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The Pell-Fibonacci Sequence Modulo *m*

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Abstract. In [6], Deveci defined the Pell-Fibonacci sequence as follows:

$$P-F(n+4)=3P-F(n+3)-3P-F(n+1)-P-F(n)$$

for $n\geq 0$ with initial constants $P-F(0)=P-F(1)=P-F(2)=0, P-F(3)=1$. Also, he derived the permanental and determinantal representations of the Pell-Fibonacci numbers and he obtained miscellaneous properties of the Pell-Fibonacci numbers by the aid of the generating function and the generating matrix of the Pell-Fibonacci sequence. The linear recurrence sequences appear in modern research in many fields from mathematics, physics, computer, architecture to nature and art; see, for example, [2, 4, 13, 18]. In this paper, we obtain the cyclic groups which are produced by generating matrix of the Pell-Fibonacci sequence when read modulo m. Furthermore, we research the Pell-Fibonacci sequence modulo m, and then we derive the relationship between the order of the cyclic groups obtained and the periods of the Pell-Fibonacci sequence modulo m.

1. Introduction

In [6], Deveci defined the Pell-Fibonacci sequence which is directly related to the Pell and Fibonacci numbers as follows:

$$P-F(n+4)=3P-F(n+3)-3P-F(n+1)-P-F(n) \tag{1}$$

for $n\geq 0$ with initial constants $P-F(0)=P-F(1)=P-F(2)=0, P-F(3)=1$.

Then by an inductive argument, he gave the generating matrix of Pell-Fibonacci sequence as follows:

$$M_3=\begin{bmatrix} 3 & 0 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix M_3 is said to be Pell-Fibonacci matrix.Then, he obtained that

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$$(M_3)^n = \begin{bmatrix} x^3_{n+4} & F_{n+2} + x^3_{n+3} - x^3_{n+4} & F_{n+3} + x^3_{n+4} - x^3_{n+5} & -x^3_{n+3} \\ x^3_{n+3} & F_{n+1} + x^3_{n+2} - x^3_{n+3} & F_{n+2} + x^3_{n+3} - x^3_{n+4} & -x^3_{n+2} \\ x^3_{n+2} & F_{n} + x^3_{n+1} - x^3_{n+2} & F_{n+1} + x^3_{n+2} - x^3_{n+3} & -x^3_{n+1} \\ x^3_{n+1} & F_{n-1} + x^3_{n} - x^3_{n+1} & F_{n} + x^3_{n+1} - x^3_{n+2} & -x^3_{n} \end{bmatrix} \quad (2)$$

for $n \geq 1$. It is important to note that $\det M_3 = 1$.

The linear recurrence sequences appear in modern research in many fields from mathematics, physics, computer, architecture to nature and art; see, for example, [2, 4, 13, 18]. Many authors have studied some special linear recurrence sequences in algebraic structures. Some of these proved that the lengths of the periods of the recurring sequences obtained by the reducing sequences by a modulo m are equal to the lengths of the ordinary recurrences in cyclic groups; see for example, [1, 3, 5, 7–15, 17, 20]. Wall [19] proved that the lengths of the periods of the recurring sequences obtained by reducing Fibonacci sequences by a modulo m are equal to the lengths of the ordinary 2-step Fibonacci recurrences in cyclic groups. Lü and Wang [16] obtained the rules for the orders of the cyclic groups generated by reducing the k-generalized Fibonacci matrix modulo m . Ozkan et al. [17] proved two original theorem concerning Wall number of the 3-step Fibonacci sequences and they gave conjectures concerning 3-step Fibonacci sequence.In this paper, we obtain the cyclic groups which are produced by generating matrix of the Pell-Fibonacci sequence when read modulo m. Also, we study the Pell-Fibonacci sequence modulo m. Finally, we derive the relationship between the order of the cyclic groups obtained and the periods of the Pell-Fibonacci sequence modulo m.

