcal{D}(R^*) \subset \mathcal{D}(C_\phi)$

$$\begin{aligned}\langle Bf|u\rangle &= \langle RMf|u\rangle = \langle Mf|R^*u\rangle_2 \\ &= \int_X \langle f|\psi_x\rangle\langle\phi_x|u\rangle \,\mathrm{d}\mu(x).\end{aligned}$$

(iii)⇒(i): It suffices to take, for every fixed ϕ-extension B of A, $a_f : x \in X \mapsto a_x(f) = \langle f|\psi_x\rangle \in \mathbb{C}$ for all $f \in \mathcal{D}(B)$. Indeed, $a_f \in L^2(X, \mu)$ and, for some $\gamma > 0$, we have $\int_X |a_x(f)|^2 \,\mathrm{d}\mu(x) = \int_X |\langle f|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \gamma\|f\|^2$, since ψ is a Bessel function. Moreover, by definition of weak B-dual, we have $\langle Bf|u\rangle = \int_X a_f(x)\langle\phi_x|u\rangle \,\mathrm{d}\mu(x)$, for $f \in \mathcal{D}(C_\psi)\bigcap\mathcal{D}(B) = \mathcal{D}(B), u \in \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$. Indeed, we note that $\mathcal{D}(C_\psi) = \mathcal{H}$ since ψ is a Bessel function. □

Remark 4.6. *We don't know if ψ is a weak upper A-semi-frame, in the sense of Definition 5.7, indeed ψ need not be μ-total, that is, $\int_X |\langle f|\psi_x\rangle|^2 \neq 0$ for every $f \in \mathcal{H}$, $f \neq 0$.*

5. Duality and Weak upper A-semi-frames

If $C \in GL(\mathcal{H})$, a *frame controlled by the operator* C or C*- controlled frame* [9] is a family of vectors $\phi = (\phi_n \in \mathcal{H} : n \in \Gamma)$, such that there exist two constants $\mathsf{m}_A > 0$ and $\mathsf{M}_A < \infty$ satisfying

$$\mathsf{m_A} \, \|f\|^2 \leq \sum_n \langle f|\phi_n\rangle\langle C\phi_n|f\rangle \leq \mathsf{M_A} \, \|f\|^2 \, , \forall \, f \in \mathcal{H} \tag{5.1}$$

or, to put it in a continuous form:

$$\mathsf{m_A} \, \|f\|^2 \leq \int_X \langle f|\phi_x\rangle \, \langle C\phi_x|f\rangle \, \mathrm{d}\mu(x) \leq \mathsf{M_A} \, \|f\|^2 \, , \ \forall \, f \in \mathcal{H}. \tag{5.2}$$

According to Proposition 3.2 of [9], an A-controlled frame is in fact a classical frame when the controlling operator belongs to $GL(\mathcal{H})$. A similar result holds true for a weak lower A-semi-frame if A is bounded as we show in Proposition 5.5. From there it follows that, if A is bounded, a weak lower A-semi-frame has an upper semi-frame dual to it.

Remark 5.7. *We recall that a bounded operator A is surjective if and only if A^* is injective and $\mathcal{R}(A^*)$ is norm closed (if and only if A^* is injective and $\mathcal{R}(A)$ is closed)* [17, Theor. 4.14 and 4.15].

Proposition 5.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and ϕ be a weak lower A-semi-frame. Assume that anyone of the following assumptions is satisfied:*

(i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
(ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
(iii) *A surjective.*

Then,

(a) *ϕ is a lower semi-frame of $\mathcal{H}$ in the sense of* (1.5),
(b) *there exists an upper semi-frame ψ dual to ϕ.*

Proof. (a) By Remark 5.7, it suffices to prove (iii). By Theorem 4.15 in [17], A is surjective if and only if there exists $\gamma > 0$ such that $\|A^* f\| \geq \gamma \|f\|$, for every $f \in \mathcal{H}$, then

$$\gamma^2 \alpha \|f\|^2 \leq \alpha \|A^* f\|^2 \leq \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x), \qquad \forall \, f \in \mathcal{H}.$$

(b) The thesis follows from (a) and Proposition 2.1 (ii) in [7] (with $\{e_n\}$ an ONB of $\mathcal{H}$).

