Journal of New Theory
ISSN: 2149-1402

34 (2021) 37-44
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
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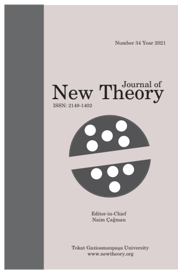

On Evolutes of Null Cartan Curves in Minkowski 4-Space

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Article History
Received: 07 Dec 2020
Accepted: 28 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — This paper aims to discuss the theory of evolutes of null Cartan curves in Minkowski 4-space. In the second part, we present the basic concepts of curves in Minkowski 4-space with its Frenet equations. In the next section, the definition of evolutes of null Cartan curves in Minkowski 4-space is given, and we derive some theorems related to casual characters of those evolute curves. The last part provides an example for the theorems in the preceding section.

Keywords — *Evolutes, involutes, null Cartan curves, Minkowski space*

Mathematics Subject Classification (2020) — 53A04, 51B20

1. Introduction

Applications of geometry in many aspects of human life have motivated many mathematicians to find and develop many new theories of local and general properties of curves and surfaces in geometry. Many theories in classical differential geometry are extended to non-classical differential geometry, such as Lorentzian manifold. It started at the beginning of the twentieth century, when Einstein's theory opened a door for the use of new geometries. One of the theories in classical differential geometry which can be extended to Lorentzian space is the theory of involute-evolute of curves. The concept of involute and evolute of curves in Riemannian manifold was firstly introduced by Huygens in 1973 when he tried to create an accurate clock called isochronous pendulum clock [1]. There are many books and research articles providing explanations about the involute and evolute of curves both in Riemannian space and semi-Riemannian space [2–10].

In Lorentz-Minkowski space, a curve can locally be time-like, space-like or null depending on the casual character of the tangent vector of the curves. For non-null curves (time-like, space-like) it can easily analogue with the curve in Euclidean space. However, geometry of null curves is different from that of non-null curves since the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. The theory of null curves in Minkowski space has been studied by many mathematicians such as Ferrandez, Gimenez and Lucas [11], Inoguchi and Lee [12], and Qian and Kim [13]. Application of null curves has been studied by Duggal [14] and Mohajan [15].

In this study, we will discuss the theory of evolute curves of null Cartan curves in Minkowski 4-space. In the second part, we focus on the basic concepts of curves in Minkowski 4-space with its Frenet equations. In the next section, we introduce and give the general formula of evolute curves of null Cartan curves in Minkowski 4-space. We also provide some theorems and corollaries related to the casual characteristics of the evolute curves which are derived from the null Cartan curves. In the last part, an example is given as an application of the theorems in the previous section.

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2. Preliminary

Minkowski space $\mathbb{E}_1^4$ is the real vector space $\mathbb{R}^4$ equipped with the standard indefinite metric $\langle , \rangle$ defined as

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \tag{1}$$

for any vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. In Minkowski space, any vector $v \neq 0$ is said to be time-like if $\langle v, v\rangle < 0$, space-like if $\langle v, v\rangle > 0$ or $v = 0$ and null if $\langle v, v\rangle = 0$ and $v \neq 0$. The norm of a vector in $\mathbb{E}_1^3$ is defined by $\|v\| = \sqrt{|\langle v, v\rangle|}$.

Let $\alpha : I \to \mathbb{E}_1^3$ be a curve in Minkowski space. Locally, α can be time-like, space-like or null if its tangent vector is time-like, space-like or null, respectively. For non-null curves, the arc length s is defined by $s = \int_0^t \sqrt{|\langle\alpha', \alpha'\rangle|}dt$. If $\langle\alpha', \alpha'\rangle = 1$ the non-null curve is called the curve parametrized by the arc length. For null curves, since $\langle\alpha', \alpha'\rangle = 0$, the pseudo-arc length is defined by $s = \int_0^t \langle\alpha'', \alpha''\rangle^{\frac{1}{4}} dt$, and if $\langle\alpha'', \alpha''\rangle = 1$, then the null curve is parametrized by pseudo-arc length.

