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Construction of Degenerate q-Daehee Polynomials with Weight α and its Applications

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Abstract

The fundamental aim of the present paper is to deal with introducing a new family of Daehee polynomials which is called degenerate q-Daehee polynomials with weight α by using p-adic q-integral on $\mathbb{Z}_p$. From this definition, we obtain some new summation formulae and properties. We also introduce the degenerate q-Daehee polynomials of higher order with weight α and obtain some new interesting identities.

1. Introduction

The notion of p-adic numbers was firstly considered by Kurt Hensel (1861-1941). Motivated by this fruitful idea, many scientists begun to study new scientific tools using good and useful properties of them. Diverse effects of these new researches have emerged in mathematical physics in which they are used in the theory of ultrametric calculus, p-adic quantum mechanics, the p-adic mechanics, etc.

The one useful tool of p-adic analysis is Volkenborn integral (or so-called p-adic integral). Intense research activities in such an area as p-adic integral are principally motivated by their importance in special polynomials, especially the Bernoulli polynomials and their various generalizations. The other useful tool of p-adic analysis is q -analogue of p-adic invariant integral which is invented by Kim [10] . He showed that the Carlitz's q-Bernoulli polynomials and their different generalizations can be represented as a p-adic q-invariant integral which is called Witt's formula. Therefore, in recent years, p-adic integral and its various generalizations have been considered and extensively studied by many mathematicians, *cf.* [3], [4], [5], [7], [11], [12], [16], [20], [21], [22], [29].

We now begin with recalling some basic notations as follows.

Throughout this paper we use the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

The parameter p stands for the first letter of p-adic being a fixed prime number. The symbols denoted by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ mean p-adic integers field, p-adic rational numbers field and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. The known

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p-adic norm denoted by $|\,.\,|_p$ is normalized by the equality $|p|_p = p^{-1}$. The $UD(\mathbb{Z}_p)$ means the space of $\mathbb{C}_p$-valued uniformly differentiable functions over $\mathbb{Z}_p$. The p-adic q-integral on $\mathbb{Z}_p$ of a function $f \in UD(\mathbb{Z}_p)$ is originally given by Kim [10] as follows:

$$I_q(f) := \int_{\mathbb{Z}_p} f(y) d\mu_q(y) = \lim_{N\to\infty} \sum_{k=0}^{p^N-1} f(k)\mu_q(k+p^N\mathbb{Z}_p), \tag{1.1}$$

where $\mu_q(k+p^N\mathbb{Z}_p) = \frac{q^k}{[p^N]_q}$ is Kim's q-Haar distribution. It follows from (1.1) that

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1)f(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

In the year 2011, Kim [11] defined weighted q-Bernoulli polynomials (or known as q-Bernoulli polynomials with weight α) which can be represented by the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q}^{(\alpha)}(x) \frac{z^n}{n!} = \int_{\mathbb{Z}_p} e^{z[x+y]_{q^\alpha}} d\mu_q(y), \tag{1.3}$$

or equivalently by

$$\beta_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_{q^\alpha}^{n-l} \beta_{l,q}^{(\alpha)}, \quad (n \geq 0). \tag{1.4}$$

The pioneering of degenerate versions of Bernoulli and Euler polynomials was Carlitz who considered $(1+\lambda z)^{\frac{1}{\lambda}}$ instead of e^z in their generating functions. When $\lambda \to 0$, it returns to classical one. Actually, Carlitz [1], [2] gave the generating function of degenerate Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_n (x \mid \lambda) \frac{z^n}{n!} = \frac{z}{(1+\lambda z)^{\frac{1}{\lambda}} - 1} (1+\lambda z)^{\frac{x}{\lambda}}.$$

When $x = 0$, $\beta_n (0 \mid \lambda) := \beta_n (\lambda)$ are called degenerate Bernoulli numbers. It is noteworthy that

$$\lim_{\lambda\to 0} \beta_n (x \mid \lambda) = B_n (x),$$

where $B_n (x)$ are the Bernoulli polynomials, see [5], [30], [31], [32].

