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A Two-step Hybrid Block Method with Four Off-step Points on Singular Initial and Boundary Value Problems

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Abstract. This work contains the numerical studies of singular initial and boundary value problems of ordinary differential equations. The importance of these problems arose from the recent attentions Mathematicians have drawn to them due to their occurrence naturally and repeatedly in physical models. The numerical solutions to these problems are obtained by a class of block hybrid methods. The behaviour of basis functions differs and as such has an influence on numerically derived methods. Previous studies eulogize the use of a single basis function. This study establishes the advantage of using multiple basis functions as the derived methods combine multiple properties of basis function as this approach improves the stability of the resulting methods. This work combines two basis functions namely: a newly Constructed orthogonal polynomial and shifted Chebyshev orthogonal polynomial for the development of some continuous hybrid schemes in collocation and interpolation technique. To make the continuous schemes self-starting, some block methods of discrete hybrid form were derived. The schemes were analyzed using appropriate existing definitions to investigate their stability, consistency, and convergence. The investigation shows that the developed schemes are consistent, zero stable, and hence convergent. Comparison with the exact solutions and existing methods show that the proposed methods are effective numerical methods for the solutions of singular initial and boundary value problems.

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1. Introduction

Singular problems arose in the formulation or modelling of real-life situation. These problems usually pose difficult to solve using known analytical means and as such effective numerical solutions are required. Mathematicians develop mathematical models to help them understand the physical phenomena in real life problems. These models often lead to equations involving some derivatives of an unknown function of single or several variables which are called differential equations. Differential equations have various application in many fields such as engineering, science and social sciences. Recent occurrence of these type of problems have drawn the attention of researchers to their approximations. Numerous methods have also been developed in recent times and however, improvements on these methods for the

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solutions of the problems are required. This study uses the collocation and interpolation approach to estimate the solution of a class of these problems. The approximation was sought in the combination of two basis functions.
Many applied Mathematicians and Physicists focused their attention on the studies of second order singular problems in Ordinary Differential Equations (ODEs). Such kind of equations have been discussed by various researchers among these are; Qu and Agwarwal [14] who employed collocation method for solving a class of singular non-linear two-point BVP. Wazwaz [18, 19] presented series and exact solution of Lane-Emden and Emden-Fowler type of problems based on Adomian decomposition and modified Adomian decomposition methods. Koch *et al.* [8] evaluated the approximate solution of the singular IVP by implicit Euler method and finally used an acceleration technique known as the iterated defect correction to improve the approxiamtions. Hassan and Zhu [4, 5] solved such a singular IVP by Taylor series and modified Adomian decomposition method. Changqing and Jianhua [2] developed a numerical method for Lane-Emden equations using chebyshev polynomials and collocation method. A reliable algorithm based on chebyshev polynomial and collocation method in order to obtain approximate solution for the Lane-Emden equation was thereby presented. Kumar and Singh [9] applied modified Adomian decomposition method and computer implementation for solving singular BVP arising from various physical problems. AbdulKaled *et al.* [1] constructed a numerical approach for solving a class of singular boundary value problems arising from physiology. Hassan *et al.* [3, 4] derived an implicit method for solving first and second order singular IVPs, which give more accurate results than those obtained by implicit Euler and second order implicit Runge-Kutta methods. Iryna *et al.* [6] applied residual power series method in order to obtain efficient analytical numerical scheme for a class of non-linear system of IVPs with infinitely many singularities; the method provide analytical solution in terms of rapid convergence series with easily computable components. Shiraleshette *et al.* [17] employed a new method, named Haar wavelet collocation method, numerical solution of singular initial value problems or integral equations, ordinary and partial differential equations were obtained by the method. Roul [15] presented a numerical of singular two-point BVP using Adomian decomposition homotopy perturbation approach. Najafi and Yaghouba [11] constructed a non-standard finite difference schemes for numerical solution of non-linear Lane-Emden type equations, the use of the method and its approximate play as important role for the formulation of stable numerical methods. Roul [16] presented a fast converging iterative approach for the solution of doubly singular BVP with derivative dependent source foundation. A special class of the form of (1.1) is called the Lane-Emden equation and this problem was investigated and solved using a class of Runge-kutta method by Ogunniran *et al.* [13].
The general form of second order singular differential equations is denoted as:

$$y''(x) + \frac{P(x)}{Q(x)}y'(x) + f(x, y) = g(x, y); \quad a \leq x \leq b. \tag{1.1}$$

In order to solve equation (1.1), any of the conditions stated below need to be imposed.

$$y(a) = \alpha, y'(a) = \beta \tag{1.2}$$

$$y(a) = \alpha_1, y'(b) = \beta_1. \tag{1.3}$$

Equation (1.1) together with (1.2) are called Initial Value Problem (IVP) while equation and (1.3) are called Boundary Value Problem (BVP).
Equation (1.1) is singular at $Q(x) = 0$ and $f(x, y)$, $g(x, y)$ are non-linear continuous functions. It is well known that some of these problems have proved to be either difficult to solve or cannot be solved analytically due to its singularity as the approximate solutions lost their accuracy in the neighbourhood of the singular points; hence the need to derive numerical methods for the solution of such problems becomes necessary.

1.1. **Orthogonal Polynomials.** Orthogonal polynomials have found widespread use in all areas of science and engineering. Typically, they are used as trial functions to expand other more complicated functions in which, many at times, arise from initial or boundary value problems.
Consider the equation:

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = h_n\sigma_{mn}$$

with

$$\sigma_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases},$$

where the weight function $w(x)$ is continuous and positive on $[a, b]$ such that the moments:

$$\mu = \int_a^b w(x)x^n dx, \quad n = 0, 1, 2, \cdots$$

exist and h_n is a non-zero constant. Then the integral

$$< \phi_m, \phi_n >= \int_a^b w(x)\phi_m(x)\phi_n(x)dx$$

denotes an inner product of the polynomials ϕ_m and ϕ_n.
For orthogonality,

$$< \phi_m, \phi_n >= \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n.$$

The orthogonal polynomial valid in the interval $[0, 1]$ and with respect to weight function $w(x) = x^2$ are given as:

$$\left.\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= 4x - 3 \\ \phi_2(x) &= 15x^2 - 20x + 6 \\ \phi_3(x) &= 56x^3 - 105x^2 + 600x - 10 \\ \phi_4(x) &= 210x^4 - 504x^3 + 420x^2 - 140x + 15 \\ \phi_5(x) &= 729x^5 - 2310x^4 + 2520x^3 - 1260x^2 + 280x - 21 \\ \phi_6(x) &= 3003x^6 - 10296x^5 + 13860x^4 - 9240x^3 + 3150x^2 - 504x + 28 \\ \phi_7(x) &= 11440x^7 - 45045x^6 + 72072x^5 + 60060x^4 + 27720x^3 - 6930x^2 + 840x - 36. \end{aligned}\right\}$$