Let $\{T(s), N(s), B_1(s), B_2(s)\}$ be the Frenet frame along the curve $\alpha(s)$ in $\mathbb{E}_1^4$. T, N, B_1 and B_2 are the tangent, principal normal, first binormal and second binormal vector fields, respectively. If α is a pseudo-null unit speed curve i.e., a space-like curve with light-like principal normal vector field parametrized by arc length s in $\mathbb{E}_1^4$, the Frenet equations of α are given by

$$T' = \kappa N, \quad N' = \tau B_1, \quad B_1' = \sigma N - \tau B_2, \quad B_2' = -\kappa T - \sigma B_1 \tag{2}$$

where κ and σ denote the curvature and bitorsion of α, respectively. T, N, B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$\begin{aligned} \langle T, T\rangle = \langle B_1, B_1\rangle = 1, \quad \langle N, N\rangle = \langle B_2, B_2\rangle = 0, \quad \langle N, B_2\rangle = 1, \\ \langle T, N\rangle = \langle T, B_1\rangle = \langle T, B_2\rangle = \langle N, B_1\rangle = \langle B_1, B_2\rangle = 0 \end{aligned} \tag{3}$$

The curvature κ in (2) has value 0 when α is a straight line and 1 in all other cases [16].

Let $\gamma : I \to (s)$ be an arbitrary null Cartan curve in $\mathbb{E}_1^4$. Then, there exists a unique Cartan frame $\{T, N, B_1, B_2\}$ given by

$$T = \frac{\gamma'}{\varphi}, \quad N = \left(\frac{1}{\varphi}\right)'\gamma' + \frac{1}{\varphi}\gamma'', \quad B_1 = -\frac{1}{\varphi}\gamma''' - \frac{\langle\gamma''', \gamma'''\rangle}{2\varphi^3}\gamma', \quad B_2 = \frac{1}{\varphi^3}(\gamma' \times \gamma'' \times \gamma''') \tag{4}$$

for any given $\varphi = \sqrt{\langle\varphi'', \varphi''\rangle} > 0$. The Frenet equations of the null curve γ is given by

$$T' = N, \quad N' = -k_1T - B_1, \quad B' = -k_1N + k_2B_2, \quad B_2' = -k_2T \tag{5}$$

where

$$k_1 = \frac{1}{2\varphi^2}(\langle\gamma''', \gamma'''\rangle + 2\varphi\varphi'' - 4(\varphi')^2), \quad k_2 = -\frac{1}{\varphi^4}\det\left(\gamma', \gamma'', \gamma''', \gamma^4\right) \tag{6}$$

Here, k_1 and k_2 are called the first and the second null curvatures of γ. The Cartan Frame $\{T, N, B_1, B_2\}$ satisfies the equations

$$\begin{aligned} \langle T, T\rangle = \langle B_1, B_2\rangle = 0, \quad \langle T, N\rangle = \langle N, N\rangle = \langle B_2, B_2\rangle = 1, \\ \langle T, N\rangle = \langle T, B_2\rangle = \langle B_1, N\rangle = \langle B_1, B_2\rangle = \langle B_2, N\rangle = 0, \end{aligned} \tag{7}$$

and

$$N \times T \times B_1 = B_2, \quad N \times B_2 \times T = T, \quad N \times B_1 \times B_2 = B_1, \quad T \times B_2 \times B_1 = N \tag{8}$$

(see [17]).

A null curve lies on pseudo-sphere in $\mathbb{E}_1^4$ with radius r if and only if $k_2 = \pm\frac{1}{r}$ [18]. In addition, a null curve which has non-zero constant k_1 and k_2 in $\mathbb{E}_1^4$ are called null helices [14]. Furthermore, a null Cartan curve in $\mathbb{E}_1^4$ is a Bertrand null curve if and only if k_1 is non-zero constant and k_2 is zero [19]. In Euclidean case, if β is an evolute of α, then for a given point P on β and the corresponding point P' on α the principal normal line of β at P is parallel to the tangent line of α at P' [8].

3. Evolutes of Null Cartan Curves

Definition 3.1. The curve $\gamma^*(s)$ is the evolute of null Cartan curve $\gamma(s)$ if and only if for all $s \in I \subseteq \mathbb{R}$, the the tangent line of $\gamma^*(s)$ intersects $\gamma(s)$ orthogonally.

