Mathematical Sciences and Applications E-Notes
https://doi.org/10.36753/mathenot.839111
9 (1) 1-8 (2021) - Research Article
ISSN: 2147-6268

Degenerate Poly-Type 2-Bernoulli Polynomials

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Abstract
Recently, Kim-Kim [10] have studied type 2-Changhee and Daehee polynomials. They have also introduced the type 2-Bernoulli polynomials in order to express the central factorial numbers of the second kind by making use of type 2-Bernoulli numbers of negative integral orders. Inspired by their work, we consider a new class of generating functions of type 2-Bernoulli polynomials. We give some identities for these polynomials including type 2-Euler polynomials and Stirling numbers of the second kind.

Keywords: *Bernoulli numbers and polynomials; Type 2-Bernoulli numbers and polynomials; Polylogarithm; Generating function; Degenerate.*

AMS Subject Classification (2020): *Primary: 11B83; 11B73 ; Secondary: 05A19.*

1. Introduction

Special numbers and polynomials have significant roles in various branches of mathematics, theoretical physics, and engineering. The problems arising in mathematical physics and engineering are framed in terms of differential equations. Most of these equations can only be treated by using various families of special polynomials which provide new means of mathematical analysis. They are widely used in computational models of scientific and engineering problems. Also, they lead to the derivation of different useful identities in a fairly straight forward way and motivate to consider possible extensions of new families of special polynomials.

The motivation for the degenerate polynomials related with type 2-Bernoulli polynomials is because of their intrinsic scientific significance and to the way that some polynomials may be demonstrated to be natural solutions of a certain set of (partial) differential equations under some conditions which often appear in the treatment of the electromagnetic wave propagation, quantum beam life-ime in storage rings, etc.

Recent investigations involving the type 2 degenerate Bernoulli and Euler numbers [3], type 2 degenerate central Fubini polynomials [7], type 2 degenerate poly-Bernoulli numbers and polynomials arising from degenerate polyexponential function [9], type 2 degenerate Bernoulli polynomials [14], generalized type 2 degenerate Euler numbers [15], type 2 Daehee and Changhee polynomials derived from p-adic integrals on $\mathbb{Z}_p$ [16], type 2 poly-Apostol-Bernoulli polynomials [17], type 2 degenerate poly-Euler Polynomials [18] and type 2 degenerate Euler and Bernoulli polynomials [20] have been investigated extensively.

We begin with the following definitions of some special polynomials.

Let $B_n(\theta)$ be the Bernoulli polynomials given by

Received : *11-12-2020,* ***Accepted :*** *13-02-2021*

$$\sum_{n=0}^{\infty} B_n(\theta)\frac{\gamma^n}{n!} = \frac{\gamma}{e^{\gamma}-1}e^{\theta\gamma} \qquad (|\gamma|<2\pi)\,. \tag{1.1}$$

In the case when $\theta = 0$, $B_n =: B_n(0)$ are called the Bernoulli numbers, *cf.* [3, 8, 9, 13, 19].

In [11], type 2-Bernoulli and Euler polynomials are defined by means of the following generating function, respectively:

$$\sum_{n=0}^{\infty} b_n(\theta)\frac{\gamma^n}{n!} = \frac{\gamma}{e^{\gamma}-e^{-\gamma}}e^{\theta\gamma} \quad (|\gamma|<\pi) \text{ and } \sum_{n=0}^{\infty} E_n^*(\theta)\frac{\gamma^n}{n!} = \frac{2}{e^{\gamma}+e^{-\gamma}}e^{\theta\gamma} \quad \left(|\gamma|<\frac{\pi}{2}\right). \tag{1.2}$$

When $\theta = 0$, $b_n(0) := b_n$ and $E_n^*(0) := E_n^*$ are called type 2-Bernoulli and Euler numbers.