As explained above, the notion of duality given in (3.3) is too general. Therefore, in what follows ψ will be said to be *dual to* ϕ if one has

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle \, \mathrm{d}\mu(x), \qquad \forall f \in \mathcal{D}(C_\phi), \, g \in \mathcal{D}(C_\psi). \tag{5.3}$$

□

An interesting question is to identify a weak A-dual of a weak lower A-semi-frame. We expect one should generalize to the present situation the notion of upper semi-frame. We first consider the next definition and examine its consequences.

Definition 5.7. *Let A be a densely defined operator on $\mathcal{H}$. A* weak upper A-semi-frame *for $\mathcal{H}$ is a function $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ such that, for all $f \in \mathcal{D}(A)$, the map $x \mapsto \langle f|\psi_x\rangle$ is measurable on X and there exists a closed extension F of A and a constant $\alpha > 0$ such that*

$$\int_X |\langle u|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \alpha \|F^* u\|^2, \qquad \forall \, u \in \mathcal{D}(F^*). \tag{5.4}$$

Remark 5.8.

(i) *From Definition 5.7, it is clear that $\mathcal{D}(F^*) \subset \mathcal{D}(C_\psi)$.*

(ii) *If $A \in \mathcal{B}(\mathcal{H})$, then ψ it is clearly a Bessel family.*

Corollary 5.1. *Let ψ be a Bessel mapping of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$. Assume that anyone of the following statements is satisfied:*

(i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
(ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
(iii) *A surjective.*

Then, ψ is a weak upper A-semi-frame.

Proof. By Remark 5.7, it suffices to prove (iii). By Theorem 4.15 in [17], we have just to note that

$$\int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \gamma \|f\|^2 \leq \alpha^2\gamma\|A^* f\|^2, \qquad \forall\, f \in \mathcal{H}.$$

□

Remark 5.9. *The previous result is true* a fortiori *if ψ is an upper semi-frame of $\mathcal{H}$.*

Summarizing Proposition 5.5, Corollary 5.1 together with the preceding results we have that:

Corollary 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Assume that anyone of the following assumptions is satisfied:*

(i) *A^* injective, with $\mathcal{R}(A^*)$ norm closed or*
(ii) *A^* injective, with $\mathcal{R}(A)$ closed or*
(iii) *A surjective*

and let ϕ be a weak lower A-semi-frame. Then, there exists a weak upper A-semi-frame ψ dual to ϕ.

Theorem 5.3. *Let (X,μ) be a locally compact, σ-compact measure space, A a densely defined operator and $\psi : x \in X \mapsto \psi_x \in \mathcal{H}$ a map such that, for every $f \in \mathcal{D}(A)$, the function $x \mapsto \langle f|\psi_x\rangle$ is measurable on X. Then, the following statements are equivalent:*

(i) ψ is a weak upper A-semi-frame for $\mathcal{H}$;
(ii) For every closed, densely defined extension F of A such that (5.4) holds true, there exists a closed, densely defined extension Q of C_ψ^* such that $Q = FN$ for some $N \in \mathcal{B}(L^2(X,\mu),\mathcal{H})$.

Proof.
(i)⇒(ii): Let ψ be a weak upper A-semi-frame, then for every closed extension F of A for which (5.4) holds true, consider the operator $E = C_\psi \upharpoonright \mathcal{D}(F^*)$. It is densely defined, closable since C_ψ is closed. Define an operator O on $R(F^*) \subseteq \mathcal{H}$ as $OF^*f = Ef \in L^2(X,\mu)$. Then, O is a well-defined bounded operator by (5.4). Now, we extend O to the closure of $R(F^*)$ by continuity and define it to be zero on $R(F^*)^\perp$. Therefore $O \in \mathcal{B}(\mathcal{H}, L^2(X,\mu))$ and $OF^* = E$, i.e., $E^* = FO^*$ and the statement is proved by taking $Q = E^*$ and $N = O^*$.
(ii)⇒(i): From $Q = FN$, with Q a densely defined closed extension of C_ψ^*, we have that $Q^* = N^*F^* \subset C_\psi$. For every $u \in \mathcal{D}(F^*) = \mathcal{D}(N^*F^*) = \mathcal{D}((FN)^*) \subset \mathcal{D}(C_\psi)$,

$$\|C_\psi u\|_2^2 = \int_X |\langle u|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) = \|N^*F^*u\|_2^2 \leq \alpha\|F^*u\|^2$$

for some $\alpha > 0$. □

We can now prove the following duality result, which suggests that Definition 5.7 is convenient in this context.

