



Global Analysis of a $(1, 2)$ -Type System of Non-Linear Difference Equations

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Abstract

This paper deals with the study of global analysis of following $(1, 2)$ -type system of non-linear difference equations:

$$u_{n+1} = \frac{\alpha v_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 u_n^p u_{n-2}^q}, \quad n = 0, 1, \dots$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q$ and the initial conditions $u_i, v_i, i = -2, -1, 0$ are non negative real numbers.

Keywords: $(1, 2)$ -type system, Difference equations, Equilibrium, Global stability, Rate of convergence.

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1. Introduction

Difference equations (also called recursive sequences) appear in a lot of fields of pure and applied mathematics, both as discrete analogs of continuous behavior (analysis, numerical approximations) and as independent models for discrete behavior (population dynamics, economics, biology, ecology, etc.), [1]. In recent years, many models, especially in mathematical biology, are based on non-linear ones, [11]. Difference equation theory, especially nonlinear ones, is very fertile subject for scientists and is one of the important subjects of applied mathematics. So, many researchers have dealt with the qualitative behavior of nonlinear higher order rational difference equations and systems, see [1]-[33].

In [9], El-Owaidy *et al.* studied the global analysis of the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_n^p}, \quad n = 0, 1, \dots \quad (1.1)$$

with non-negative parameters and non-negative initial values.

In [5], Ahmed investigated the global asymptotic behavior and the periodic character for the rational difference equation

$$x_{n+1} = \frac{bx_{n-1}}{A + Bx_n^p x_{n-2}^p}, \quad n = 0, 1, \dots \quad (1.2)$$

where the parameters b, A, B, p, q are non-negative numbers and the initial values x_{-2}, x_{-1}, x_0 are arbitrary non-negative real numbers.

In [15], Gümüş and Soykan investigated the local asymptotic stability of equilibria, the periodic nature of solutions, the existence of unbounded solutions and the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-(k+1)}}{\beta + \gamma x_{n-k}^p x_{n-(k+2)}^q}, \quad n = 0, 1, \dots \tag{1.3}$$

where the parameters $\alpha, \beta, \gamma, p, q$ are non-negative numbers and the initial values $x_{-(k+2)}, x_{-(k+1)}, \dots, x_{-1}, x_0 \in \mathbb{R}^+$.

In [13], Gümüş and Soykan studied the dynamical behavior of positive solutions for a system of rational difference equation following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^p}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^p}, \quad n = 0, 1, \dots \tag{1.4}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p$ and the initial values u_{-i}, v_{-i} for $i = 0, 1, 2$ are positive real numbers.

In [14], Gümüş and Öcalan studied the dynamical behavior of positive solutions for a system of rational difference equations following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}, \quad n = 0, 1, \dots \tag{1.5}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q, p_1, q_1$ are positive real numbers and the initial values u_{-i}, v_{-i} are non-negative real numbers for $i = 0, 1, 2$.

In [28], Khan *et al.* investigated the asymptotic behavior of following anti-competitive system of rational difference equations

$$\left. \begin{aligned} x_{n+1} &= \frac{\alpha y_n}{\beta + \gamma x_n^r}, \\ y_{n+1} &= \frac{\alpha_1 x_n}{\beta_1 + \gamma_1 y_n^r}, \end{aligned} \right\} n = 0, 1, \dots$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, r \in (0, \infty)$ and $x_0, y_0 \in (0, \infty)$.

In [29], Qureshi and Din investigated the qualitative asymptotic behavior of positive solution for an anti-competitive system of third-order rational difference equations

$$\left. \begin{aligned} x_{n+1} &= \frac{y_{n-2}}{\beta + \gamma x_n x_{n-1} x_{n-2}}, \\ y_{n+1} &= \frac{\alpha_1 x_{n-2}}{\beta_1 + \gamma_1 y_n y_{n-1} y_{n-2}}, \end{aligned} \right\} n = 0, 1, \dots$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and $x_0, x_{-1}, x_{-2}, y_0, y_{-1}, y_{-2}$ are positive real numbers.