Let $\gamma(s)$ be a null Cartan curve parametrized by pseudo-arc length s and γ^* be its evolute curves. If x^* be the point of contact on the evolute to the tangent line which intersects γ at $x(s)$, then $x^* - x$ lies on the tangent line of γ^* and perpendicular to the tangent vector of γ. Since γ is a null Cartan curve, $x^* - x$ can be represented as linear combination of the principal normal vector $N(s)$ and the second binormal vector $B_2(s)$ of curve $\gamma(s)$. Therefore, it can be written as

$$x^* = x(s) + p(s)(s)N(s) + q(s)B_2(s) \tag{9}$$

Next, we will find the function $p(s)$ and $q(s)$ by considering the causal characters of the curves.

Theorem 3.2. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then,

$$\gamma^*(s) = \gamma(s) + \frac{1}{k_2}B_2(s) \tag{10}$$

Proof. If we take the derivative of equation (9), we have

$$\begin{aligned}(\gamma^*)' =& T + p'N + p(-k_1T - B_1) + q'B_2 + q(-k_2T)\\ =&(1 - pk_1 - qk_2)T + p'N - pB_1 + q'B_2\end{aligned} \tag{11}$$

Since $(\gamma^*)'(s)$ is the tangent of $\gamma^*(s)$, which is perpendicular to the tangent vector T of γ, $\gamma^*(s)$ is proportional to $\gamma^*(s) - \gamma(s) = pN + qB_2$. Therefore, we get

$$1 - pk_1 - qk_2 = 0 \quad p' = \lambda p, \quad p = 0, \quad q' = \lambda q \tag{12}$$

for some smooth real function λ in $\mathbb{E}^4_1$. Consequently, From Equation (12) we have

$$p = 0, \qquad q = \frac{1}{k_2} \tag{13}$$

Substituting these value into Equation (9) obtains Equation (10). □

Theorem 3.3. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, the distance between $\gamma^*(s)$ and $\gamma(s)$ is $\frac{1}{k_2}$.

Proof. From equation (10), we have

$$\gamma^*(s) - \gamma(s) = \frac{1}{k_2}B_2(s) \tag{14}$$

Therefore,

$$\|\gamma^*(s) - \gamma(s)\| = \sqrt{\left\langle \frac{1}{k_2}B_2(s), \frac{1}{k_2}B_2(s)\right\rangle} = \frac{1}{k_2}$$

□

Theorem 3.4. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, $\gamma^*(s)$ is a space-like curve.

Proof. From Equations (11) and (13) we have

$$(\gamma^*(s))' = -\frac{k_2'}{k_2^2}B_2 \tag{15}$$

Therefore,

$$\langle(\gamma^*(s))', (\gamma^*(s))'\rangle = \left\langle -\frac{k_2'}{k_2^2}B_2, -\frac{k_2'}{k_2^2}B_2\right\rangle = \left(\frac{k_2'}{k_2^2}\right)^2 > 0$$

Thus, the proof is completed □

Theorem 3.5. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s and $\{T^*, N^*, B_1^*, B_2^*\}$ be the Frenet frame of $\gamma^*(s)$. If $\{T, N, B_1, B_2\}$ and k_2 are the Frenet frame and the non-constant second null curvature of $\gamma(s)$, then

$$T^* = -B_2, \quad N^* = \frac{k_2^3}{k_2'}T, \quad B_1^* = \frac{3(k_2')^2 - k_2k_2''}{k_2k_2'}T + N, \quad B_2^* = \frac{k_2'}{k_2^3}B_1 \tag{16}$$

PROOF. Let s^* be the arc length parameter of γ^*. Therefore, by Equation (15) we have

$$\frac{d\gamma^*}{ds^*} \cdot \frac{ds^*}{ds} = -\frac{k_2'}{k_2^2}B_2 \Longrightarrow T^*\frac{ds^*}{ds} = -\frac{k_2'}{k_2^2}B_2$$

Taking the norm of the Equation above yields $\frac{ds^*}{ds} = \pm\frac{k_2'}{k_2^2}$. As a result we get

$$T^* = -B_2 \tag{17}$$

Differentiating (17) towards parameter s yields

$$\frac{dT^*}{ds^*}\frac{ds^*}{ds} = k_2T \Longrightarrow \kappa N^* = \frac{k_2^3}{k_2'}T \tag{18}$$

From (18) we find that N^* is a null principal normal vector field since T is a null vector. Therefore, γ^* is a pseudo-null curve in $\mathbb{E}_1^4$. Take $\kappa = 1$ by assuming that γ^* is a non-straight line.