1.2. **Shifted Chebyshev Polynomials.** The shifted polynomial of the first kind are orthogonal on the support interval [0,1] with weight function:

$$w(x) = \frac{1}{\sqrt{x - x^2}}$$

and normalized by the requirement that $T_n^*(1) = 1$. $T_n^*(1) = 1$ satisfies the three-term recurrence relation:

$$T_{n+1}^*(x) = 2(2x - 1)T_n^*(x) - T_{n-1}^*(x); \quad \text{for } n \geq 1$$

with starting values:

$$\begin{aligned} T_0^*(x) &= 1 \\ T_1^*(x) &= 2x - 1. \end{aligned}$$

2. Methodology

2.1. **Introduction.** Let the approximate solution be given as a combination of two basis functions in the form:

$$Y(x) = \sum_{r=0}^{p} a_r\phi_r(x) + \sum_{r=p+1}^{k} b_r T_r^*(x) \approx y(x), \quad x_n \leq x \leq x_{n+m}, \tag{2.1}$$

where $x \in [x_n, x_{n+m}]$, $m = 1, 2$, $a's$ and $b's$ are real coefficients to be determined, $p = u + v - 3, k = u + v - 1$. u is the number of collocation points, v is the number of interpolation points, $\phi_r(x)$ is the constructed polynomial with weight function $w(x) = x^2$ over an interval $[0, 1]$ and $T_r^*(x)$ is a shifted Chebyshev polynomial, $h = x_{n+1} - x_n$ is a constant step size of the partition of interval $[a, b]$ which is given by

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Differentiating (2.1) twice gives:

$$y'' = \sum_{r=0}^{p} a_r\phi_r''(x) + \sum_{r=p+1}^{k} b_r T_r^{*''}(x).$$

2.2. **Development of Two-step Method with Four Off-step Points.** A transformation

$$t = \frac{2x - 2x_n - mh}{mh}, \tag{2.2}$$

where m is the step number, will be used throughout the formulation of this method. To derive this, four points are introduced and $m = 3$. These off-step points are carefully selected to guarantee zero stability conditions. The four off-steps are $x_{n+\frac{6}{5}}, x_{n+\frac{7}{5}}, x_{n+\frac{8}{5}}$ and $x_{n+\frac{9}{5}}$ using (2.1) with $u = 7, v = 2$ gives a polynomial of degree $u + v - 1$ as follows:

$$y(x) = \sum_{r=0}^{6} a_r \phi_r(x) + \sum_{r=7}^{8} b_r T_r^*, \tag{2.3}$$

where $x \in [x_n, x_{n+2}]$. The second derivative is given by:

$$y''(x) = \sum_{r=0}^{6} a_r \phi_r''(x) + \sum_{r=7}^{8} b_r T_r^{*\prime\prime}. \tag{2.4}$$

Substituting (2.4) into (1.1) gives:

$$f(x, y, y') = \sum_{r=0}^{6} a_r \phi_r''(x) + \sum_{r=7}^{8} b_r T_r^{*\prime\prime}.$$

Now, interpolating equation (2.3) at $x_{n+\frac{6}{5}}, x_{n+\frac{7}{5}}$ and collocating (2.4) at all points in selected interval $x_n, x_{n+1}, x_{n+\frac{6}{5}}, x_{n+\frac{7}{5}}, x_{n+\frac{8}{5}}, x_{n+\frac{9}{5}}$ and x_{n+2} gives the system of equations:

$$Ax = b, \tag{2.5}$$

where

$$A = \begin{bmatrix}
1 & -\frac{11}{5} & \frac{13}{5} & -\frac{219}{125} & \frac{13}{215} & \frac{4117}{3125} & -\frac{25727}{15625} & -\frac{378843}{78125} & \frac{2844233}{390625} \\
1 & -7/5 & 2/5 & \frac{98}{125} & -\frac{17}{25} & -\frac{1081}{3125} & \frac{11332}{15625} & -\frac{1247689}{78125} & -\frac{5426047}{390625} \\
0 & 0 & 30\,h^{-2} & -546\,h^{-2} & 6384\,h^{-2} & -61200\,h^{-2} & 524070\,h^{-2} & -1988928\,h^{-2} & 15252768\,h^{-2} \\
0 & 0 & 30\,h^{-2} & -210\,h^{-2} & 840\,h^{-2} & -2520\,h^{-2} & 6300\,h^{-2} & -1984\,h^{-2} & 3920\,h^{-2} \\
0 & 0 & 30\,h^{-2} & -\frac{714}{5\,h^2} & 336\,h^{-2} & -\frac{11952}{25\,h^2} & \frac{45198}{125\,h^2} & -\frac{956736}{3125\,h^2} & -\frac{15064112}{15625\,h^2} \\
0 & 0 & 30\,h^{-2} & -\frac{378}{5\,h^2} & \frac{168}{5\,h^2} & \frac{2664}{25\,h^2} & -\frac{17172}{125\,h^2} & \frac{5853248}{3125\,h^2} & -\frac{10017328}{15625\,h^2} \\
0 & 0 & 30\,h^{-2} & -\frac{42}{5\,h^2} & -\frac{336}{5\,h^2} & -\frac{144}{25\,h^2} & \frac{13518}{125\,h^2} & \frac{1634752}{3125\,h^2} & \frac{96542672}{15625\,h^2} \\
0 & 0 & 30\,h^{-2} & 294\,h^{-2} & \frac{168}{5\,h^2} & -\frac{1368}{25\,h^2} & -\frac{17172}{125\,h^2} & -\frac{46851264}{3125\,h^2} & -\frac{248344112}{15625\,h^2} \\
0 & 0 & 30\,h^{-2} & 126\,h^{-2} & 336\,h^{-2} & 720\,h^{-2} & 1350\,h^{-2} & -48704\,h^{-2} & 138928\,h^{-2}
\end{bmatrix}$$

$$x = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ b_7 \\ b_8 \end{bmatrix}, \text{ and } b = \begin{bmatrix} y_{n+\frac{6}{5}} \\ y_{n+\frac{7}{5}} \\ f_n \\ f_{n+1} \\ f_{n+\frac{6}{5}} \\ f_{n+\frac{7}{5}} \\ f_{n+\frac{8}{5}} \\ f_{n+\frac{9}{5}} \\ f_{n+2} \end{bmatrix}.$$