In [27], Qureshi and Khan studied the global dynamics of following (1,2)-type systems of difference equations

$$x_{n+1} = \frac{\eta y_{n-1}}{1 + \mu x_{n-2}^p}, \quad y_{n+1} = \frac{\mu x_{n-1}}{1 + \eta y_{n-2}^p}, \quad n = 0, 1, \dots \tag{1.6}$$

$$x_{n+1} = \frac{\eta y_{n-1}}{1 + \mu y_{n-2}^p}, \quad y_{n+1} = \frac{\mu x_{n-1}}{1 + \eta x_{n-2}^p}, \quad n = 0, 1, \dots$$

where η, μ, p and initial conditions $x_l, y_l, l = -2, -1, 0$ are non negative real numbers.

In the present paper, we will investigate of some properties, such as the local asymptotic stability, the global asymptotic stability, the existence of periodic solutions, the rate of converges etc., for (1,2)-type system of difference equations in the title, which has been investigated different versions of it in the known literature. We first note down critical error for the results of the article [27]. Namely, to put it briefly, they can not obtain the equations they claim with their transformations. Using the transformations, they could get equations in this form;

$$\left. \begin{aligned} x_{n+1} &= \frac{\eta y_{n-1}}{1 + \mu y_{n-2}^p}, \\ y_{n+1} &= \frac{\eta_1 x_{n-1}}{1 + \mu_1 x_{n-2}^p}, \end{aligned} \right\} n = 0, 1, \dots$$

with

$$\eta = \frac{\alpha}{\beta}, \quad \eta_1 = \frac{\alpha_1}{\beta_1}$$

and

$$\mu_1 = \frac{\beta}{\gamma}, \quad \mu = \frac{\beta_1}{\gamma_1}.$$

The same applies to the other equation. Let us also note that the theoretical results they obtained in their article are correct. However, an error was made only at the beginning.

The aim of this paper is to investigate the equilibrium points, the local asymptotic stability of these points, the global behavior of positive solutions, the existence of the prime two-periodic solutions and the rate of convergence of positive solutions of the following system

$$\left. \begin{aligned} u_{n+1} &= \frac{\alpha v_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \\ v_{n+1} &= \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 u_n^p u_{n-2}^q}, \end{aligned} \right\} n = 0, 1, \dots \tag{1.7}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1, p, q$ are positive and initial condition $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0 \in (0, \infty)$. Our results extend and complement some results in the literature.

If the initials conditions $u_i = v_i$ in the system (1.7) for $i \in \{-2, -1, 0\}$ and $\alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1$ then one obtain that $u_n = v_n$ for all $n \geq -2$, hence, the system (1.7) reduces to the difference equation

$$v_{n+1} = \frac{\alpha v_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad n = 0, 1, \dots$$

which was studied by [4]. Therefore, to avoid degenerate situations, here we discuss the case $u_i \neq v_i$ for $i \in \{-2, -1, 0\}$ and we investigate the system (1.7) basing on this condition.

It is clear that the system (1.7) can be reduced to the following system of difference equations

$$\left. \begin{aligned} x_{n+1} &= \frac{r y_{n-1}}{1 + s_1 y_n^p y_{n-2}^q}, \\ y_{n+1} &= \frac{r_1 x_{n-1}}{1 + s x_n^p x_{n-2}^q}, \end{aligned} \right\} n = 0, 1, \dots \tag{1.8}$$

by the change of variables

$$u_n = \left(\frac{\beta \beta_1}{\gamma \gamma_1} \right)^{1/p+q} x_n$$

and

$$v_n = \left(\frac{\beta \beta_1}{\gamma \gamma_1} \right)^{1/p+q} y_n$$

with

$$r = \frac{\alpha}{\beta}, \quad r_1 = \frac{\alpha_1}{\beta_1}$$

and

$$s = \frac{\beta}{\gamma}, \quad s_1 = \frac{\beta_1}{\gamma_1}.$$

So, in order to study the system (1.7), we investigate the system (1.8).