Taking the derivative of N^* towards s^* yields,

$$\frac{dN}{ds^*} = \frac{dN^*}{ds}\frac{ds}{ds^*} = \left(\frac{3k_2^2(k_2')^2 - k_2^3k_2''}{(k_2')^2}T + \frac{k_2^3}{k_2'}N\right)\frac{k_2^2}{k_2'} = \frac{3k_2^4(k_2')^2 - k_2^5k_2''}{(k_2')^3}T + \frac{k_2^5}{(k_2')^2}N$$

As a consequence, we have

$$\|\frac{dN}{ds^*}\| = \frac{k_2^5}{(k_2')^2} \tag{19}$$

Using Equation (2) we find

$$B_1^* = \frac{\frac{dN}{ds^*}}{\|\frac{dN}{ds^*}\|} = \frac{3(k_2')^2 - k_2k_2''}{k_2k_2'}T + N \tag{20}$$

and

$$B_2^* = \frac{k_2'}{k_2^3}B_1 \tag{21}$$

satisfying Equation (3). □

Theorem 3.5 results in the following corollary.

Corollary 3.6. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. Then, $\gamma^*(s)$ is a pseudo-null curve i.e., a space-like curve with light-like principal normal vector field.

Theorem 3.7. Let $\gamma^*(s)$ be the evolute of a null Cartan curve $\gamma(s)$ parametrized by pseudo-arc length s. If γ^* is a non-straight line, then the curvature, torsion and bitorsion of γ^* are given by

$$\kappa = 1, \quad \tau = \frac{k_2^5}{(k_2')^2}, \quad \sigma = -\frac{1}{k_2}\left(\frac{3(k_2')^2 - k_2k_2''}{k_2k_2'}\right)^2 - k_1 \tag{22}$$

PROOF. Since $\gamma^*(s)$ is a pseudo-null in $\mathbb{E}_1^4$ and a non-straight line, from Equations (2) and (19), we have

$$\kappa = 1, \quad \tau = \|\frac{dN^*}{ds^*}\| = \frac{k_2^5}{(k_2')^2}$$

Differentiating Equation (21) towards parameter s^*, we have

$$\frac{dB_2^*}{ds^*} = \frac{dB_2^*}{ds}\frac{ds}{ds^*} = \left(\frac{k_2^3 k_2'' - 3k_2^2(k_2')^2}{k_2^6}B_1 + \frac{k_2'}{k_2^2}(-k_1 N + k_2 B_2)\right)\frac{k_2^2}{k_2'} = -k_1 N + \frac{k_2 k_2'' - 3(k_2')^2}{k_2^2 k_2'}B_1 + k_2 B_2$$

Therefore, using (2), we find

$$\begin{aligned}\sigma &= -\left\langle B_1^*, \frac{B_2^*}{ds^*}\right\rangle \\ &= \left\langle \frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}T + N, -k_1 N + \frac{k_2 k_2'' - 3(k_2')^2}{k_2^2 k_2'}B_1 + k_2 B_2\right\rangle \\ &= -\frac{1}{k_2}\left(\frac{3(k_2')^2 - k_2 k_2''}{k_2 k_2'}\right)^2 - k_1\end{aligned}$$

□

Theorem 3.7 results in some corollaries as follow:

Corollary 3.8. If $\gamma(s)$ lies on pseudo-sphere in $\mathbb{E}_1^4$ with radius r, $\gamma(s)$ has no evolute curve.

Corollary 3.9. Let $\gamma(s)$ be a planar null Cartan curve. Then, there is no evolute of $\gamma(s)$.

Corollary 3.10. If $\gamma(s)$ is a Bertrand null curve, $\gamma(s)$ has no evolute curve.

The proof of corollaries 3.8, 3.9, and 3.10 is clear since k_2 is a constant, which implies the tangent vectors of γ^* vanish everywhere.

4. Example

In this section, an example of the evolute of the null Cartan curves is provided as an application of the theorems in the previous section.