The Stirling numbers of the second kind are defined by

$$\frac{1}{\ell!}(e^{\gamma}-1)^{\ell} = \sum_{m=\ell}^{\infty} S_2(m,\ell)\frac{\gamma^m}{m!} \qquad \text{(see [3, 6, 7, 9])}. \tag{1.3}$$

The degenerate exponential function, $e_{\lambda}^{\theta}(\gamma)$, may be interpreted without the limit case. Namely, it is given by

$$e_{\lambda}^{\theta}(\gamma) := (1+\lambda\gamma)^{\frac{\theta}{\lambda}} = \sum_{m=0}^{\infty} (\theta)_{m,\lambda}\frac{\gamma^m}{m!} \text{ and } e_{\lambda}^{\theta}(\gamma)\,e_{\lambda}^{\nu}(\gamma) = e_{\lambda}^{\theta+\nu}(\gamma)\,, \tag{1.4}$$

with the assumption $e_{\lambda}^{1}(\gamma) := e_{\lambda}(\gamma)$, where $(\theta)_{m,\lambda}$ is the λ-falling factorial sequence given by

$$(\theta)_{m,\lambda} = \prod_{\ell=0}^{m-1}(\theta-\ell\lambda) \qquad (m\geq 1), \tag{1.5}$$

with the assumption $(\theta)_{0,\lambda} := 1$, *cf.* [9].

The pioneering of this idea was Carlitz [2] who considered for Bernoulli and Euler polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}(\theta)\frac{\gamma^n}{n!} = \frac{\gamma}{e_{\lambda}(\gamma)-1}e_{\lambda}^{\theta}(\gamma) \quad \text{and} \quad \sum_{n=0}^{\infty} E_{n,\lambda}(\theta)\frac{\gamma^n}{n!} = \frac{2}{e_{\lambda}(\gamma)+1}e_{\lambda}^{\theta}(\gamma) \quad (\lambda\in\mathbb{R}), \tag{1.6}$$

respectively. At the point $\theta = 0$ in (1.6), $\beta_{n,\lambda} =: \beta_{n,\lambda}(0)$ and $\beta_{n,\lambda} =: \beta_{n,\lambda}(0)$ are called, respectively, the degenerate Bernoulli and Euler numbers, see [2].

In [20], Jang and Kim introduced type 2 degenerate Euler polynomials by the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(\theta)\frac{\gamma^n}{n!} = \frac{2}{e_{\lambda}(\gamma)+e_{\lambda}^{-1}(\gamma)}e_{\lambda}^{\theta}(\gamma)\,.$$

The polylogarithm function is defined by

$$\mathrm{Li}_k(\zeta) = \sum_{m=1}^{\infty}\frac{\zeta^m}{m^k} \quad (k\in\mathbb{Z}, \zeta\in\mathbb{C} \text{ with } |\zeta|<1)\,, \text{ see [4]}. \tag{1.7}$$

Since, for $k = 1$, $\mathrm{Li}_1(\zeta) = -\log(1-\zeta)$, we have $\mathrm{Li}_1\left(1-e^{-\zeta}\right) = \zeta$. In the case $k \leq 0$, $\mathrm{Li}_k(\zeta)$ are the rational functions:

$$\mathrm{Li}_0(\zeta) = \frac{\zeta}{1-\zeta}, \mathrm{Li}_{-1}(\zeta) = \frac{\zeta}{(1-\zeta)^2}, \mathrm{Li}_{-2}(\zeta) = \frac{\zeta^2+\zeta}{(1-\zeta)^3}, \mathrm{Li}_{-3}(\zeta) = \frac{\zeta^3+4\zeta^2+\zeta}{(1-\zeta)^4}, \cdots.$$

See [4] for details.

One of the various extensions of Bernoulli numbers is poly-Bernoulli numbers defined in [6] as follows:

$$\sum_{m=0}^{\infty} \beta_m^{(\ell)}\frac{\gamma^m}{m!} = \frac{\mathrm{Li}_{\ell}(1-e^{-\gamma})}{1-e^{-\gamma}}.$$

For $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$, Kim and Kim [12] also introduced the degenerate poly-Bernoulli polynomials given by

$$\sum_{m=0}^{\infty} \beta_m^{(\ell)}(\theta \mid \lambda) \frac{\gamma^m}{m!} = \frac{\mathrm{Li}_\ell(1-e^{-\gamma})}{e_\lambda(\gamma)-1} e_\lambda^\theta(\gamma).$$

When $\theta = 0$, $\beta_n^{(\ell)}(0 \mid \lambda) := \beta_n^{(\ell)}(\lambda)$ are called the degenerate poly-Bernoulli numbers.