2. Preliminaries

For the completeness in the paper, we find useful to remind some basic concepts of the difference equations theory as follows:
 Let us introduce the six-dimensional discrete dynamical system

$$\begin{aligned} x_{n+1} &= f_1(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \\ y_{n+1} &= f_2(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}), \end{aligned} \quad (2.1)$$

$n \in \mathbb{N}$ where $f_1 : I_1^3 \times I_2^3 \rightarrow I_1$ and $f_2 : I_1^3 \times I_2^3 \rightarrow I_2$ are continuously differentiable functions and I_1, I_2 are some intervals of real numbers. Then, for every initial conditions $(x_i, y_i) \in I_1 \times I_2$, for $i = -2, -1, 0$ the system (2.1) has a unique solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$.

Definition 2.1. An equilibrium point of system (2.1) is a point (\bar{x}, \bar{y}) that satisfies

$$\begin{aligned} \bar{x} &= f_1(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}), \\ \bar{y} &= f_2(\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}), \end{aligned}$$

Together with system (2.1), if we consider the associated vector map

$$F = (f_1, x_n, x_{n-1}, x_{n-2}, f_2, y_n, y_{n-1}, y_{n-2}),$$

then the point (\bar{x}, \bar{y}) is also called a fixed point of the vector map F .

Definition 2.2. If (\bar{x}, \bar{y}) be an equilibrium point of a map

$$F = (f_1, x_n, x_{n-1}, x_{n-2}, f_2, y_n, y_{n-1}, y_{n-2})$$

where f_1 and f_2 are continuously differentiable functions at (\bar{x}, \bar{y}) . The linearized system (2.1) about the equilibrium point (\bar{x}, \bar{y}) is

$$X_{n+1} = F(X_n) = BX_n$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and B is a Jacobian matrix of the system (2.1) about the equilibrium point (\bar{x}, \bar{y}) .

Theorem 2.3. For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \dots$, of difference equations such that \bar{X} is a fixed point of F . If all eigenvalues of the Jacobian matrix B about \bar{X} lie inside the open unit disk $|\lambda| < 1$, then \bar{X} is locally asymptotically stable. If one of them has a modulus greater than one, then \bar{X} is unstable.

3. Stability Character Of Equilibrium

In this section we will prove the stability nature of the zero equilibrium point. In the following theorem we will give the equilibrium points of system (1.8).

Theorem 3.1. For all parameters r, r_1, s, s_1 , system (1.8) have a unique zero equilibrium point.

Proof. It is clear from the equilibrium definition. □

Before we give the following stability theorems about the local asymptotic stability of the zero equilibrium point, we build the corresponding linearized form of the system (1.8) and consider the following transformation;

$$(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) \rightarrow (f, f_1, f_2, g, g_1, g_2)$$

where

$$\begin{aligned} f &= \frac{ry_{n-1}}{1 + s_1 y_n^p y_{n-2}^q}, \\ f_1 &= x_n, \\ f_2 &= x_{n-1}, \\ g &= \frac{r_1 x_{n-1}}{1 + s x_n^p x_{n-2}^q}, \\ g_1 &= y_n, \\ g_2 &= y_{n-1}. \end{aligned}$$

The Jacobian matrix about the fixed point (\bar{x}, \bar{y}) under the above transformation is as follows:

$$B(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{rs_1 p \bar{y}^{p+q}}{(1 + s_1 \bar{y}^{p+q})^2} & \frac{r}{1 + s_1 \bar{y}^{p+q}} & -\frac{rs_1 q \bar{y}^{p+q}}{(1 + s_1 \bar{y}^{p+q})^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{r_1 s p \bar{x}^{p+q}}{(1 + s \bar{x}^{p+q})^2} & \frac{r_1}{1 + s \bar{x}^{p+q}} & -\frac{r_1 s q \bar{x}^{p+q}}{(1 + s \bar{x}^{p+q})^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where $r, s, r_1, s_1, p, q \in (0, \infty)$.

Theorem 3.2. For system (1.8) the following properties hold:

- (i) The zero equilibrium point is locally asymptotically stable if $rr_1 < 1$.
- (ii) The zero equilibrium point is locally unstable if $rr_1 > 1$.