Example 4.1. Let $\gamma : I \to E_1^3$ be a null Cartan curve parametrized by pseudo-arc length s and given as

$$\begin{aligned}\gamma(s) = \Bigg(&\frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right), \frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right), \\ &\frac{2s^2}{9\sqrt{14}}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \\ &\frac{2s^2}{9\sqrt{14}}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg)\end{aligned}$$

By direct calculation using (4), we find

$$T = \left(\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{s} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{s}\right), \frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{s} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{s}\right), \frac{s\sqrt{14}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right), \frac{s\sqrt{14}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$\begin{aligned}N = \Bigg(&\frac{\sqrt{14}}{56}\left(\frac{(2+3\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}}{2s^2} + \frac{(2-3\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}}{2s^2}\right), \frac{\sqrt{14}}{56}\left(\frac{(2+3\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}}}{2s^2} - \frac{(2-3\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}}{2s^2}\right), \\ &-\frac{\sqrt{14}}{28}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{14}}{28}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),\end{aligned}$$

$$B_1 = \Bigg(-\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}} + (5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}} - (5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right)$$

$$\frac{\sqrt{7}}{28s}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{7}}{28s}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

$$B_2 = \Bigg(-\frac{\sqrt{7}}{28s^2}\left(-s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{7}}{28s^2}\left(-s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{21}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right),$$

$$\frac{3\sqrt{21}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg)$$

By using (6), we find

$$k_1 = -\frac{6}{s^2}, \qquad k_2 = -\frac{3\sqrt{3}}{2s^2}$$

Substituting k_2 and B_2 into (9), we find the evolute curve of γ as

$$\gamma^*(s) = \Bigg(\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right) + \frac{\sqrt{21}}{126}\left(s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right)$$

$$+\frac{\sqrt{21}}{126}\left(s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{s^2\sqrt{14}}{63}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$

$$+\frac{s^2\sqrt{7}}{7}\sin\left(\frac{\sqrt{2}}{2}\ln s\right), \frac{s^2\sqrt{14}}{63}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)$$

$$-\frac{s^2\sqrt{7}}{7}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg)$$

By using Equation (16), we get the Frenet frame of $\gamma^*(s)$ as follows:

$$T^* = \left(\frac{\sqrt{7}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right), \frac{\sqrt{7}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right), \frac{3\sqrt{21}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)}{14}, -\frac{3\sqrt{21}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)}{14} \right),$$

$$N^* = \left(\frac{s\sqrt{14}}{112}\left(s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{s\sqrt{14}}{112}\left(s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{s^3\sqrt{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)}{56}, \frac{s^3\sqrt{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)}{56} \right),$$

$$B_1^* = \Bigg(-\frac{\sqrt{14}}{168}\left(\frac{(3\sqrt{6}-4)s^{2+\frac{3\sqrt{6}}{2}}}{s^2} - \frac{(3\sqrt{6}+4)s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right), \frac{\sqrt{14}}{56}\left(\frac{(3\sqrt{6}-4)s^{2+\frac{3\sqrt{6}}{2}}}{s^2} + \frac{(3\sqrt{6}+4)s^{2-\frac{3\sqrt{6}}{2}}}{s^2}\right),$$

$$-\frac{\sqrt{7}}{14}\left(2\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + \sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{7}}{14}\left(-2\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) + \cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

$$B_2^* = \Bigg(\frac{\sqrt{14}}{63}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}} + (5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{14}}{63}\left((5+\sqrt{6})s^{2+\frac{3\sqrt{6}}{2}} - (5-\sqrt{6})s^{2-\frac{3\sqrt{6}}{2}}\right)$$

$$-\frac{2s^2\sqrt{7}}{189}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$-\frac{2s^2\sqrt{7}}{189}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

Finally, by using Equation (20), we get

$$\kappa = 1, \quad \tau = -\frac{81\sqrt{3}}{32s^4}, \quad \sigma = 2\sqrt{3} + \frac{3}{s^2}$$

5. Conclusion

Based on the definitions, theorems, and an example in the previous section, we find that the evolute of the null Cartan curve in Minkowski 4-space is a pseudo-null curve - i.e., a space-like curve with light-like principal normal vector field. Furthermore, there is no evolute of null Cartan helices, null Bertrand curves, and null curves lying on the pseudo-sphere in $\mathbb{E}_1^4$.

Conflicts of Interest

The authors declare no conflict of interest.

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