Proposition 5.6. *Let A be a densely defined operator and ψ a weak upper A-semi-frame. Let F be a closed extension of A satisfying (5.4) for some $\alpha > 0$. Assume that $\phi \subset \mathcal{D}(A)$ is a weak F-dual of ψ such that*

(a) *$F^*\mathcal{D}(F^*) \subset \mathcal{D}(C_\phi)$,*

(b) *the function* $x \to \|A\phi_x\|$ *is in* $L^2(X,\mu)$.

Then, $F \in \mathcal{E}_{A\phi}(A)$ *and* $A\phi$ *is a weak lower* A*-semi-frame with* F *as* $(A\phi)$*-extension and lower bound* α^{-1}*, i.e.,*

$$\alpha^{-1}\,\|F^*u\|^2 \le \int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x), \qquad \forall\, u \in \mathcal{D}(F^*) \cap \mathcal{D}(C_\phi). \tag{5.5}$$

Proof. For every $u \in \mathcal{D}(FF^*)$,

$$\begin{aligned}
\|F^*u\|^2 &= \langle F^*u|F^*u\rangle = \langle FF^*u|u\rangle \\
&= \int_X \langle F^*u|\phi_x\rangle\langle\psi_x|u\rangle\,\mathrm{d}\mu(x), \ \text{ by weak } F\text{-duality} \\
&\le \left(\int_X |\langle u|\psi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}\left(\int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\le \alpha^{1/2}\,\|F^*u\|\left(\int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}.
\end{aligned}$$

The right hand side of the previous inequality is finite because of (a). Hence,

$$\|F^*u\| \le \alpha^{1/2}\left(\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}, \quad \forall u \in \mathcal{D}(FF^*).$$

Now, we take into account that $\mathcal{D}(FF^*)$ is a core for F^* by von Neumann's theorem [16, Theorem 3.24]. Therefore, for every $u \in \mathcal{D}(F^*)$, there exists a sequence $\{u_n\} \subset \mathcal{D}(FF^*)$ such that $\|u_n - u\| \to 0$ and $\|F^*u_n - F^*u\| \to 0$. This implies, of course, that $\langle F^*u_n|\phi_x\rangle \to \langle F^*u|\phi_x\rangle$, for every $x \in X$. Moreover, since $\{u_n\}$ is bounded, we have

$$|\langle F^*u_n|\phi_x\rangle| = |\langle u_n|F\phi_x\rangle| \le M\|F\phi_x\|$$

for some $M > 0$ and for every $x \in X$. The assumption that $x \to \|A\phi_x\|$ is in $L^2(X,\mu)$ allows us to apply the dominated convergence theorem and conclude that

$$\|F^*u\| \le \alpha^{1/2}\left(\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x)\right)^{1/2}, \quad \forall u \in \mathcal{D}(F^*).$$

The right hand side of the latter inequality is finite again by (a), hence $\mathcal{D}(F^*) \subset \mathcal{D}(C_{A\phi})$. This fact also implies that $F \in \mathcal{E}_{A\phi}(A)$ since, if $u \in \mathcal{D}(F^*)$, we get

$$\int_X |\langle u|A\phi_x\rangle|^2\,\mathrm{d}\mu(x) = \int_X |\langle F^*u|\phi_x\rangle|^2\,\mathrm{d}\mu(x) = \|C_{A\phi}u\|^2 < \infty.$$

□

Remark 5.10.

(1) *Note* (5.5) *can obviously be also written*

$$\alpha^{-1}\,\|h\|^2 \le \int_X |\langle h|\phi_x\rangle|^2\,\mathrm{d}\mu(x), \qquad \forall\, h \in \mathcal{R}(F^*).$$

(2) *For every* $f \in \mathcal{D}(F^*)$*, with our new definition, by*

$$\alpha^{-1}\int_X |\langle f|\psi_x\rangle|^2\,\mathrm{d}\mu(x) \le \ \|A^*f\|^2 \le \alpha\int_X |\langle f|A\phi_x\rangle|^2\,\mathrm{d}\mu(x),$$

it follows that $\|C_\psi f\| \le \alpha\|C_{A\phi}f\|$*, for every* $f \in \mathcal{D}(F^*)$*. Since* C_ψ *is closed, then* $\mathcal{D}(C_\psi^*)$ *is dense and* (5.4) *and* (5.5) *imply that* $\mathcal{D}(C_{A\phi}) \subseteq \mathcal{D}(F^*) \subseteq \mathcal{D}(C_\psi)$*, hence the latter is dense too.*