Kim also applied the idea of degenerate version to various special functions, polynomials and numbers, *cf.* [12], [13], [14], [15]. For example, Kim considered a new class of q-Bernoulli polynomials which is called degenerate q-Bernoulli polynomials given by

$$\int_{\mathbb{Z}_p} (1+\lambda z)^{\frac{[x+y]_q}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda) \frac{z^n}{n!}, \tag{1.5}$$

where the parameters are assumed that $\lambda, z, q \in \mathbb{C}_p$ with $|\lambda z|_p < p^{-\frac{1}{p-1}}$ and $|1-q|_p < p^{-\frac{1}{p-1}}$, see [12].

Let $D_n(x)$ be Daehee polynomials given by

$$\sum_{n=0}^{\infty} D_n(x) \frac{z^n}{n!} = \frac{\log(1+z)}{z} (1+z)^x.$$

In the case when $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers, *cf.* [5], [7], [17], [18], [19], [21], [22], [24], [25], [26], [28], [29]. The degenerate version of Carlitz's type q-Daehee polynomials is considered by

$$\sum_{n=0}^{\infty} D_{n,q,\lambda} (x) \frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda \log(1+z))^{\frac{[x+y]_q}{\lambda}} d\mu_q (y), \tag{1.6}$$

where the case $x = 0$, $D_{n,q,\lambda} (0) := D_{n,q,\lambda}$ stands for the degenerate of q-analogue Carlitz's type Daehee numbers, see [23]. Clearly that

$$D_{n,q,\lambda} \to D_{n,q} \text{ as } \lambda \to 0.$$

The Stirling numbers of first and second kinds are given, respectively, by

$$\sum_{n=k}^{\infty} S_1(n,k)\frac{z^n}{n!} = \frac{(\log(z+1))^k}{k!} \quad \textit{and} \quad \sum_{n=k}^{\infty} S_2(n,k)\frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}, \tag{1.7}$$

satisfying

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \text{ see [8], [30], [32].}$$

Motivated by the works of [3], [4] and [12], we consider the degenerate q-Daehee polynomials with weight α as follows:

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!}.$$

From this definition, we obtain explicit identities and properties. We also introduce the degenerate q-Daehee polynomials of higher order with weight α.

2. The degenerate q-Daehee polynomials with weight α

We begin with the following definition.

Definition 2.1. *Let $\lambda, z, q \in \mathbb{C}_p$ with $|\lambda z|_p < p^{-\frac{1}{p-1}}$ and $|1-q|_p < p^{-\frac{1}{p-1}}$. The degenerate q-Daehee polynomials $D_{n,q;\alpha,\lambda}(x)$ are given by*

$$\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y). \tag{2.1}$$

Remark 2.2. *Putting $\alpha = 1$ in Eq. (2.1) reduces to Eq. (1.6).*

Remark 2.3. *Traditionally, in the case $x = 0$, the polynomial reduces to its number. So, when $x = 0$ in (2.1), $D_{n,q;\alpha,\lambda}(0) := D_{n,q;\alpha,\lambda}$ will be called the degenerate q-Daehee numbers with weight α.*

It follows from Eq. (2.1) that

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!} &= \int_{\mathbb{Z}_p} (1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y) \\
&= \int_{\mathbb{Z}_p} \sum_{j=0}^{\infty} \binom{\frac{[x+y]_{q^\alpha}}{\lambda}}{j} \lambda^j (\log(1+z))^j d\mu_q(y) \\
&= \sum_{j=0}^{\infty} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} \sum_{m=j}^{\infty} S_1(m,j)\, d\mu_q(y)\frac{z^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} S_1(m,j)\, d\mu_q(y)\right)\frac{z^m}{m!},
\end{aligned}$$

where we have used

$$\left([\gamma+\zeta]_{q^\alpha}\right)_{j,\lambda} = [\gamma+\zeta]_{q^\alpha}\left([\gamma+\zeta]_{q^\alpha}-\lambda\right)\left([\gamma+\zeta]_{q^\alpha}-2\lambda\right)\cdots\left([\gamma+\zeta]_{q^\alpha}-(j-1)\lambda\right).$$

Thus we obtain the following theorem.