By the motivation of the works of Kim-Kim [11, 15, 20], we first define degenerate poly-type 2-Bernoulli polynomials. We investigate some new properties of these polynomials and derive some new identities and relations between the degenerate type 2-Euler, type 2-Bernoulli polynomials and Stirling numbers of the second kind.

2. Degenerate poly-type 2-Bernoulli polynomials

In this section, we begin with the following definition.

Definition 2.1. Let $\mathcal{B}_{m,\lambda}^{(k)}(x)$ be degenerate poly-type 2-Bernoulli polynomials. The generating function of these polynomials is given by

$$\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(\ell)}(\theta) \frac{\gamma^m}{m!} = \frac{\mathrm{Li}_\ell(1-e^{-\gamma})}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)} e_\lambda^\theta(\gamma). \tag{2.1}$$

Remark 2.1. Letting λ to 0 in (2.1) yields

$$\sum_{m=0}^{\infty} \mathcal{B}_{m}^{(\ell)}(\theta) \frac{\gamma^m}{m!} = \frac{\mathrm{Li}_\ell(1-e^{-\gamma})}{e^\gamma-e^{-\gamma}} e^{\theta\gamma}, \tag{2.2}$$

where $\mathcal{B}_m^{(\ell)}(\theta)$ are poly-type 2-Bernoulli polynomials.

Remark 2.2. Taking $\ell = 1$ in (2.1) yields

$$\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(1)}(\theta) \frac{\gamma^m}{m!} = \frac{\gamma}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)} e_\lambda^\theta(\gamma),$$

where $B_{n,\lambda}^{(1)}(\theta)$, which is called degenerate type 2-Bernoulli polynomials, can be considered as $b_{n,\lambda}(\theta)$ due to Eq. (1.2).

Remark 2.3. When $\ell = 1$ and $\lambda \to 0$ in Eq. (2.1), we have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(1)}(\theta) \frac{\gamma^m}{m!} = \sum_{m=0}^{\infty} b_m(\theta)\frac{\gamma^m}{m!} = \frac{\gamma}{e^\gamma-e^{-\gamma}} e^{\theta\gamma}, \tag{2.3}$$

where $b_m(\theta)$ are called type 2-Bernoulli polynomials.

By (1.7), it is easy to see that

$$\begin{aligned}
\frac{d}{d\gamma}\mathrm{Li}_\ell\left(1-e^{-\gamma}\right) &= \frac{d}{d\gamma}\sum_{n=1}^{\infty}\frac{\left(1-e^{-\gamma}\right)^n}{n^\ell} \\
&= \sum_{n=1}^{\infty}\frac{n\left(1-e^{-\gamma}\right)^{n-1}e^{-\gamma}}{n^\ell} \\
&= \frac{1}{e^\gamma-1}\mathrm{Li}_{l-1}\left(1-e^{-\gamma}\right).
\end{aligned}$$

It becomes

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(\ell)}(\theta) \frac{\gamma^m}{m!} &= \frac{e_\lambda^\theta(\gamma)}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}\int_0^\gamma \frac{1}{e^{\gamma_1}-1}\int_0^{\gamma_1}\frac{1}{e^{\gamma_2}-1}\cdots\int_0^{\gamma_{\ell-2}}\frac{1}{e^{\gamma_{\ell-1}}-1}\mathrm{Li}_1\left(1-e^{-\gamma_{\ell-1}}\right)\prod_{i=1}^{\ell-1}dt_i \\
&= \frac{e_\lambda^\theta(\gamma)}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}\int_0^\gamma \frac{1}{e^{\gamma_1}-1}\int_0^{\gamma_1}\frac{1}{e^{\gamma_2}-1}\cdots\int_0^{\gamma_{\ell-2}}\frac{\gamma_{\ell-1}}{e^{\gamma_{\ell-1}}-1}\prod_{i=1}^{\ell-1}dt_i.
\end{aligned}$$