Proof. (i) The linearized system of system (1.8) about the equilibrium point

$$(\bar{x}_0, \bar{y}_0) = (0, 0)$$

is given by

$$X_{n+1} = B(\bar{x}_0, \bar{y}_0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \end{pmatrix}$$

and

$$B(\bar{x}_0, \bar{y}_0) = \begin{pmatrix} 0 & 0 & 0 & 0 & r & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $B(\bar{x}_0, \bar{y}_0)$ is as follows:

$$P(\lambda) = \lambda^6 - (rr_1)\lambda^2 = 0.$$

The roots of $P(\lambda)$ are

$$\begin{aligned}\lambda_{1,2} &= 0, \\ \lambda_{3,4} &= \pm\sqrt[4]{rr_1}, \\ \lambda_{5,6} &= \pm i\sqrt[4]{rr_1}.\end{aligned}$$

Since all eigenvalues of the Jacobian matrix B about

$$(\bar{x}_0, \bar{y}_0) = (0, 0)$$

lie inside the open unit disk

$$|\lambda| < 1,$$

the zero equilibrium point is locally asymptotically stable.

(ii) It is easy to see that if $rr_1 > 1$, then the zero equilibrium point of system (1.8) is unstable. □

Now, we will study the global asymptotic stability of system (1.8) about the zero equilibrium point.

Theorem 3.3. *The zero equilibrium point of system (1.8) is globally asymptotically stable when $r < 1$ and $r_1 < 1$.*

Proof. In view of Theorem 3.2, it suffices to prove that

$$\lim_{n \rightarrow \infty} (x_n, y_n) = (0, 0).$$

It is evident from (1.8) that

$$0 \leq x_{n+1} = \frac{ry_{n-1}}{1 + s_1 y_n^p y_{n-2}^q} < ry_{n-1} < y_{n-1}.$$

This implies that

$$x_{4n+1} < y_{4n-1}$$

and

$$x_{4n+5} < y_{4n+3}.$$

Besides this,

$$0 \leq y_{n+1} = \frac{r_1 x_{n-1}}{1 + s x_n^p x_{n-2}^q} < r_1 x_{n-1} < x_{n-1}$$

This implies that

$$y_{4n+1} < x_{4n-1}$$

and

$$y_{4n+5} < x_{4n+3}.$$

So

$$x_{4n+5} < y_{4n+3} < x_{4n+2}$$

and

$$y_{4n+5} < x_{4n+3} < y_{4n+2}.$$

Hence, the subsequences

$$\{x_{4n+1}\}, \{x_{4n+2}\}, \{x_{4n+3}\}, \{x_{4n+4}\}$$

and

$$\{y_{4n+1}\}, \{y_{4n+2}\}, \{y_{4n+3}\}, \{y_{4n+4}\}$$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are monotonic which are decreasing. Hence

$$\lim_{n \rightarrow \infty} x_n = 0$$

and

$$\lim_{n \rightarrow \infty} y_n = 0.$$

This completes the proof. □

4. Prime Periodic Two-Solutions 1.8

In this section we will investigate the periodic nature of system (1.8).

Theorem 4.1. *System (1.8) has no prime period two solutions.*

Proof. Assuming

$$\dots, (a, b), (c, d), (a, b), (c, d), \dots$$

is prime period two solutions of the system (1.8) such that

$$a, b, c, d \neq 0$$

and

$$a \neq c, b \neq d.$$

Then we have

$$a = \frac{rb}{1 + s_1 d^{p+q}}, b = \frac{r_1 a}{1 + s c^{p+q}} \tag{4.1}$$

and

$$c = \frac{rd}{1 + s_1 b^{p+q}}, d = \frac{r_1 c}{1 + s a^{p+q}} \tag{4.2}$$

After some tedious calculations from (4.1) and (4.2), we can obtain the following equalities;

$$(a + c)^2 - 4ac = 0,$$

and

$$(b + d)^2 - 4bd = 0.$$

But they are contrary to our assumption and therefore system (1.8) has no prime period-two solutions. This completes the proof. □

5. Rate of Convergence

In this section, we will give exact results about the rate of convergence of positive solutions that converge to the equilibrium point of the system (1.8), in the regions of parameters described in Theorem (3.3).