Solving the resulting system in (2.5), the values of the unknowns parameters $a_r, r = 0(1)6$ and $b_r, r = 7, 8$ are obtained and given as below:

$$
\left.\begin{array}{c}
a_0 = -\frac{7}{4}y_{n+\frac{6}{5}} + \frac{11}{4}y_{n+\frac{7}{5}} + \frac{1383902809}{436988805120000}h^2 f_n - \frac{10515988529}{17340825600000}h^2 f_{n+1} + \frac{19260609929}{3468165120000}h^2 f_{n+2} \\
+\frac{73125319373}{2080899072000}h^2 f_{n+\frac{6}{5}} + \frac{6654122843}{1213857792000}h^2 f_{n+\frac{7}{5}} + \frac{127239583357}{1387266048000}h^2 f_{n+\frac{8}{5}} - \frac{7503029699}{445906944000}h^2 f_{n+\frac{9}{5}} \\
. \\
a_1 = -\frac{5}{4}y_{n+\frac{6}{5}} + \frac{5}{4}y_{n+\frac{7}{5}} + \frac{60495623}{10924720128000}h^2 f_n - \frac{732859577}{65028069000}h^2 f_{n+1} + \frac{2781777359}{260112384000}h^2 f_{n+2} \\
+\frac{1016503967}{17340825600}h^2 f_{n+\frac{6}{5}} - \frac{1498918909}{22759833600}h^2 f_{n+\frac{7}{5}} + \frac{578340089}{4161798144}h^2 f_{n+\frac{8}{5}} - \frac{1540718621}{39016857600}h^2 f_{n+\frac{9}{5}} \\
. \\
a_2 = \frac{4043}{462422016}h^2 f_n - \frac{122978557}{8174960640}h^2 f_{n+1} + \frac{47068585}{3269984256}h^2 f_{n+2} \\
+\frac{34598525}{467140608}h^2 f_{n+\frac{6}{5}} - \frac{787495825}{5722472448}h^2 f_{n+\frac{7}{5}} + \frac{1024755425}{6539968512}h^2 f_{n+\frac{8}{5}} - \frac{96738025}{1634992128}h^2 f_{n+\frac{9}{5}} \\
. \\
a_3 = \frac{4286551}{400573071360}h^2 f_n - \frac{20955931}{1589575680}h^2 f_{n+1} + \frac{39417311}{3179151360}h^2 f_{n+2} \\
+\frac{119513675}{1907490816}h^2 f_{n+\frac{6}{5}} - \frac{143637775}{1112702976}h^2 f_{n+\frac{7}{5}} + \frac{13915225}{115605504}h^2 f_{n+\frac{8}{5}} - \frac{152103575}{2861236224}h^2 f_{n+\frac{9}{5}} \\
. \\
a_4 = \frac{993743}{118351134720}h^2 f_n - \frac{28840687}{4226826240}h^2 f_{n+1} + \frac{4110899}{650280960}h^2 f_{n+2} \\
+\frac{56339825}{1690730496}h^2 f_{n+\frac{6}{5}} - \frac{179575675}{2958778368}h^2 f_{n+\frac{7}{5}} + \frac{185831075}{3381460992}h^2 f_{n+\frac{8}{5}} - \frac{3271925}{120766464}h^2 f_{n+\frac{9}{5}} \\
. \\
a_5 = \frac{26975}{7510745088}h^2 f_n - \frac{19325}{11176704}h^2 f_{n+1} + \frac{312275}{178827264}h^2 f_{n+2} \\
+\frac{419975}{59609088}h^2 f_{n+\frac{6}{5}} + \frac{415175}{32514048}h^2 f_{n+\frac{8}{5}} - \frac{1977475}{156473856}h^2 f_{n+\frac{7}{5}} - \frac{482675}{67060224}h^2 f_{n+\frac{9}{5}} \\
. \\
a_6 = \frac{1325}{2219083776}h^2 f_n - \frac{575}{96864768}h^2 f_{n+1} + \frac{3275}{14902272}h^2 f_{n+2} + \frac{112375}{581188608}h^2 f_{n+\frac{6}{5}} + \frac{476375}{387459072}h^2 f_{n+\frac{8}{5}} \\
-\frac{266375}{339026688}h^2 f_{n+\frac{7}{5}} - \frac{742375}{871782912}h^2 f_{n+\frac{9}{5}} \\
. \\
b_7 = -\frac{625}{8257536}h^2 f_{n+1} + \frac{625}{8257536}h^2 f_{n+2} + \frac{3125}{8257536}h^2 f_{n+\frac{6}{5}} - \frac{3125}{4128768}h^2 f_{n+\frac{7}{5}} + \frac{3125}{4128768}h^2 f_{n+\frac{8}{5}-\frac{3125}{8257536}h^2 f_{n+\frac{9}{5}}} \\
. \\
b_8 = -\frac{625}{11098128384}h^2 f_n - \frac{625}{44040192}h^2 f_{n+1} + \frac{625}{88080384}h^2 f_{n+2} + \frac{15625}{264241152}h^2 f_{n+\frac{6}{5}} \\
-\frac{15625}{154140672}h^2 f_{n+\frac{7}{5}} + \frac{15625}{176160768}h^2 f_{n+\frac{8}{5}} - \frac{15625}{396361728}h^2 f_{n+\frac{9}{5}}.
\end{array}\right\}
$$

The values of $a'_r s, r = 0(1)6$ and $b'_r s, r = 7, 8$ are substituted back into equation (2.3) to give implicit continuous hybrid method of the form:

$$
y(x) = \alpha_{\frac{6}{5}} y_{n+\frac{6}{5}} + \alpha_{\frac{7}{5}} y_{n+\frac{7}{5}} + h^2 \left[\beta_0 f_n + \beta_1 f_{n+1} + \beta_{\frac{6}{5}} f_{n+\frac{6}{5}} + \beta_{\frac{7}{5}} f_{n+\frac{7}{5}} + \beta_{\frac{8}{5}} f_{n+\frac{8}{5}} + \beta_{\frac{9}{5}} f_{n+\frac{9}{5}} + \beta_2 f_{n+2}\right]. \tag{2.6}
$$