Here, in particular $\ell = 2$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} \mathcal{B}_{m,\lambda}^{(2)}(\theta) \frac{\gamma^m}{m!} &= \frac{e_\lambda^\theta(\gamma)}{e_\lambda(\gamma) - e_\lambda^{-1}(\gamma)} \int_0^\gamma \frac{\gamma_1}{e^{\gamma_1} - 1} d\gamma_1 \\ &= \left(\sum_{m=0}^{\infty} b_{m,\lambda}(\theta) \frac{\gamma^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{B_n}{n+1} \frac{\gamma^n}{n!}\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{\ell=0}^{m} \binom{m}{\ell} b_{m-\ell,\lambda}(\theta) \frac{B_\ell}{\ell+1}\right) \frac{\gamma^m}{m!}.\end{aligned}$$

Thus we have the following theorem.

Theorem 2.1. *The following identitiy holds true:*

$$\mathcal{B}_{m,\lambda}^{(2)}(\theta) = \sum_{\ell=0}^{m} \binom{m}{\ell} b_{m-\ell,\lambda}(\theta) \frac{B_\ell}{\ell+1}.$$

Theorem 2.2. *Let n be a nonnegative number. Then, the following equality holds*

$$\mathcal{B}_{n,\lambda}^{(\ell)}(\theta) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n,\lambda}^{(\ell)} \ (\theta)_{n-m,\lambda}. \tag{2.4}$$

Proof. It is proved by using (1.4) and (2.1) that

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{(\ell)}(\theta) \frac{\gamma^n}{n!} = \frac{\mathrm{Li}_\ell(1 - e^{-\gamma})}{e_\lambda(\gamma) - e_\lambda^{-1}(\gamma)} e_\lambda^\theta(\gamma) \tag{2.5}$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{(\ell)} \frac{\gamma^n}{n!}\right)\left(\sum_{n=0}^{\infty} (\theta)_{n,\lambda} \frac{\gamma^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{m,\lambda}^{(\ell)} \ (\theta)_{n-m,\lambda}\right) \frac{\gamma^n}{n!}. \tag{2.6}$$

Therefore, by equating the left-hand side of Eq. (2.5) and the right-hand side of Eq. (2.6) of the coefficients $\frac{\gamma^n}{n!}$, we arrive at the desired result. □

Theorem 2.3. *Let n be nonnegative number. Then the identity*

$$\mathcal{B}_{m,\lambda}^{(k)}(\theta) = \frac{1}{2} \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_\ell^{(k)}(\lambda) \mathcal{E}_{m-\ell,\lambda}\left(x + \frac{1}{2}\right)$$

holds true. In particular, we have

$$\mathcal{B}_{m,\lambda}^{(k)} = \frac{1}{2} \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_\ell^{(k)}(\lambda) \mathcal{E}_{m-\ell,\lambda}\left(\frac{1}{2}\right).$$

Proof. Recalling from (2.1) that

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{(k)}(\theta) \frac{\gamma^n}{n!} = \frac{\mathrm{Li}_k(1 - e^{-\gamma})}{e_\lambda(\gamma) - e_\lambda^{-1}(\gamma)} e_\lambda^\theta(\gamma). \tag{2.7}$$