Consider the following system of difference equations

$$\left. \begin{aligned} x_{n+1} &= f_1(x_n, y_n), n = 0, 1, \dots \\ y_{n+1} &= f_2(x_n, y_n), n = 0, 1, \dots \end{aligned} \right\} \tag{5.1}$$

where f_1, f_2 are continuous functions that maps some set I into I . The set I is an interval of real numbers. System (5.1) is competitive if $f_1(x, y)$ is non-decreasing in x and non-increasing in y and $f_2(x, y)$ is non-increasing in x and non-decreasing in y . System (5.1) is called anti-competitive system, if the functions f_1 and f_2 have monotonic character opposite to the monotonic character in competitive system.

We state that the following theorems give precise information about the asymptotics of linear non-autonomous difference equations. Consider the scalar m th-order linear difference equation

$$y_{n+m} + p_1(n)y_{n+m-1} + p_m(n)y_n = 0 \tag{5.2}$$

where m is a positive integer and $p_i : \mathbb{Z}^+ \rightarrow \mathbb{C}$ for $i \in \{1, \dots, m\}$. Suppose that

$$q_i = \lim_{n \rightarrow \infty} p_i(n), \text{ for } i = 1, 2, \dots, m, \tag{5.3}$$

exist in \mathbb{C} . For the following limiting equation of (5.2)

$$y_{n+m} + q_1 y_{n+m-1} + \dots + q_m y_n = 0, \tag{5.4}$$

the asymptotics of solutions of (5.2) are given the following results. See [25].

Theorem 5.1. (Poincaré’s Theorem) Consider (5.2) based on the condition (5.3). Let λ_i for $i = 1, \dots, m$ be the roots of the characteristic equation

$$\lambda^m + q_1 \lambda^{m-1} + \dots + q_m = 0 \tag{5.5}$$

of the limiting equation (5.4) under the condition that $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. If x_n is a positive solution of (5.2), then either $x_n = 0$ for all large n or there exists an index $j \in \{1, \dots, m\}$ such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The related results were obtained by Perron, and one of Perron’s results was improved by Pituk, see [25].

Theorem 5.2. Assume that (5.3) holds. If x_n is a positive solution of (5.2), then either eventually $x_n = 0$ or

$$\limsup_{n \rightarrow \infty} (|x_{n+j}|)^{1/n} = |\lambda_j|,$$

where $\lambda_1, \dots, \lambda_m$ are the roots (not necessarily distinct) of the characteristic equation (5.5).

Consider

$$Y_{n+1} = [A + B(n)]Y_n \tag{5.6}$$

where Y_n is an m -dimensional vector, $A \in \mathbb{C}^{m \times m}$ is a constant matrix and

$$B : \mathbb{Z}^+ \rightarrow \mathbb{C}^{m \times m}$$

is a matrix function satisfying

$$\|B(n)\| \rightarrow 0, \text{ when } n \rightarrow \infty, \tag{5.7}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm $\|\cdot\|$. See [20].

Theorem 5.3. (Pituk) Suppose that condition (5.7) holds for system (5.6). If Y_n is a solution of (5.6), then either

$$Y_n = 0$$

for all large n or

$$\theta = \lim_{n \rightarrow \infty} \|Y_n\|^{1/n}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A .

Theorem 5.4. (Pituk) Suppose that condition (5.7) holds for system (5.6). If Y_n is a solution of (5.6), then either

$$Y_n = 0$$

for all large n or

$$\theta = \lim_{n \rightarrow \infty} \frac{\|Y_{n+1}\|}{\|Y_n\|}$$

exists and θ is equal to the modulus of one the eigenvalues of the matrix A .

Using Theorem (5.3) and (5.4), we obtain the following rate of convergence result.

Theorem 5.5. Suppose that $r < 1$ and $r_1 < 1$. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (1.8) such that

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &= \bar{x}_1, \\ \lim_{n \rightarrow \infty} y_n &= \bar{x}_2 \end{aligned}$$

where $M = (\bar{x}_1, \bar{x}_2)$ and M is globally asymptotically stable. Then, the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_{n-2}^1 \\ e_n^2 \\ e_{n-1}^2 \\ e_{n-2}^2 \end{pmatrix}_{6 \times 1} = \begin{pmatrix} x_n - \bar{x}_1 \\ x_{n-1} - \bar{x}_1 \\ x_{n-2} - \bar{x}_1 \\ y_n - \bar{x}_2 \\ y_{n-1} - \bar{x}_2 \\ y_{n-2} - \bar{x}_2 \end{pmatrix}_{6 \times 1}$$

of every positive solution of the system (1.8) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n \rightarrow \infty} \|E_n\|^{1/n} &= |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \dots, 6 \\ \lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, \dots, 6 \end{aligned}$$

where

$$|\lambda_i J_F(M)|$$

is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M .