Writing the α_r and β_r as continuous function of t using (2.2) for $m = 2$ implies that $t = \frac{x-x_n-h}{h}$. The parameters are obtained as:

$$
\left.\begin{array}{c}
\alpha_{\frac{6}{5}}(t) = -5t + 2 \\
\alpha_{\frac{7}{5}}(t) = 5t - 1 \\
\beta_0(t) = h^2\left[\frac{625}{338688}t^8 - \frac{625}{84672}t^7 + \frac{425}{36288}t^6 - \frac{25}{2688}t^5 + \frac{137}{36288}t^4 - \frac{1}{1512}t^3 + \frac{479}{42336000}t - \frac{221}{317520000}\right] \\
\beta_1(t) = h^2\left[-\frac{625}{1344}t^8 + \frac{625}{504}t^7 - \frac{25}{72}t^6 - \frac{25}{12}t^5 + \frac{851}{288}t^4 - \frac{125}{72}t^3 + \frac{1}{2}t^2 - \frac{33617}{504000} + \frac{4001}{1260000}\right] \\
\beta_{\frac{6}{5}}(t) = h^2\left[\frac{15625}{8064}t^8 - \frac{3125}{672}t^7 + \frac{125}{846}t^6 + \frac{1675}{192}t^5 - \frac{8125}{864}t^4 + \frac{125}{36}t^3 - \frac{5553}{22400}t^2 + \frac{51563}{1512000}t + \right] \\
\beta_{\frac{7}{5}}(t) = h^2\left[-\frac{15625}{4704}t^8 + \frac{3125}{441}t^7 + \frac{125}{84}t^6 - \frac{1175}{84}t^5 + \frac{11875}{1008}t^4 - \frac{125}{42}t^3 + \frac{12947}{352800}t + \frac{1693}{882000}\right] \\
\beta_{\frac{8}{5}}(t) = h^2\left[\frac{15625}{5376}t^8 - \frac{3125}{576}t^7 - \frac{1375}{576}t^6 + \frac{4175}{384}t^5 - \frac{4375}{576}t^4 + \frac{125}{72}t^3 - \frac{503}{16128}t^2 + \frac{1247}{1008000}t + \right] \\
\beta_{\frac{9}{5}}(t) = h^2\left[-\frac{15625}{12096}t^8 + \frac{3125}{1512}t^7 + \frac{875}{648}t^6 - \frac{25}{6}t^5 + \frac{6875}{2592}t^4 - \frac{125}{216}t^3 + \frac{125}{216}t^2 + \frac{3209}{302400}t - \frac{1163}{2268000}\right] \\
\beta_2(t) = h^2\left[\frac{625}{2688}t^8 - \frac{625}{2016}t^7 - \frac{25}{96}t^6 + \frac{125}{192}t^5 - \frac{113}{288}t^4 + \frac{1}{12}t^3 - \frac{1543}{10080000}t^2 + \frac{199}{2520000}t + \right].
\end{array}\right\} \tag{2.7}
$$

Evaluating (2.6) together with (2.7) at the interpolating points $x_n, x_{n+1}, x_{n+\frac{8}{5}}, x_{n+\frac{9}{5}}, x_{n+2}$ implying that $t = -1, 0, \frac{3}{5}, \frac{4}{5}$ and 1 respectively yields the following discrete methods:

$$\left.\begin{array}{r}
y_n+6y_{n+\frac{7}{5}}-7y_{n+\frac{6}{5}}=h^2[\frac{524119}{15120000}f_n+\frac{105837}{20000}f_{n+1}-\frac{101161}{120000}f_{n+2}-\frac{1050157}{72000}f_{n+\frac{6}{5}}+\frac{831203}{42000}f_{n+\frac{7}{5}}\\
-\frac{227631}{16000}f_{n+\frac{8}{5}}+\frac{580867}{108000}f_{n+\frac{9}{5}}]\\
y_{n+1}+y_{n+\frac{7}{5}}-2y_{n+\frac{6}{5}}=h^2[\frac{-221}{31752000}f_n+\frac{4001}{1260000}f_{n+1}+\frac{199}{2520000}f_{n+2}+\frac{51563}{1512000}f_{n+\frac{6}{5}}+\frac{1693}{882000}f_{n+\frac{7}{5}}\\
+\frac{1247}{1008000}f_{n+\frac{8}{5}}-\frac{1163}{2268000}f_{n+\frac{9}{5}}]\\
y_{n+\frac{8}{5}}-2y_{n+\frac{7}{5}}+y_{n+\frac{6}{5}}=h^2[\frac{31}{317520000}f_n-\frac{241}{1260000}f_{n+1}+\frac{31}{2520000}f_{n+2}+\frac{6203}{1512000}f_{n+\frac{6}{5}}+\frac{28363}{882000}f_{n+\frac{7}{5}}\\
+\frac{4187}{1008000}f_{n+\frac{8}{5}}-\frac{533}{2268000}f_{n+\frac{9}{5}}]\\
y_{n+\frac{9}{5}}-3y_{n+\frac{7}{5}}+2y_{n+\frac{6}{5}}=h^2[\frac{31}{105840000}f_n-\frac{57}{140000}f_{n+1}-\frac{109}{840000}f_{n+2}+\frac{4103}{504000}f_{n+\frac{6}{5}}+\frac{20033}{294000}f_{n+\frac{7}{5}}\\
+\frac{4569}{112000}f_{n+\frac{8}{5}}+\frac{267}{756000}f_{n+\frac{9}{5}}]\\
y_{n+2}-4y_{n+\frac{7}{5}}+3y_{n+\frac{6}{5}}=h^2[\frac{-11}{52920000}f_n-\frac{59}{210000}f_{n+1}+\frac{1109}{420000}f_{n+2}+\frac{2633}{252000}f_{n+\frac{6}{5}}+\frac{15833}{147000}f_{n+\frac{7}{5}}\\
+\frac{12937}{168000}f_{n+\frac{8}{5}}+\frac{16057}{378000}f_{n+\frac{9}{5}}].
\end{array}\right\}$$