Another possibility is to mimic the notion of controlled frame (5.1) or (5.2), introduced in [9, Definition 3.1]. Because the operator A is supposed to belong to $GL(\mathcal{H})$, we end up with a generalized frame (and actually a genuine frame). It would be interesting to extend the definition to an unbounded operator or at least to operators less regular than elements of $GL(\mathcal{H})$. A different strategy is to investigate the following generalization. Let B be a linear operator with domain $\mathcal{D}(B)$. Suppose that $\psi_n \in \mathcal{D}(B)$ for all n. Put

$$\Omega_B(f,g) = \sum_n \langle f|\psi_n\rangle\langle B\psi_n|g\rangle, \quad \forall\, f,g \in \mathcal{D}(\Omega_B),$$

where $\mathcal{D}(\Omega_B)$ is some domain of the sesquilinear form defined formally on the rhs. Following [13, Sec. 4], we may consider the form Ω_B as the form generated by two sequences, $\{\psi_n\}$ and $\{B\psi_n\}$. Then, the operator associated to the form Ω_B is precisely B, since one has $\langle Bf|g\rangle = \Omega_B(f,g)$. A continuous version of (5.1) would be

$$\mathsf{m}_A \|f\|^2 \le \int_X \langle f|\psi_x\rangle\langle A\psi_x|f\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}_A \|f\|^2, \quad \text{for all } f \in \mathcal{H},$$

and the sesquilinear form becomes

$$\Omega_A(f,g) = \int_X \langle f|\psi_x\rangle\langle A\psi_x|g\rangle \,\mathrm{d}\mu(x) \le \mathsf{M}_A \|f\|^2, \quad \text{for all } f,g \in \mathcal{D}(\Omega_A).$$

From the last relation, we might infer two alternative possible definitions of an *upper A-semi-frame*, namely:

$$\int_X |\langle Af|\psi_x\rangle|^2 \,\mathrm{d}\mu(x) \le \mathsf{M} \|f\|^2, \quad \forall\, f \in \mathcal{D}(A),$$

$$\int_X \langle f|\psi_x\rangle\, \langle \psi_x|Af\rangle \,\mathrm{d}\mu(x) \le \mathsf{M} \|f\|^2, \quad \forall\, f \in \mathcal{D}(A). \tag{5.6}$$

Actually the definition (5.6) leads to that of an *A-Bessel map*, provided that $\psi_x \in \mathcal{D}(A^*)$, for all $x \in X$:

$$\int_X \langle f|\psi_x\rangle\, \langle A^*\psi_x|f\rangle \,\mathrm{d}\mu(x) \le \mathsf{M} \|f\|^2, \quad \forall\, f \in \mathcal{D}(A).$$

Further, study will hopefully reveal which of the three definitions of an upper A-semi-frame is the most natural one.

6. Examples

(1) A reproducing kernel Hilbert space. We start from the example of a lower semi-frame in a reproducing kernel Hilbert space described in increasing generality in [6, 8]. Let $\mathcal{H}_K$ be a reproducing kernel Hilbert space of (nice) functions on a measure space (X,μ), with kernel function $k_x, x \in X$, that is, $f(x) = \langle f|k_x\rangle_K, \forall f \in \mathcal{H}_K$. Choose a (real valued, measurable) weight function $m(x) > 1$ and consider the unbounded self-adjoint multiplication operator $(Mf)(x) = m(x)f(x), \forall x \in X$, with dense domain $\mathcal{D}(M)$. For each $n \in \mathbb{N}$, define $\mathcal{H}_n = \mathcal{D}(M^n)$, equipped with its graph norm, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual). Then, we have the Hilbert scale $\{\mathcal{H}_n,\, n \in \mathbb{Z}\}$:

$$\ldots \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 = \mathcal{H}_K \;\subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \;\subset \mathcal{H}_{\overline{n}} \ldots .$$