Proof. Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be any positive solution of the system (1.8) such that

$$\lim_{n \rightarrow \infty} x_n = \bar{x}_1$$

and

$$\lim_{n \rightarrow \infty} y_n = \bar{x}_2.$$

To find the error terms, we have

$$\begin{aligned} x_{n+1} - \bar{x}_1 &= \sum_{i=0}^2 A_i (x_{n-i} - \bar{x}_1) + \sum_{i=0}^2 B_i (y_{n-i} - \bar{x}_2) \\ y_{n+1} - \bar{x}_2 &= \sum_{i=0}^2 C_i (x_{n-i} - \bar{x}_1) + \sum_{i=0}^2 D_i (y_{n-i} - \bar{x}_2). \end{aligned}$$

Set

$$\begin{aligned} e_n^1 &= x_n - \bar{x}_1, \\ e_n^2 &= y_n - \bar{x}_2; \end{aligned}$$

therefore, it follows that

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^2 A_i e_{n-i}^1 + \sum_{i=0}^2 B_i e_{n-i}^2 \\ e_{n+1}^2 &= \sum_{i=0}^2 C_i e_{n-i}^1 + \sum_{i=0}^2 D_i e_{n-i}^2 \end{aligned}$$

where

$$\begin{aligned}
 A_0 &= 0, \quad A_1 = 0, \quad A_2 = 0, \\
 B_0 &= -\frac{rs_1 p \bar{y}(y_n^p y_{n-2}^q - \bar{y}^{p+q})}{(1 + s_1 y_n^p y_{n-2}^q)^2}, \\
 B_1 &= \frac{r}{1 + s_1 y_n^p y_{n-2}^q}, \\
 B_2 &= -\frac{rs_1 q \bar{y}(y_n^p y_{n-2}^q - \bar{y}^{p+q})}{(1 + s_1 y_n^p y_{n-2}^q)^2}, \\
 C_0 &= -\frac{r_1 s p \bar{x}(x_n^p x_{n-2}^q - \bar{x}^{p+q})}{(1 + s x_n^p x_{n-2}^q)^2}, \\
 C_1 &= \frac{r_1}{1 + s x_n^p x_{n-2}^q}, \\
 C_2 &= -\frac{r_1 s q \bar{x}(x_n^p x_{n-2}^q - \bar{x}^{p+q})}{(1 + s x_n^p x_{n-2}^q)^2}, \\
 D_0 &= 0, \quad D_1 = 0, \quad D_2 = 0.
 \end{aligned}$$

Taking the limits, it is clear that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} A_0 &= 0, \quad \lim_{n \rightarrow \infty} A_1 = 0, \quad \lim_{n \rightarrow \infty} A_2 = 0, \\
 \lim_{n \rightarrow \infty} B_0 &= -\frac{rs_1 p \bar{y}(y_n^p y_{n-2}^q - \bar{y}^{p+q})}{(1 + s_1 y_n^p y_{n-2}^q)^2}, \\
 \lim_{n \rightarrow \infty} B_1 &= \frac{r}{1 + s_1 y_n^p y_{n-2}^q}, \\
 \lim_{n \rightarrow \infty} B_2 &= -\frac{rs_1 q \bar{y}(y_n^p y_{n-2}^q - \bar{y}^{p+q})}{(1 + s_1 y_n^p y_{n-2}^q)^2}, \\
 \lim_{n \rightarrow \infty} C_0 &= -\frac{r_1 s p \bar{x}(x_n^p x_{n-2}^q - \bar{x}^{p+q})}{(1 + s x_n^p x_{n-2}^q)^2}, \\
 \lim_{n \rightarrow \infty} C_1 &= \frac{r_1}{1 + s x_n^p x_{n-2}^q}, \\
 \lim_{n \rightarrow \infty} C_2 &= -\frac{r_1 s q \bar{x}(x_n^p x_{n-2}^q - \bar{x}^{p+q})}{(1 + s x_n^p x_{n-2}^q)^2}, \\
 \lim_{n \rightarrow \infty} D_0 &= 0, \quad \lim_{n \rightarrow \infty} D_1 = 0, \quad \lim_{n \rightarrow \infty} D_2 = 0.
 \end{aligned}$$