Differentiating equation (2.7) where $\frac{dt}{dx}=\frac{1}{h}$, gives rise to:

$$\left.\begin{array}{r}
\alpha'_{\frac{6}{5}}(t)=\frac{-5}{h}\\
\alpha'_{\frac{7}{5}}(t)=\frac{5}{h}\\
\beta'_0(t)=h\left[\frac{625}{42336}t^7-\frac{625}{12096}t^6+\frac{425}{6048}t^5-\frac{125}{2688}t^4+\frac{137}{9072}t^3-\frac{1}{504}t^2+\frac{479}{42336000}\right]\\
\beta'_1(t)=h\left[-\frac{625}{168}t^7+\frac{625}{72}t^6-\frac{25}{12}t^5-\frac{125}{12}t^4+\frac{851}{72}t^3-\frac{125}{24}t^2+t-\frac{33617}{504000}\right]\\
\beta'_{\frac{6}{5}}(t)=h\left[\frac{15625}{1008}t^7-\frac{3125}{96}t^6+\frac{125}{144}t^5+\frac{8375}{192}t^4-\frac{8125}{216}t^3+\frac{125}{12}t^2-\frac{5553}{22400}\right]\\
\beta'_{\frac{7}{5}}(t)=h\left[-\frac{15625}{588}t^7+\frac{3125}{63}t^6+\frac{125}{14}t^5-\frac{5875}{84}t^4+\frac{11875}{25}t^3-\frac{125}{14}t^2+\frac{12947}{352800}\right]\\
\beta'_{\frac{8}{5}}(t)=h\left[\frac{15625}{672}t^7-\frac{21875}{576}t^6-\frac{1375}{96}t^5+\frac{2087}{384}t^4-\frac{4375}{144}t^3+\frac{125}{24}t^2-\frac{503}{16128}\right]\\
\beta'_{\frac{9}{5}}(t)=h\left[-\frac{15625}{1512}t^7+\frac{3125}{216}t^6+\frac{875}{108}t^5-\frac{125}{6}t^4+\frac{6875}{648}t^3-\frac{125}{72}t^2+\frac{3209}{302400}\right]\\
\beta'_2(t)=h\left[\frac{625}{336}t^7-\frac{625}{288}t^6-\frac{25}{16}t^5+\frac{625}{192}t^4-\frac{113}{72}t^3+\frac{1}{4}t^2-\frac{1543}{10080000}\right].
\end{array}\right\}\tag{2.8}$$

Evaluating the derivative of (2.6) together with (2.8) at all points in selected intervals i.e.

$$x_n, x_{n+1}, x_{n+\frac{6}{5}}, x_{n+\frac{7}{5}}, x_{n+\frac{8}{5}}, x_{n+\frac{9}{5}} \text{ and } x_{n+2}$$

implying that $t=-1,0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$ and 1 respectively yields the following discrete derivative methods:

$$\left.\begin{array}{r}
hy'_n+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{127008000}[-25437313f_n-1781543484f_{n+1}+330902082f_{n+2}\\
+5395701990f_{n+\frac{6}{5}}-7456339080f_{n+\frac{7}{5}}+5463257625f_{n+\frac{8}{5}}-2091652220f_{n+\frac{9}{5}}]\\
hy'_{n+1}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{42336000}[479f_n-2823828f_{n+1}-64806f_{n+2}\\
-10495170f_{n+\frac{6}{5}}+1553640f_{n+\frac{7}{5}}-1320375f_{n+\frac{8}{5}}+449260f_{n+\frac{9}{5}}]\\
hy'_{n+\frac{6}{5}}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{127008000}[-289f_n+342468f_{n+1}+64386f_{n+2}\\
-8125530f_{n+\frac{6}{5}}-6309000f_{n+\frac{7}{5}}+1822905f_{n+\frac{8}{5}}-495740f_{n+\frac{9}{5}}]\\
hy'_{n+\frac{7}{5}}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{127008000}[253f_n-270396f_{n+1}-61362f_{n+2}\\
+3680250f_{n+\frac{6}{5}}+10743480f_{n+\frac{7}{5}}-1874565f_{n+\frac{8}{5}}+483140f_{n+\frac{9}{5}}]\\
hy'_{n+\frac{8}{5}}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{42336000}[-43f_n+6636f_{n+1}+28182f_{n+2}\\
+546210f_{n+\frac{6}{5}}+8526120f_{n+\frac{7}{5}}+3890355f_{n+\frac{8}{5}}-296660f_{n+\frac{9}{5}}]\\
hy'_{n+\frac{9}{5}}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{18144000}[59f_n-44388f_{n+1}-46206f_{n+2}\\
+509430f_{n+\frac{6}{5}}+2864520f_{n+\frac{7}{5}}+4364685f_{n+\frac{8}{5}}+1423900f_{n+\frac{9}{5}}]\\
hy'_{n+2}+5y_{n+\frac{6}{5}}-5y_{n+\frac{7}{5}}=\frac{h^2}{127008000}[-1313f_n+600516f_{n+1}+7838082f_{n+2}\\
-1298010f_{n+\frac{6}{5}}+31660920f_{n+\frac{7}{5}}+13757625f_{n+\frac{8}{5}}+36347780f_{n+\frac{9}{5}}].
\end{array}\right\}\tag{2.9}$$

Combining equations (2.7) and (2.9) will produce a block of the form:

$$AY_m=BR_1+h^2\left(CR_2+DR_3\right),\tag{2.10}$$

where

$$A=\begin{bmatrix}
0 & -7 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 3 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0\\
0 & 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h
\end{bmatrix},\quad Y_m=\begin{bmatrix}
y_{n+1}\\ y_{n+\frac{6}{5}}\\ y_{n+\frac{7}{5}}\\ y_{n+\frac{8}{5}}\\ y_{n+\frac{9}{5}}\\ y_{n+2}\\ y'_{n+1}\\ y'_{n+\frac{6}{5}}\\ y'_{n+\frac{7}{5}}\\ y'_{n+\frac{8}{5}}\\ y'_{n+\frac{9}{5}}\\ y'_{n+2}
\end{bmatrix},$$

$$B=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},\quad R_1=\begin{bmatrix}
y_n\\ y'_n\\ y_{n-1}\\ y_{n-2}\\ y_{n-3}\\ y_{n-4}\\ y_{n-5}\\ y'_{n-1}\\ y'_{n-2}\\ y'_{n-3}\\ y'_{n-4}\\ y'_{n-5}
\end{bmatrix},$$

$$C=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{524119\,h^2}{15120000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{-221h^2}{31752000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{31h^2}{317520000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{31h^2}{105840000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{-11\,h^2}{52920000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-25437313\,h^2}{127008000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{479\,h^2}{42336000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{-289h^2}{127008000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{253\,h^2}{127008000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-43\,h^2}{42336000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{59\,h^2}{18144000}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1313\,h^2}{127008000}
\end{bmatrix},\quad R_2=\begin{bmatrix}
f_{n-1}\\ f_{n-2}\\ f_{n-3}\\ f_{n-4}\\ f_{n-5}\\ f_{n-6}\\ f_{n-7}\\ f_{n-8}\\ f_{n-9}\\ f_{n-10}\\ f_{n-11}\\ f_n
\end{bmatrix},$$