Theorem 2.4. *Let $m \in \mathbb{N}_0$. The degenerate q-Daehee polynomials with weight α satisfy*

$$D_{m,q;\alpha,\lambda}(x) = \sum_{j=0}^{m} \int_{\mathbb{Z}_p} \left([x+y]_{q^\alpha}\right)_{j,\lambda} S_1(m,j)\, d\mu_q(y).$$

Let $B_{k,q;\alpha,\lambda}(x)$ be degenerate q-Bernoulli polynomials with weight α which may be given by

$$\sum_{n=0}^{\infty} B_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda z)^{\frac{[x+y]_{q^\alpha}}{\lambda}} d\mu_q(y). \tag{2.2}$$

Replacing z by e^z-1 in Eq. (2.1) gives

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{(e^z-1)^m}{m!} &= \int_{\mathbb{Z}_p}(1+\lambda z)^{\frac{[x+y]_{q^\alpha}}{\lambda}}\,d\mu_q(y),\\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{(e^z-1)^m}{m!} &= \sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty} S_2(n,m)\frac{z^n}{n!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_{n,q;\alpha,\lambda}(x)S_2(m,n)\right)\frac{z^m}{m!}.\end{aligned} \tag{2.4}$$

Thus, from (2.3) and (2.4), we have the following theorem.

Theorem 2.5. *Let $m\in\mathbb{N}_0$. The following identity holds*

$$B_{m,q;\alpha,\lambda}(x)=\sum_{n=0}^{m} D_{n,q;\alpha,\lambda}(x)S_2(m,n).$$

Changing z to $\log(1+z)$ in Eq. (2.2) yields

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}\,d\mu_q(y)\\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\frac{(\log(1+z))^m}{m!}\\ &= \sum_{m=0}^{\infty} B_{m,q;\alpha,\lambda}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_{m,q;\alpha,\lambda}(x)S_1(n,m)\right)\frac{z^n}{n!}.\end{aligned}$$

By comparing coefficitents of $\frac{z^n}{n!}$ on the both sides of the above, we procure the following theorem.

Theorem 2.6. *Let $m\in\mathbb{N}_0$. The following summation formula satisfies*

$$D_{m,q;\alpha,\lambda}(x)=\sum_{m=0}^{n} B_{m,q;\alpha,\lambda}(x)S_1(n,m).$$

Since

$$q^x=e^{x\log q},$$

we have

$$\begin{aligned}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} &= e^{\frac{[x+y]_{q^\alpha}}{\lambda}\log(1+\lambda\log(1+z))}\\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)^n\frac{(\log(1+\lambda\log(1+z)))^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\frac{[x+y]_{q^\alpha}}{\lambda}\right)^n\sum_{m=n}^{\infty} S_1(m,n)\lambda^m\frac{(\log(1+z))^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{l=0}^{j}[x+y]_{q^\alpha}^l\,\lambda^{j-l}S_1(j,l)S_1(n,j)\right)\frac{z^n}{n!}.\end{aligned} \tag{2.5}$$

Taking p-adic q-integral on $\mathbb{Z}_p$ on both sides of (2.5) becomes

$$\begin{aligned}\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}(x)\frac{z^m}{m!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}d\mu_q(y) \qquad (2.6)\\ &= \int_{\mathbb{Z}_p}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{l=0}^{j}[x+y]_{q^\alpha}^{l}\ \lambda^{j-l}S_1(j,l)S_1(m,j)\right)d\mu_q(y)\frac{z^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{l=0}^{j}\lambda^{j-l}S_1(j,l)S_1(m,j)\beta_{l,q}^{(\alpha)}(x)\right)\frac{z^m}{m!}. \qquad (2.7)\end{aligned}$$

By (2.6) and (2.7), we arrive at the following theorem.

Theorem 2.7. *Let $m\in\mathbb{N}_0$. The following relation holds*

$$D_{m,q;\alpha,\lambda}(x)=\sum_{j=0}^{m}\sum_{l=0}^{j}\lambda^{j-l}S_1(j,l)S_1(m,j)\beta_{l,q}^{(\alpha)}(x).$$

It is easy to check that

$$[x+y]_{q^\alpha}=\frac{1-q^{\alpha(x+y)}}{1-q^\alpha}=\frac{1-q^{\alpha x}}{1-q^\alpha}+\frac{q^{\alpha x}(1-q^{\alpha y})}{1-q^\alpha}=[x]_{q^\alpha}+q^{\alpha x}[y]_{q^\alpha}.$$