That is

$$\begin{aligned}
 B_0 &= -\frac{rs_1 p \bar{y}(y_1^p y_2^q - \bar{y}^{p+q})}{(1 + s_1 \bar{y}_2^{p+q})^2} + \alpha_n, \\
 B_1 &= \frac{r}{1 + s_1 \bar{y}_2^{p+q}} + \beta_n, \\
 B_2 &= -\frac{rs_1 q \bar{y}(y_1^p y_2^q - \bar{y}^{p+q})}{(1 + s_1 \bar{y}_2^{p+q})^2} + \gamma_n, \\
 C_0 &= -\frac{r_1 s p \bar{x}(x_2^p x_1^q - \bar{x}^{p+q})}{(1 + s \bar{x}_1^{p+q})^2} + \delta_n, \\
 C_1 &= \frac{r_1}{1 + s \bar{x}_1^{p+q}} + \eta_n, \\
 C_2 &= -\frac{r_1 s q \bar{x}(x_2^p x_1^q - \bar{x}^{p+q})}{(1 + s \bar{x}_1^{p+q})^2} + \theta_n,
 \end{aligned}$$

where $\alpha_n \rightarrow 0, \beta_n \rightarrow 0, \gamma_n \rightarrow 0, \delta_n \rightarrow 0, \eta_n \rightarrow 0, \theta_n \rightarrow 0$ for $n \rightarrow \infty$.

Thus, the limiting system of error terms about the equilibrium M can be written as follows:

$$E_{n+1} = (C + D(n))E_n,$$

where $E_n = (e_n^1, e_{n-1}^1, e_{n-2}^1, e_n^2, e_{n-1}^2, e_{n-2}^2)^T$,

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & r & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{6 \times 6},$$

$$D_n = \begin{pmatrix} 0 & 0 & 0 & \alpha_n & \beta_n & \gamma_n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_n & \eta_n & \theta_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{6 \times 6}$$

and $\|D(n)\| \rightarrow 0$, when $n \rightarrow \infty$. As desired. □

Corollary 5.6. Assume that $rr_1 < 1$. Then, the error vector of every non-trivial solution of system 1.8 satisfies both of the following asymptotic relations:

$$\lim_{n \rightarrow \infty} \|E_n\|^{1/n} = |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, 3, 4, 5, 6,$$

$$\lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_i J_F(M)|, \text{ for some } i = 1, 2, 3, 4, 5, 6$$

where $|\lambda_i J_F(M)|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point M , i.e. $\{\lambda_{1,2} = 0, \lambda_{3,4} = \pm \sqrt[4]{rr_1}, \lambda_{5,6} = \pm i \sqrt[4]{rr_1}\}$.

6. Conclusions

In the present paper, we described the qualitative behaviors of solutions of the system (1.8) of nonlinear difference equations. More precisely, we studied the equilibrium points, the local asymptotic stability, the global asymptotic stability of zero equilibrium, the existence of the prime two-periodic solutions and the rate of convergence of positive solutions of the aforementioned system. Also, we gave a correction about an article in the literature. Our system generalized the systems studied in [13, 14, 27].

The results in this paper can be extend to the following system of difference equations;

$$\left. \begin{aligned} u_{n+1} &= \frac{\alpha v_{n-1}}{\beta + \gamma \sum_{i=0}^k v_{n-2i}^{p_i}}, \\ v_{n+1} &= \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 \sum_{i=0}^k u_{n-2i}^{p_i}}, \end{aligned} \right\}, n = 0, 1, \dots$$

7. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this manuscript.

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