As an operator on the scale, which is a partial inner product space [2], the operator M has continuous representatives $M_{n+1} \to M_n, n \in \mathbb{Z}$. Fix some $n > 1$ and define the measurable functions $\phi_x = k_x m^n(x), \psi_x = k_x m^{-n}(x)$, for every $x \in X$. Then $\psi_x \in \mathcal{H}_n$, for every $x \in X$, and ψ is an upper semi-frame, whereas $\phi_x \in \mathcal{H}_{\overline{n}}$, for every $x \in X$, and ϕ is a lower semi-frame. Also, $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously. One has indeed, for every

$g \in \mathcal{H}_K, \langle \psi_x | g \rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_n$ and $\langle \phi_x | g \rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$. Next, choose a real valued, measurable function $x \mapsto a(x)$ such that $a(x) \leq m^n(x), \forall x \in X$, and define $A = A^*$ as the multiplication operator by $a : (Af)(x) = a(x)f(x), \forall\, x \in X$. Let $\mathcal{D}(A) = \mathcal{H}_n$. Then $A \in \mathcal{B}(\mathcal{H}_n)$ since $\|af\| \leq \|m^n f\| < \infty$, for every $f \in \mathcal{H}_n$ and since $a(x)m^{-n}(x) < 1$ for every $x \in X$ and for every $f \in \mathcal{H}_n$, then $\mathcal{R}(A) \subset \mathcal{D}(M^n) = \mathcal{H}_n$. As an operator on the scale, A has continuous representatives $A_{p,n+p} : \mathcal{H}_{n+p} \to \mathcal{H}_p$. Then, we have, $\forall f \in \mathcal{D}(A) = \mathcal{H}_n \subset \mathcal{D}(C_\phi)$,

$$\|Af\|^2 = \int_X |f(x)|^2\, a(x)^2 \,\mathrm{d}\mu(x) \leq \int_X |f(x)|^2\, m^{2n}(x) \,\mathrm{d}\mu(x) = \int_X |\langle f | \phi_x \rangle_K|^2 \,\mathrm{d}\mu(x) < \infty,$$

that is, ϕ is a weak A-frame for $\mathcal{H}_K$. The same holds for every self-adjoint operator A' which is the multiplication operator by the measurable function $x \mapsto a'(x)$ such that $a'(x) \leq m^n(x), \forall x \in X$, and $\mathcal{D}(A') = \mathcal{H}_n$.

Let now the closed operator B be a ϕ-extension of A, that is,

$$A \subset B \text{ and } \mathcal{H}_n = \mathcal{D}(A) \subset \mathcal{D}(B^*) \subset \mathcal{D}(C_\phi)$$

and

$$\|B^* f\|^2 \leq \int_X |\langle f | \phi_x \rangle_K|^2 \,\mathrm{d}\mu(x) < \infty,\ \forall\, f \in \mathcal{D}(B^*).$$

Then, ϕ is a weak lower A-semi-frame for $\mathcal{H}_K$.

(2) A discrete example. A more general situation may be derived from the discrete example of Section 5.2 of [6]. Take a weight sequence $m := \{|m_n|\}_{n\in\mathbb{N}}, m_n \neq 0$, where $m \in \ell^\infty$ has a subsequence converging to zero (or $m \in c_0$). Then consider the space ℓ^2_m with norm $\|\xi\|_{\ell^2_m} := \sum_{n\in\mathbb{N}} |m_n \xi_n|^2$. Thus, we have the following triplet

$$\ell^2_{1/m} \subset \ell^2 \subset \ell^2_m.$$

Next, for each $n \in \mathbb{N}$, define $\psi_n = m_n e_n$, where $e := \{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in ℓ^2. Then ψ is an upper semi-frame and $C_\psi : \mathcal{H} \to \ell^2_{1/m}$, continuously. On the other hand, $\phi := \{(1/\overline{m_n})e_n\}_{n\in\mathbb{N}}\}$ is a lower semi-frame and $C_\phi : \mathcal{H} \to \ell^2_m$, continuously. In other words, $\psi = Me$ and $\phi = M^{-1}e$, where M is the diagonal operator $M_n = m_n, n \in \mathbb{N}$. In order to define a weak lower A-semi-frame for ℓ^2, we take another diagonal operator $A = \{a_n\}$ such that, for each $n \in \mathbb{N}$ one has $|a_n| \leq |m_n|^{-1}$. Then, $\forall f \in \mathcal{D}(A)$,

$$\begin{aligned}\|Af\|^2 &= \sum_{n\in\mathbb{N}} |a_n|^2 |f_n|^2 = \sum_{n\in\mathbb{N}} |a_n|^2 |\langle f | e_n \rangle|^2 \leq \sum_{n\in\mathbb{N}} |m_n|^{-2} |\langle f | e_n \rangle|^2 \\ &= \sum_{n\in\mathbb{N}} |\langle f | \phi_n \rangle|^2.\end{aligned}$$