From here, we see that

$$\begin{aligned}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}} &= (1+\lambda\log(1+z))^{\frac{[x]_{q^\alpha}+q^{\alpha x}[y]_{q^\alpha}}{\lambda}}\\ &= (1+\lambda\log(1+z))^{\frac{[x]_{q^\alpha}}{\lambda}}(1+\lambda\log(1+z))^{\frac{q^{\alpha x}[y]_{q^\alpha}}{\lambda}}\\ &= \left(\sum_{j=0}^{\infty}\binom{\frac{[x]_{q^\alpha}}{\lambda}}{j}\lambda^j(\log(1+z))^j\right)\left(\sum_{m=0}^{\infty}\frac{q^{m\alpha x}[y]_{q^\alpha}^m}{\lambda^m}\frac{(\log(1+\lambda\log(1+z)))^m}{m!}\right)\\ &= \left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\left([x]_{q^\alpha}\right)_{j,\lambda}S_1(n,j)\right)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\lambda^{k-l}q^{\alpha lx}[y]_{q^\alpha}^{l}S_1(k,l)S_1(n,k)\right)\frac{z^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n}{j}\lambda^{k-l}\left([x]_{q^\alpha}\right)_{m,\lambda}q^{\alpha lx}[y]_{q^\alpha}^{l}S_1(k,l)S_1(j,k)S_1(n,m)\right)\frac{z^n}{n!}.\end{aligned}$$

Thus we have the following theorem.

Theorem 2.8. *Let n be nonnegative integer. The following implicit summation formula satisfies*

$$D_{n,q;\alpha,\lambda}(x)=\sum_{j=0}^{n}\sum_{m=0}^{n-j}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n}{j}\lambda^{k-l}\left([x]_{q^\alpha}\right)_{m,\lambda}q^{\alpha lx}S_1(k,l)S_1(j,k)S_1(n,m)\beta_{l,q}^{(\alpha)}.$$

Now we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}(x)\frac{z^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}\,d\mu_q(y)\\
&= \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}q^y\\
&= \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{y=0}^{dp^N-1}(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}}q^y\\
&= \lim_{N\to\infty}\frac{1}{[d]_q[p^N]_{q^d}}\sum_{a=0}^{d-1}\sum_{y=0}^{p^N-1}q^a(1+\lambda\log(1+z))^{\frac{[x+a+dy]_{q^\alpha}}{\lambda}}q^{dy}\\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{y=0}^{p^N-1}(1+\lambda\log(1+z))^{\frac{[d]_{q^\alpha}\left[\frac{x+a}{d}+y\right]_{q^{d\alpha}}}{\lambda}}q^{dy}\\
&= \frac{1}{[d]_q}\sum_{a=0}^{d-1}q^a\lim_{N\to\infty}\frac{1}{[p^N]_{q^d}}\sum_{y=0}^{p^N-1}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{j=0}^{k}S_1(k,j)S_1(n,k)[d]_{q^\alpha}^{j}\left[\frac{a+x}{d}+y\right]_{q^\alpha}^{j}\lambda^{n-k}\right)q^{dy}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{[d]_q}\sum_{a=0}^{d-1}\sum_{k=0}^{n}\sum_{j=0}^{k}q^aS_1(k,j)S_1(n,k)[d]_{q^\alpha}^{j}\beta_{j,q^d}^{(\alpha)}\left(\frac{a+x}{d}\right)\lambda^{n-k}\right)\frac{z^n}{n!}.
\end{aligned}
$$

Thus we get the following theorem.

Theorem 2.9. *Let n be nonnegative integer. The following distribution formula for degenerate q-Daehee polynomials with weight α holds*

$$
D_{n,q;\alpha,\lambda}(x)=\frac{1}{[d]_q}\sum_{a=0}^{d-1}\sum_{k=0}^{n}\sum_{j=0}^{k}q^aS_1(k,j)S_1(n,k)[d]_{q^\alpha}^{j}\beta_{j,q^d}^{(\alpha)}\left(\frac{a+x}{d}\right)\lambda^{n-k}.
$$

Recall from Eq. (1.2) that

$$
qI_q(f_1)-I_q(f)=\frac{q-1}{\log q}f'(0)+(q-1)f(0).
$$

Let us now consider the following function

$$
f(y)=(1+\lambda\log(1+z))^{\frac{[x+y]_{q^\alpha}}{\lambda}},
$$

then we find the following difference equation for degenerate q -Daehee polynomials with weight α as follows:

$$
\begin{aligned}
qD_{n,q;\alpha,\lambda}(x+1)-D_{n,q;\alpha,\lambda}(x) &= (q-1)\sum_{k=0}^{n}\left([x]_{q^\alpha}\right)_{k,\lambda}S_1(n,k)+n\frac{\alpha}{[\alpha]_q}(q-1)^2q^{\alpha x}\\
&\times\sum_{j=0}^{n-1}\sum_{k=0}^{j}\binom{n}{j+1}(-1)^k\lambda^k k!D_{n-1-j,q;\alpha,\lambda}(x+1)S_1(j+1,k+1).
\end{aligned}
$$