By the simple calculation, it becomes

$$\begin{aligned}
\frac{\mathrm{Li}_k\left(1-e^{-\gamma}\right)}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}e_\lambda^\theta(\gamma) &= \frac{\mathrm{Li}_k\left(1-e^{-\gamma}\right)}{\left(e_\lambda(\gamma)+1\right)\left(e_\lambda(\gamma)-1\right)}e_\lambda^{\theta+1}(\gamma) \\
&= \frac{1}{2}\frac{\mathrm{Li}_k\left(1-e^{-\gamma}\right)}{e_\lambda(\gamma)-1}\frac{2}{e_\lambda(\gamma)+1}e_\lambda^{\theta+1}(\gamma) \\
&= \frac{1}{2}\frac{\mathrm{Li}_k\left(1-e^{-\gamma}\right)}{e_\lambda(\gamma)-1}\frac{2}{e_\lambda^{\frac{1}{2}}(\gamma)+e_\lambda^{-\frac{1}{2}}(\gamma)}e_\lambda^{\theta+\frac{1}{2}}(\gamma) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\beta_n^{(k)}(\lambda)\frac{\gamma^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}\left(x+\frac{1}{2}\right)\frac{\gamma^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}\beta_\ell^{(k)}(\lambda)\mathcal{E}_{n-\ell,\lambda}\left(x+\frac{1}{2}\right)\right)\frac{\gamma^n}{n!}. \qquad (2.8)
\end{aligned}$$

By comparing the coefficients of the same powers in γ of (2.7) and (2.8), we complete the proof of Theorem. □

The following identity is well known from [10] that

$$\frac{1}{\gamma}\mathrm{Li}_k\left(1-e^{-\gamma}\right)=\sum_{n=0}^{\infty}\left\{\sum_{\ell=1}^{n+1}\frac{(-1)^{n+\ell+1}}{\ell^k}l!\frac{S_2(n+1,\ell)}{n+1}\right\}\frac{\gamma^n}{n!}. \qquad (2.9)$$

We now state the following Theorem which is the sums of the products of degenerate poly-type 2-Bernoulli polynomials and Stirling numbers of the second kind.

Theorem 2.4. *Let $n\in\mathbb{N}_0$. Then the sums for the products of $b_{m,\lambda}(x)$ and $S_2(n,m)$ holds*

$$\mathcal{B}_{n,\lambda}^{(k)}(\theta)=\sum_{m=0}^{n}\sum_{\ell=1}^{m+1}\frac{(-1)^{m+\ell+1}}{\ell^k}\ell!\frac{S_2(m+1,\ell)}{m+1}b_{n-m,\lambda}(\theta).$$

Proof. By making use of (2.1), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{B}_{n,\lambda}^{(k)}(\theta)\frac{\gamma^n}{n!} &= \frac{\mathrm{Li}_k\left(1-e^{-\gamma}\right)}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}e_\lambda^\theta(\gamma) \\
&= \frac{1}{t}\mathrm{Li}_k\left(1-e^{-\gamma}\right)\frac{\gamma}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}e_\lambda^\theta(\gamma) \qquad (2.10) \\
&= \left(\sum_{n=0}^{\infty}\left\{\sum_{\ell=1}^{n+1}\frac{(-1)^{n+\ell+1}}{\ell^k}\ell!\frac{S_2(n+1,\ell)}{n+1}\right\}\frac{\gamma^n}{n!}\right)\left(\sum_{n=0}^{\infty}b_{n,\lambda}(\theta)\frac{\gamma^n}{n!}\right) \qquad (2.11) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{\ell=1}^{m+1}\frac{(-1)^{m+\ell+1}}{\ell^k}\ell!\frac{S_2(m+1,\ell)}{m+1}b_{n-m,\lambda}(\theta)\right)\frac{\gamma^n}{n!}. \qquad (2.12)
\end{aligned}$$

Thus we complete the proof of the Theorem. □

For $k=1$ in (2.1), we have the symmetric property of degenerate poly-type 2-Bernoulli polynomials as follows:

$$b_{m,\lambda}(\theta)=(-1)^m b_{m,-\lambda}(-\theta)$$

since

$$\begin{aligned}
\sum_{m=0}^{\infty}b_{m,\lambda}(\theta)\frac{\gamma^m}{m!} &= \frac{\gamma}{e_\lambda(\gamma)-e_\lambda^{-1}(\gamma)}e_\lambda^\theta(\gamma) \\
&= \frac{\gamma}{(1+\lambda\gamma)^{\frac{1}{\lambda}}-(1+\lambda\gamma)^{-\frac{1}{\lambda}}}(1+\lambda\gamma)^{\frac{\theta}{\lambda}} \\
&= \frac{-\gamma}{(1+(-\lambda)(-\gamma))^{\frac{1}{-\lambda}}-(1+\lambda\gamma)^{-\frac{1}{(-\lambda)}}}(1+(-\lambda)(-\gamma))^{\frac{-\theta}{-\lambda}} \\
&= \sum_{m=0}^{\infty}b_{m,-\lambda}(-\theta)(-1)^m\frac{\gamma^m}{m!}.
\end{aligned}$$