Thus, ϕ is a weak A-frame for ℓ^2. As in Example (1), we get a weak lower A-semi-frame for ℓ^2 if we have a ϕ-extension B of A. The same result holds true if one replaces the ONB $\{e_n\}$ by a frame $\{\theta_n\}_{n\in\mathbb{N}}$:

$$\alpha \left\|f\right\|^2 \leq \sum_n |\langle f | \theta_n \rangle|^2 \leq \beta \left\|f\right\|^2,\ \forall f \in \mathcal{H},$$

for some $\alpha, \beta > 0$. Since $m \in \ell^\infty$, we can as well assume that $|m_n| < \delta$, $\forall n \in \mathbb{N}$ for some $\delta > 0$. Thus $|1/\overline{m_n}| > 1/\delta$, $\forall n$. Then, for every $f, g \in \mathcal{H}$, we have

$$\sum_n |\langle f|\psi_n\rangle|^2 = \sum_n |m_n|^2 |\langle f|\theta_n\rangle|^2 \leq \delta^2 \sum_n |\langle f|\theta_n\rangle|^2 \leq \delta^2 \beta \|f\|^2 ,$$

$$\sum_n |\langle g|\phi_n\rangle|^2 = \sum_n \left|\frac{1}{m_n}\right|^2 |\langle g|\theta_n\rangle|^2 \geq \frac{1}{\delta^2} \sum_n |\langle g|\theta_n\rangle|^2 \geq \frac{1}{\delta^2} \alpha \|g\|^2 .$$

Thus, indeed, ψ is an upper semi-frame and ϕ is a lower semi-frame. The rest of the construction follows.

(3) A standard construction. As explained in Section 2, a standard construction of lower semi-frames stems from the consideration of a metric operator induced by an unbounded operator. Given a closed, densely defined, unbounded operator S with dense domain $\mathcal{D}(S)$, define the metric operator $G = I + S^*S$, which is unbounded with bounded inverse. Then, if we take an ONB $\{e_n\}$ of $\mathcal{D}(G^{1/2}) = \mathcal{D}(S)$, contained in $\mathcal{D}(S^*S)$, then $\{\phi_n\} = \{Ge_n\} = \{(I + S^*S)e_n\}$ is a lower semi-frame of $\mathcal{H}$ on $\mathcal{D}(S)$. Now, if A is a densely defined operator that satisfies the equation

$$\alpha \|A^* f\| \leq \|f\|_{C_\phi} , \ \forall\, f \in \mathcal{D}(A^*)$$

instead of (2.1), then ϕ is a weak A-frame for $\mathcal{H}$. As for the equivalent of (2.2), it is of course trivial.

7. Conclusion

In the search of expansions of certain functions into simpler ones, the usual strategy is to pass from orthonormal bases (e.g. Fourier series or integral) to frames, and then to semi-frames, lower or upper. In each case, one obtains more flexibility. The aim of this paper is to apply the same philosophy to more recent structures.

Given a densely defined linear operator A on a Hilbert space $\mathcal{H}$, the notion of weak A-frame was introduced in [10], as explained in Def. 3.2. Following the strategy described above, we obtain the notion of *weak lower A-semi-frame* given in Def. 3.4 and discussed in Section 3. A parallel notion is that of weak atomic system for A, also introduced in [10], from which we obtain that of *lower atomic system for A*. As explained in Section 4, the two original structures go hand in hand and the equivalence extends to the new ones as well. In the same way, one defines a *weak upper A-semi-frame* (Def. 5.7), although this definition is only tentative. Finally, there is a recurrent property of duality, namely, a weak upper A-semi-frame generates by duality a lower one.

The conclusion is that, in each case, one can obtain more flexibility for expansions by passing from frames to semi-frames, as illustrated by the examples provided. Of course, more work is needed. The results presented here are in fact a first step toward a generalization of the notions of weak A-frames and weak atomic system to lower semi-frames.

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