2. The Pell-Fibonacci Sequence Modulo *m*

For given a matrix $A = \left[a_{ij}\right]$ of integers, $A\,(\text{mod } m)$ means that the entries of A are reduced modulo m, that is, $A\,(mod\ m) = \left(a_{ij}\ (mod\ m)\right)$. Let us consider the set $\langle A \rangle_m = \left\{ A^i\,(modm) \middle|\ i \geq 0 \right\}$. If $\gcd(m, \det A) = 1$, then the set $\langle A \rangle_m$ is a cyclic group. Let the notation $|\langle A \rangle_m|$ denote the order of the set $\langle B \rangle_m$.

Since $\det M_3 = 1$, it is clear that the set $\langle M_3 \rangle_m$ is a cyclic group for every positive integer m.

Theorem 2.1. ***(Wall [19]).*** *The number $k(s, p^n)$ divides $k(s, p^n)p^{n-1}$, and the two quantities are equal provided $k(s, p) = k(s, p^2)$*

Theorem 2.2. *Let p be a prime and let $\langle M_3 \rangle_{p^m}$ be a cyclic groups. If u is the largest positive integer such that $\left|\langle M_3 \rangle_p\right| = \left|\langle M_3 \rangle_{p^u}\right|$, then $\left|\langle M_3 \rangle_{p^v}\right| = p^{v-u}.\left|\langle M_3 \rangle_p\right|$ for every $v \geq u$. In particular, if $\left|\langle M_3 \rangle_p\right| \neq \left|\langle M_3 \rangle_{p^2}\right|$, then $\left|\langle M_3 \rangle_{p^v}\right| = p^{v-1}.\left|\langle M_3 \rangle_p\right|$ for every $v \geq 2$.*

Proof. Let us consider the cyclic group $\langle M_3 \rangle_{p^m}$. Suppose that s is a positive integer and $\left|\langle M_3 \rangle_{p^m}\right|$ is denoted by $L_{P-F}\,(p^m)$. If $(M_3)^{L_{P-F}\left(p^{s+1}\right)} \equiv I\left(modp^{s+1}\right)$, then, we can write $(M_3)^{L_{P-F}\left(p^{s+1}\right)} \equiv I\,(modp^s)$ where I is a 4×4 identity matrix. Thus we get that $L_{P-F}\,(p^s)$ divides $L_{P-F}\left(p^{s+1}\right)$. Furthermore, if we denote $(M_3)^{L_{P-F}\left(p^s\right)} = I + \left(m^{(s)}_{ij} \cdot p^s\right)$, then by the binomial expansion, we may write

$$(M_3)^{L_{P-F}\left(p^s\right) \cdot p} = \left(I + \left(m^{(s)}_{ij} \cdot p^s\right)\right)^p = \sum_{i=0}^{p} \left(\begin{array}{c} p \\ i \end{array} \right) \left(m^{(s)}_{ij} \cdot p^s\right)^i \equiv I\left(modp^{s+1}\right)$$

This yields that $L_{P-F}\left(p^{s+1}\right)$ divides $L_{P-F}\left(p^{s}\right)\cdot p$. Thus, $L_{P-F}\left(p^{s+1}\right)=L_{P-F}\left(p^{s}\right)$ or $L_{P-F}\left(p^{s+1}\right)=L_{P-F}\left(p^{s}\right)\cdot p$. It is easy to see that the latter holds if and only if there is an $m_{ij}^{(s)}$ which is not divisible by p. Since u is the largest positive integer such that $L_{P-F}\left(p^{s}\right)=L_{P-F}\left(p^{u}\right)$, we have $L_{P-F}\left(^{u}\right)\neq L_{P-F}\left(p^{u+1}\right)$. Then there is an $m_{ij}^{(u+1)}$ which is not divisible by p. Thus we get that $L_{P-F}\left(p^{u+1}\right)\neq L_{P-F}\left(p^{u+2}\right)$. The proof is finished by induction on u. □

Reducing the Pell-Fibonacci sequence $\{P-F(n)\}$ by a modulo m, we obtain the following repeating sequence:

$$\{P-F^{m}(n)\}=\{P-F^{m}(0),P-F^{m}(1),\ldots,P-F^{m}(i),\ldots\}$$

where $P-F^{m}(i)=P-F(i)\,(modm)$. It has the same recurrence relation as in (1).