Thus we note that

$$b_{m,\lambda}(1) = b_{m,-\lambda}(-1)(-1)^m. \tag{2.13}$$

Let us now give the following Theorem.

Theorem 2.5. *The following recurrence relation holds:*

$$\mathcal{B}_{n,\lambda}^{(\ell)}(1) - \mathcal{B}_{n,\lambda}^{(\ell)}(-1) = \sum_{k=1}^{n} \frac{(-1)^{k+n}}{k^\ell} k! S_2(n,k). \tag{2.14}$$

Proof. By (2.4), we consider

$$\begin{aligned}
\mathrm{Li}_\ell\left(1-e^{-\gamma}\right) &= \left(e_\lambda(\gamma) - e_\lambda^{-1}(\gamma)\right) \frac{\mathrm{Li}_\ell\left(1-e^{-\gamma}\right)}{e_\lambda(\gamma) - e_\lambda^{-1}(\gamma)} \\
&= \left\{\sum_{n=0}^{\infty} \left((1)_{n,\lambda} - (-1)_{n,\lambda}\right) \frac{\gamma^n}{n!}\right\} \left\{\sum_{n=0}^{\infty} \mathcal{B}_{n,\lambda}^{(\ell)} \frac{\gamma^n}{n!}\right\} \\
&= \sum_{n=0}^{\infty} \left\{\sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{m,\lambda}^{(\ell)} \left((1)_{n-m,\lambda} - (-1)_{n-m,\lambda}\right)\right\} \frac{\gamma^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{\mathcal{B}_{n,\lambda}^{(\ell)}(1) - \mathcal{B}_{n,\lambda}^{(\ell)}(-1)\right\} \frac{\gamma^n}{n!}.
\end{aligned} \tag{2.15}$$

On the other hand, it follows from (2.9) that

$$\mathrm{Li}_\ell\left(1-e^{-\gamma}\right) = \sum_{n=1}^{\infty} \left\{\sum_{k=1}^{n} \frac{(-1)^{n+k}}{k^\ell} k! S_2(n,k)\right\} \frac{\gamma^n}{n!}. \tag{2.16}$$

Comparing the same coefficients of γ on the both sides of Eqs. (2.15) and (2.16) completes the proof. □

When $\ell = 1$ and $\lambda \to 0$ in (2.14), we have

$$b_n(1) - b_n(-1) = \sum_{k=1}^{n} (-1)^{k+n} (k-1)! S_2(n,k).$$

From here, by (2.13), we see that

$$\sum_{k=1}^{2n} (-1)^k (k-1)! S_2(2n,k) = 0$$

and

$$b_{2n+1}(1) = -\frac{1}{2} \sum_{k=1}^{2n+1} (-1)^k (k-1)! S_2(2n+1,k).$$

3. Conclusion

The pioneering of "degenerate" notion was Carlitz in [2]. Kim and his research team have applied Carlitz's idea to many known special functions and polynomials, see [9-20]. This was a good way in order to introduce new generalizations of special functions and polynomials. In this paper, motivated by the works of Kim and his research team, we have studied a new type of degenerate version of type 2 Bernoulli numbers and polynomials. We have derived their explicit, closed and summation formulae by making use of their generating function, series manipulation and analytical means as has been shown in the paper.

Gaussian integral representation plays an important role in classical problems arising from quantum optics and quantum mechanics. They are studied to calculate the optical mode overlapping and transition rates between quantum eigenstates of the harmonic oscillator, *cf.* [21].