Now we introduce degenerate q-Daehee polynomials of higher order by using multivariate p-adic q-integral on $\mathbb{Z}_p$ defined by Kim in [16]:

$$
\sum_{n=0}^{\infty}D_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!}=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}}(1+\lambda\log(1+z))^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\,\overline{\mathbf{d}}\mu_q(y), \tag{2.8}
$$

where

$$
\mathbf{x}:=\sum_{i=1}^{v}x_i \text{ and } \overline{\mathbf{d}}\mu_q(y):=\prod_{i=1}^{v}d\mu_q(y_i).
$$

It follows from the Eq. (2.8) that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!} &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} (1+\lambda\log(1+z))^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\,\overline{\mathbf{d}}\mu_q(y) \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{l=0}^{\infty}\binom{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}{l}\lambda^l(\log(1+z))^l\,\overline{\mathbf{d}}\mu_q(y) \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{l=0}^{\infty}\left([\overline{\mathbf{x}}+y]_{q^\alpha}\right)_{l,\lambda}\sum_{n=l}^{\infty}S_1(n,l)\,\overline{\mathbf{d}}\mu_q(y)\frac{z^n}{n!} \\
&= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,l)[\overline{\mathbf{x}}+y]_{q^\alpha}^{j}\,\overline{\mathbf{d}}\mu_q(y)\right)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,l)\beta_{j,q}^{(\alpha,v)}(x)\right)\frac{z^n}{n!}.
\end{aligned}$$

From those applications, we deduce the following theorem.

Theorem 2.10. *Let $n\in\mathbb{N}_0$. The following relation*

$$D_{n,q;\alpha,\lambda}^{(v)}(x)=\sum_{l=0}^{n}\sum_{j=0}^{l}\lambda^{l-j}S_1(l,j)S_1(n,k)\beta_{j,q}^{(\alpha,v)}(x),$$

holds true.

We finalize our paper replacing z by e^z-1 in Eq. (2.8):

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}^{(v)}(x)\frac{(e^z-1)^m}{m!} &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{\text{v-times}} (1+\lambda z)^{\frac{[\overline{\mathbf{x}}+y]_{q^\alpha}}{\lambda}}\,\overline{\mathbf{d}}\mu_q(y) \\
&= \sum_{n=0}^{\infty} B_{n,q;\alpha,\lambda}^{(v)}(x)\frac{z^n}{n!} \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=0}^{\infty} D_{m,q;\alpha,\lambda}^{(v)}(x)\frac{(e^z-1)^m}{m!} &= \sum_{n=0}^{\infty} D_{n,q;\alpha,\lambda}^{(v)}(x)\sum_{n=m}^{\infty}S_2(n,m)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}D_{m,q;\alpha,\lambda}^{(v)}(x)S_2(n,m)\right)\frac{z^n}{n!}. \qquad (2.10)
\end{aligned}$$

Thus, from (2.9) and (2.10), we have the following theorem.

Theorem 2.11. *Let $n\in\mathbb{N}_0$. The following identity holds*

$$B_{n,q;\alpha,\lambda}^{(v)}(x)=\sum_{m=0}^{n}D_{m,q;\alpha,\lambda}^{(v)}(x)S_2(n,m).$$

3. Conclusion

The pioneering of degenerate idea was Carlitz, see[1] and [2], who considered for Bernoulli and Euler polynomials. This idea was one of good advantages in order to introduce new families of special polynomials. As has been listed in the references, Kim and his research team have been working this fruitful idea for new special polynomials intensively.

In this paper, motivated by the works of Kim and his research team, we have dealt mainly with new family of polynomials which are called degenerate q-Daehee polynomials with weight α and degenerate q-Daehee polynomials with weight α of higher order. We have derived their explicit and summation formulae by using p-adic q-integral on $\mathbb{Z}_p$ and analytic methods.

Seemingly that these types of polynomials will be continued to be studied for a while due to their interesting reflections in the fields of mathematics.

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