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,b,c,d,b,c,d,\ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,a,b,c,d,a,b,c,d,\ldots$ is simply periodic with period 4.

Theorem 2.3. *For every positive integer m, the Pell-Fibonacci sequence modulo m $\{P-F^{m}(n)\}$ is simply periodic.*

Proof. Let us consider set

$$X=\left\{(x_0,x_1,x_2,x_3)\,\middle|\;\text{x}_\text{i}\text{'s are integers such that}\quad 0\leq x_i\leq m-1\right\}.$$

Since $|X|=m^4$, there are m^4 distinct 4-tuples of elements of Z_m. Then it is easy to see that at least one of the 4-tuples appears twice in the sequence $\{P-F^{m}(n)\}$. Therefore, the subsequence following this 4-tuple repeats; hence, the sequence is periodic. Let

$$P-F^{m}(i+1)\equiv P-F^{m}(j+1),\ldots,P-F^{m}(i+3)\equiv P-F^{m}(j+3)$$

such that $i>j$, then $i\equiv j\,(mod4)$. From the definition of the Pell-Fibonacci sequence we can easily obtain

$$P-F^{m}(i)\equiv P-F^{m}(j),P-F^{m}(i-1)\equiv P-F^{m}(j-1),\ldots,P-F^{m}(i-j)\equiv P-F^{m}(0)$$

which implies that the $\{P-F^{m}(n)\}$ is a simply periodic sequence. □

The period of the sequence $\{P-F^{m}(n)\}$ is denoted by $h_{P-F}(m)$.

Example 2.4. *Some term of the Pell-Fibonacci sequence $\{P-F(n)\}$ are as follows:*

$$\{0,0,0,1,3,9,24,62,156,387,951,2323,5652,13716,33228,\ldots\}.$$

Reducing he Pell-Fibonacci sequence $\{P-F(n)\}$ by a modulo 2, the sequence becomes:

$$\{0,0,0,1,1,1,0,0,0,1,\ldots\}.$$

So, we obtained that the period of the sequence $\left\{P - F^2(n)\right\}$ *is* 6.

Similarly, Since the sequence becomes as shown:

$$\{0,0,0,1,0,0,0,2,0,0,0,1\ldots\}$$

for $m = 3$, *we have* $h_{P-F}(3) = 8$.

It is easily seen from equation (2) that $h_{P-F}(m) = |\langle M_3\rangle_m|$ for every positive integer m.

Theorem 2.5. *If m has the prime factorization* $m = \prod_{i=1}^{u}(p_i)^{s_i}$, $(u \geq 1)$ *where* p_i*'s are distinct primes. Then*

$$h_{P-F}(m) = lcm\left[h_{P-F}\left((p_1)^{s_1}\right), h_{P-F}\left((p_2)^{s_2}\right), \ldots, h_{P-F}\left((p_u)^{s_u}\right)\right].$$

Proof. Since $h_{P-F}\left((p_i)^{s_i}\right)$ is the length of the period of the sequence $\left\{P - F^{(p_i)^{s_i}}(n)\right\}$, the sequence repeats only after blocks of length $\lambda \cdot h_{P-F}\left((p_i)^{s_i}\right)$, $(\lambda \in \mathbb{N})$. Since $h_{P-F}(m)$, is period of the sequence $\{P - F^m(n)\}$, the sequence $\left\{h_{P-F}\left((p_i)^{s_i}\right)\right\}$ repeats after $h_{P-F}(m)$ terms for all values i. Thus $h_{P-F}(m)$ is the form $\lambda \cdot h_{P-F}\left((p_i)^{s_i}\right)$ for all values i, and since any such number gives a period of $\{P - F^m(n)\}$. So we get

$$h_{P-F}(m) = lcm\left[h_{P-F}\left((p_1)^{s_1}\right), h_{P-F}\left((p_2)^{s_2}\right), \ldots, h_{P-F}\left((p_u)^{s_u}\right)\right].$$

□

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