$$D = \begin{bmatrix}
\frac{1055837\,h^2}{20000} & \frac{-1050157h^2}{72000} & \frac{831203\,h^2}{42000} & -\frac{227631h^2}{16000} & \frac{580867h^2}{108000} & \frac{101161h^2}{120000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4001h^2}{1260000} & \frac{51563\,h^2}{1512000} & \frac{1693h^2}{882000} & -\frac{1247h^2}{1008000} & \frac{-1163h^2}{2268000} & \frac{199h^2}{2520000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-241\,h^2}{1260000} & \frac{6203\,h^2}{1512000} & \frac{28363\,h^2}{882000} & \frac{4187\,h^2}{1008000} & -\frac{533h^2}{2268000} & \frac{31h^2}{2520000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-57\,h^2}{140000} & \frac{410\,h^2}{504000} & \frac{20033\,h^2}{294000} & \frac{4569\,h^2}{112000} & \frac{2617h^2}{756000} & -\frac{109}{840000} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{59\,h^2}{210000} & \frac{2633\,h^2}{252000} & +\frac{15833\,h^2}{147000} & \frac{12937\,h^2}{168000} & +\frac{16057\,h^2}{378000} & \frac{1109}{420000} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1781543484\,h^2}{127008000} & +\frac{5395701990\,h^2}{127008000} & \frac{-7456339080\,h^2}{127008000} & \frac{5463257625\,h^2}{127008000} & \frac{2091652220h^2}{127008000} & \frac{330902082h^2}{127008000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2823828\,h^2}{42336000} & \frac{-10495170\,h^2}{42336000} & \frac{1553640\,h^2}{42336000} & \frac{-1320375\,h^2}{42336000} & \frac{449260\,h^2}{42336000} & -\frac{64806}{42336000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{342468\,h^2}{127008000} & \frac{-8125530\,h^2}{127008000} & \frac{-6309000\,h^2}{127008000} & \frac{1822905\,h^2}{127008000} & \frac{-495740h^2}{127008000} & \frac{64386}{127008000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-270396\,h^2}{127008000} & \frac{3680250\,h^2}{127008000} & \frac{10743480\,h^2}{127008000} & \frac{-1874565\,h^2}{127008000} & \frac{483140\,h^2}{127008000} & \frac{61362}{127008000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6636h^2}{42336000} & \frac{546210\,h^2}{42336000} & \frac{8526120\,h^2}{42336000} & \frac{3890355\,h^2}{42336000} & -\frac{296660}{42336000} & \frac{28182h^2}{42336000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-44388h^2}{18144000} & \frac{509430\,h^2}{18144000} & \frac{2864520\,h^2}{18144000} & \frac{4364685\,h^2}{18144000} & \frac{1423900}{18144000} & \frac{-46206h^2}{18144000} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{600516h^2}{127008000} & \frac{-1298010\,h^2}{127008000} & \frac{31660920\,h^2}{127008000} & \frac{13757625\,h^2}{127008000} & -\frac{36347780}{127008000} & \frac{7838082h^2}{127008000} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$R_3 = \begin{bmatrix} f_{n+2} \\ f_{n+\frac{6}{5}} \\ f_{n+\frac{7}{5}} \\ f_{n+\frac{8}{5}} \\ f_{n+\frac{9}{5}} \\ f_{n-1} \\ f_{n-\frac{6}{5}} \\ f_{n-\frac{7}{5}} \\ f_{n-\frac{8}{5}} \\ f_{n-\frac{9}{5}} \\ f_{-2} \end{bmatrix}.$$

Multiplying equation (2.10) by A^{-1} gives the hybrid block method below:

$$IY_m = \bar{B}R_1 + h^2[\bar{C}R_2 + \bar{D}R_3]$$

written as:

$$
\left.\begin{aligned}
y_{n+1}-y_n-y_n'h &= h^2\Big[\tfrac{168277}{1016064}f_n+\tfrac{35233}{4032}f_{n+1}-\tfrac{1579}{896}f_{n+2}-\tfrac{674075}{24192}f_{n+\frac{6}{5}}+\tfrac{61205}{1568}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{66325}{2304}f_{n+\frac{8}{5}}+\tfrac{402425}{36288}f_{n+\frac{9}{5}}\Big]\\
y_{n+2}-y_n-2y_n'h &= h^2\Big[\tfrac{5809}{15876}f_n+\tfrac{478}{21}f_{n+1}-\tfrac{275}{63}f_{n+2}-\tfrac{1900}{27}f_{n+\frac{6}{5}}+\tfrac{43100}{441}f_{n+\frac{7}{5}}-\\
&\qquad \tfrac{6025}{84}f_{n+\frac{8}{5}}+\tfrac{15650}{567}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{6}{5}}-y_n-\tfrac{6}{5}hy_n' &= h^2\Big[\tfrac{5039}{24500}f_n+\tfrac{10098}{875}f_{n+1}-\tfrac{1998}{875}f_{n+2}-\tfrac{6369}{175}f_{n+\frac{6}{5}}+\tfrac{310284}{6125}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{130869}{3500}f_{n+\frac{8}{5}}+\tfrac{1798}{125}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{7}{5}}-y_n-\tfrac{7}{5}hy_n' &= h^2\Big[\tfrac{3184657}{12960000}f_n+\tfrac{573839}{40000}f_{n+1}-\tfrac{2019241}{720000}f_{n+2}-\tfrac{19392877}{432000}f_{n+\frac{6}{5}}+\tfrac{449281}{7200}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{1471813}{32000}f_{n+\frac{8}{5}}+\tfrac{11455171}{648000}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{8}{5}}-y_n-\tfrac{8}{5}hy_n' &= h^2\Big[\tfrac{708928}{2480625}f_n+\tfrac{675328}{39375}f_{n+1}-\tfrac{43648}{13125}f_{n+2}-\tfrac{1261184}{23625}f_{n+\frac{6}{5}}+\tfrac{1362944}{18375}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{85784}{1575}f_{n+\frac{8}{5}}+\tfrac{1486336}{70875}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{9}{5}}-y_n-\tfrac{9}{5}hy_n' &= h^2\Big[\tfrac{510921}{1568000}f_n+\tfrac{159651}{8000}f_{n+1}-\tfrac{430839}{112000}f_{n+2}-\tfrac{6930117}{112000}f_{n+\frac{6}{5}}+\tfrac{16846461}{196000}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{14147703}{224000}f_{n+\frac{8}{5}}+\tfrac{271809}{11200}f_{n+\frac{9}{5}}\Big]\\
y_{n+1}'-y_n' &= h\Big[\tfrac{101755}{508032}f_n+\tfrac{1759}{126}f_{n+1}-\tfrac{10511}{4032}f_{n+2}-\tfrac{516875}{12096}f_{n+\frac{6}{5}}+\tfrac{103625}{1764}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{347125}{8064}f_{n+\frac{8}{5}}+\tfrac{37375}{2268}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{6}{5}}'-y_n' &= h\Big[\tfrac{29441}{147000}f_n+\tfrac{12276}{875}f_{n+1}-\tfrac{9117}{3500}f_{n+2}-\tfrac{29783}{700}f_{n+\frac{6}{5}}+\tfrac{71856}{1225}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{60201}{1400}f_{n+\frac{8}{5}}+\tfrac{8644}{525}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{7}{5}}'-y_n' &= h\Big[\tfrac{259567}{1296000}f_n+\tfrac{15778}{1125}f_{n+1}-\tfrac{187621}{72000}f_{n+2}-\tfrac{1834021}{43200}f_{n+\frac{6}{5}}+\tfrac{52913}{900}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{1239259}{28800}f_{n+\frac{8}{5}}+\tfrac{133427}{8700}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{8}{5}}'-y_n' &= h\Big[\tfrac{99364}{496125}f_n+\tfrac{110464}{7875}f_{n+1}-\tfrac{20512}{7875}f_{n+2}-\tfrac{200672}{4725}f_{n+\frac{6}{5}}+\tfrac{649472}{11025}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{67604}{1575}f_{n+\frac{8}{5}}+\tfrac{233344}{14175}f_{n+\frac{9}{5}}\Big]\\
y_{n+\frac{9}{5}}'-y_n' &= h\Big[\tfrac{157023}{784000}f_n+\tfrac{24543}{1750}f_{n+1}-\tfrac{146043}{56000}f_{n+2}-\tfrac{475497}{11200}f_{n+\frac{6}{5}}+\tfrac{288441}{4900}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{958149}{22400}f_{n+\frac{8}{5}}+\tfrac{11583}{700}f_{n+\frac{9}{5}}\Big]\\
y_{n+2}'-y_n' &= h\Big[\tfrac{6359}{31752}f_n+\tfrac{884}{63}f_{n+1}-\tfrac{641}{252}f_{n+2}-\tfrac{32125}{756}f_{n+\frac{6}{5}}+\tfrac{26000}{441}f_{n+\frac{7}{5}}\\
&\qquad -\tfrac{21625}{504}f_{n+\frac{8}{5}}+\tfrac{9500}{567}f_{n+\frac{9}{5}}\Big].
\end{aligned}\right\} \qquad (2.11)
$$