Before finalizing the paper, we derive Gaussian integral representation of type 2 Bernoulli polynomials with the following applications:

$$T_n(\alpha,\beta,\mu) := T_n := \int_{-\infty}^{\infty} b_n(\alpha\theta) e^{-\beta\theta^2+\mu\theta} d\theta. \tag{3.1}$$

By (3.1), we have

$$\sum_{n=0}^{\infty} T_n \frac{\gamma^n}{n!} = \sum_{n=0}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_n(\alpha\theta) e^{-\beta\theta^2+\mu\theta} d\theta \right) \frac{\gamma^n}{n!}.$$

It follows from (2.3) that

$$\sum_{n=0}^{\infty} T_n \frac{\gamma^n}{n!} = \frac{\gamma}{e^{\gamma} - e^{-\gamma}} \int_{-\infty}^{\infty} e^{(\alpha\gamma+\mu)\theta-\beta\theta^2} d\theta.$$

Since

$$\int_{-\infty}^{\infty} e^{\beta\theta-\alpha\theta^2+\mu} d\theta = \frac{\sqrt{\pi}}{\sqrt{\alpha}} e^{\frac{\beta^2}{4\alpha}+\mu},$$

which represents Gaussian integral, we find

$$\sum_{n=0}^{\infty} T_n \frac{\gamma^n}{n!} = \frac{\sqrt{\pi}}{\sqrt{\beta}} \frac{\gamma}{e^{\gamma} - e^{-\gamma}} \exp\left(\frac{\alpha^2}{4\beta}\gamma^2 + \frac{\mu^2}{4\beta} + \frac{\alpha\mu}{2\beta}\gamma \right)$$

with the assumption $\exp(\gamma) := e^{\gamma}$. Recall from [21] that the 2-variable Hermite polynomials are defined by means of the following generating function:

$$\sum_{m=0}^{\infty} H_m(\theta, \eta) \frac{\gamma^m}{m!} = \exp\left(\theta\gamma + \eta\gamma^2\right). \tag{3.2}$$

By (2.3) and (3.2), we derive

$$\begin{aligned} \sum_{n=0}^{\infty} T_n \frac{\gamma^n}{n!} &= \frac{\sqrt{\pi}}{\sqrt{\beta}} e^{\frac{\mu^2}{4\beta}} \left(\sum_{n=0}^{\infty} b_n \frac{\gamma^n}{n!} \right) \left(\sum_{n=0}^{\infty} H_n\left(\frac{\alpha\mu}{2\beta}, \frac{\alpha^2}{4\beta} \right) \frac{\gamma^n}{n!} \right) \\ &= \frac{\sqrt{\pi}}{\sqrt{\beta}} e^{\frac{\mu^2}{4\beta}} \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} b_{n-\ell} H_{\ell}\left(\frac{\alpha\mu}{2\beta}, \frac{\alpha^2}{4\beta} \right) \right) \frac{\gamma^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{\gamma^n}{n!}$ on the above, we obtain

$$T_n = \frac{\sqrt{\pi}}{\sqrt{\beta}} e^{\frac{\mu^2}{4\beta}} \sum_{\ell=0}^{n} \binom{n}{\ell} b_{n-\ell} H_{\ell}\left(\frac{\alpha\mu}{2\beta}, \frac{\alpha^2}{4\beta} \right). \tag{3.3}$$

Thus, by (3.1) and (3.3), we finalize our paper with the following result:

$$\int_{-\infty}^{\infty} b_n(\alpha\theta)\, e^{-\beta\theta^2+\mu\theta} d\theta = \frac{\sqrt{\pi}}{\sqrt{\beta}} e^{\frac{\mu^2}{4\beta}} \sum_{\ell=0}^{n} \binom{n}{\ell} b_{n-\ell} H_{\ell}\left(\frac{\alpha\mu}{2\beta}, \frac{\alpha^2}{4\beta} \right).$$

Seemingly that these types of polynomials will be continued to be studied for a while due to their interesting reflections in the fields of mathematics, statistics and sciences.

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