2.3. **Analysis of the Methods.** In this section, the proposed schemes are analyzed by determining the order and error constant, consistency, zero stability and region of absolute stability of the schemes.

(a): (i) Order and Error Constant of the Main Method

Taylor series expansion of (2.11) gives order $p = 9$ and error constant as $-\frac{29}{20671875000}$.

(ii) Order and Error Constant of the Block Method

Also, Taylor series expansion of terms of equation (2.11) yields

$$
\Big[\frac{183907}{3969000000},\frac{142713}{2392578125},\frac{9233903}{126562500000},\frac{16719104}{193798828125},\frac{153248493}{153125000000},
\frac{7001}{62015625},\frac{32233}{52920000},\frac{6369}{95703125},\frac{449281}{6750000000},\frac{171968}{2583984375},
\frac{815427}{12250000000},\frac{11}{165375}\Big]^T.
$$

(b): Consistency

Consistency of the Main Method

The first and second characteristics polynomial of the main method are given by:

$$\rho(z) = z^2 - 4z^{\frac{7}{5}} + 3z^{\frac{6}{5}}$$

and

$$\sigma(z) = -\frac{11}{52920000}z^0 - \frac{59}{210000}z + \frac{2633}{25200}z^{\frac{6}{5}} + \frac{15833}{147000}z^{\frac{7}{5}} + \frac{12937}{168000}z^{\frac{8}{5}} + \frac{16057}{378000}z^{\frac{9}{5}} + \frac{1109}{420000}z^2.$$

The main method (2.11) is consistent since it satisfies the following:

(i) the order of the method is $p = 9 > 1$

(ii) α_0=1,$\alpha_{\frac{7}{5}} = -4$ and $\alpha_{\frac{6}{5}} = 3$. Thus $\sum_{j=0}^{k} \alpha_j = 1 - 4 + 3 = 0$.

(iii)

$$\rho(1) = 1 - 4 + 3 = 0$$
$$\rho'(z) = 2z - \frac{28}{5}z^{\frac{2}{5}} + \frac{18}{5}z^{\frac{1}{5}}$$
$$\rho'(z) = 2 - \frac{28}{5} + \frac{18}{5}$$
$$\therefore \rho(1) = \rho'(1) = 0.$$

(iv)

$$\rho''(z) = 2 - \frac{56}{25}z^{\frac{-3}{5}} + \frac{18}{55}z^{\frac{-4}{5}}$$
$$\Rightarrow \rho''(1) = \frac{12}{25},$$
$$\sigma(1) = -\frac{11}{52920000} - \frac{59}{210000} + \frac{2633}{252000} + \frac{15833}{147000} + \frac{12937}{168000} + \frac{16057}{378000} + \frac{1109}{420000} = \frac{6}{25},$$
$$2!\sigma(1) = 2(\frac{6}{25}) = \frac{12}{25},$$
$$\rho''(1) = 2!\sigma(1).$$

Hence the main method is consistent. Similarly, the block method of (2.11) is consistent.

(c): Stability of the Method

(i) Zero-stability of the Main Method

By definition, the first characteristics polynomial of (2.11) is given as:

$$\rho(z) = z^2 - 4z^{\frac{7}{5}} + 3z^{\frac{6}{5}}.$$

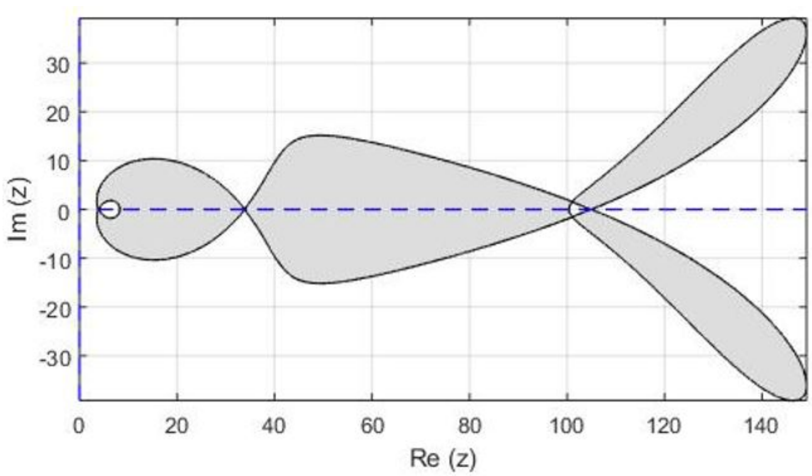
The zeros of these are: $z = 0, 1$.

(d): Convergence

The block method (2.11) is convergent since it satisfies the necessary and sufficient condition of consistency and zero-stability.

(e): Region of Absolute Stability

FIGURE 1. Region of absolute stability of $BHM_{2,4}$[1]



3. NUMERICAL RESULTS AND DISCUSSIONS

This section focuses on the performance of the proposed methods compared to that of existing methods in literature when applied to some selected problems.

[1]$BHM_{2,4}$ is the Two-step Block Hybrid Method with four off-step points.

Example 1: Roul (2018).

$$(x^2y')' = \beta x^{\alpha+\beta-2}(xy' + y(\alpha+\beta-1)); \quad 0 \le x \le 1.$$

For $\alpha = 2, \beta = 4$

$$x^2y'' + 2xy' = 4x^4(xy' + 5y)$$

with conditions $y(0) = 1, y(1) = \exp(1)$ and theoretical solution $y(x) = \exp(x^4)$.

Example 2: Jafari et al. (2012).

$$y'' + \frac{0.5}{x}y' = \exp(y)(0.5 - \exp(y)); x \in [0, 1]$$

with conditions $y(0) = \log 2, y(1) = 0$ and theoretical solution $y(x) = \log(\frac{2}{1+x^2})$.

Example 3: An Oscillatory Problem (Ndukum et al., 2015).

$$y'' = -100y + 99sinx$$

with conditions $y(0) = 1, y'(0) = 11, x \in [0, 1000]$.
The theoretical solution is $y(x) = cos(10x) + sin(10x) + sin(x)$.

Example 4: Koch et al. (2000).

$$y''(t) = -\frac{2}{t}y'(t) - n^2cos(nt) - \frac{2}{t}nsin(nt), y(0) = 2, y'(0) = 0.$$

The theoretical solution for $n = 3$ is $y(t) = 1 + cos(nt)$.

Example 5: Koch et al. (2000).

$$y''(t) = -\frac{2}{t}y'(t) - y^5(t), t \in (0, 1], \quad y(0) = 1, y'(0) = 0,$$

$$y(t) = \frac{1}{\sqrt{1 + \frac{t^2}{3}}}.$$

3.1. **Table of Results for Numerical Comparison.** Here, the tabular form of results obtained by the new methods in comparison with existing methods and exact solutions are presented below.

TABLE 1. Error of methods for Example 1

n	Kumar and Singh [10]	Roul [16]	$BHM_{2,4}$
6	4.7839×10^{-4}	6.7832×10^{-7}	4.9848×10^{-7}
8	6.7985×10^{-6}	1.6259×10^{-9}	2.0020×10^{-9}
CPU Time			$0.2500s$

3.2. **Discussion of Results.** From Table 1, the proposed method $BHM_{2,4}$ performed better in terms of accuracy than the method of Kumar and Singh [10] and the method of Roul [16]. Table 2 revealed the new method performs better than the method of Jafari *et al.* [7] and Kumar and Singh [9]. In the same vein, Tables 3, 4 and 5 showed that the performances of the proposed method are better than those of Ndukum *et al.* [12], and Koch *et al.* [8].

4. Conclusion

The hybrid block methods for solution of second order ordinary differential equations via the interpolation and collocation approach has been presented. Different types of problems of initial and boundary value types have been included to verify the applicability of the new methods. The methods are zero-stable, consistent and convergent. A comparison was made between the proposed methods and some other well-known methods. The numerical results obtained from these comparison indeed show the superiority of the new approach over some well-known method in the literature. It was also observed that the new methods are more accurate.

Table 2. Error of methods for Example 2

x	Jafari *et al.* [7]	Kumar and Singh [9]	$BHM_{2,4}$
0.1	1.27 x 10^{-4}	2.50 x 10^{-5}	3.69 x 10^{-6}
0.2	4.73 x 10^{-4}	3.87 x 10^{-4}	5.20 x 10^{-6}
0.3	9.18 x 10^{-4}	1.886 x 10^{-3}	6.08 x 10^{-6}
0.4	1.223 x 10^{-3}	5.65 x 10^{-3}	6.51 x 10^{-6}
0.5	9.72 x 10^{-4}	1.29 x 10^{-2}	6.59 x 10^{-6}
0.6	3.47 x 10^{-1}	2.48 x 10^{-2}	6.40 x 10^{-6}
0.7	3.377 x 10^{-3}	4.1896 x 10^{-2}	6.00 x 10^{-6}
0.8	8.694 x 10^{-3}	6.3701 x 10^{-2}	5.47 x 10^{-6}
0.9	9.151518 x 10^{-1}	9.862106 x 10^{-1}	4.85 x 10^{-6}
1.0	2.78647 x 10^{-2}	1.08368 x 10^{-1}	4.17 x 10^{-6}

Table 3. Error of methods for Example 3

N	h	Ndukum *et al.* [12]	$BHM_{2,4}$	CPU Time ($BHM_{2,4}$)
1000	1.0000	2.3×10^{-2}	1.0657×10^{-5}	$0.656s$
2000	0.5000	4.3×10^{-4}	6.4005×10^{-7}	$0.750s$
4000	0.2500	1.7×10^{-5}	3.9597×10^{-8}	$0.752s$
8000	0.1250	1.1×10^{-6}	2.4654×10^{-9}	$1.58s$
16000	0.0625	6.3×10^{-7}	1.6370×10^{-10}	$2.89s$

Table 4. Error of methods for Example 4

h	Koch *et al.* [8]	$BHM_{2,4}$
$\frac{1}{5}2^{0}$	8.5	6.3416×10^{-1}
$\frac{1}{5}2^{-3}$	1.1×10^{-1}	9.9434×10^{-3}
$\frac{1}{5}2^{-5}$	2.7×10^{-2}	9.9965×10^{-4}
CPU Time		$0.1917s$

Table 5. Error of methods for Example 5

h	Koch *et al.* [8]	$BHM_{2,4}$
$\frac{1}{5}2^{-2}$	7.1×10^{-2}	3.6865×10^{-3}
$\frac{1}{5}2^{-4}$	1.8×10^{-3}	2.1320×10^{-4}
$\frac{1}{5}2^{-6}$	4.7×10^{-3}	1.3090×10^{-5}
CPU Time		$0.0938s$

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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