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Fractional Order of a New 7D Hyperchaotic Lorenz-like System

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Abstract

In this paper, a new 7D hyperchaotic Lorenz-like system is proposed with perspective of fractional order. Numerical implementations of this proposed system with specific parameters are investigated and compared with the new 7D continuous hyperchaotic system. In addition to this, due to the hyperchaotic attractors do not exist lower than 0.6, the values of fractional order are analysed in range between 0.6 to 1. Stability conditions are obtained through the stability theory of fractional systems. Numerical analysis of Lyapunov exponents verifies the existence of hyperchaos for less than five orders.

Keywords: *Chaos; Fractional order derivatives; Fractional stability; Hyperchaos; Lorenz-like system; Lyapunov exponents.*
2010 Mathematics Subject Classification: *65P20; 26A33; 65P40; 34D08*

1. Introduction

In last decades, chaotic theory in differential equations have become an exciting field in science and engineering studies. Hyperchaos is firstly defined as a system with more than one positive Lyapunov exponent in the classical example of hyperchaotic systems paper, written by Rössler in 1979 [34]. Another classical example for hyperchaotic behaviour research is seen in Kapitaniak's paper [18], which proposed an electronic circuit type called the hyperchaotic Chua's circuit. Dynamical behaviours of chaotic attractors is much more limited than hyperchaos [16]. Therefore, applications of hyperchaotic systems in technological fields have been widely used in many disciplines such as control, neural networks, lasers, nonlinear circuits, communication and bio-medical [1, 3, 15, 35, 37]. It is known that the complicatedness and randomness increase with higher order hyperchaotic system behaviours and likewise predictability decreases [11, 16]. Nevertheless, the number of recent studies on high-dimensional hyperchaotic Lorenz systems have been increased [6, 40, 41, 42, 43, 44].
Fractional calculus is known as one of the generalizations of the classical calculus. Despite having a long mathematical history, there has been a growing interest in the area of fractional calculus and its applications in last years; for example, see [7, 8, 14, 25, 33]. The advances in fractional calculus focus on modern examples in differential and integral equations, dynamic systems, mathematical biology, optimal control and mechanics [5, 10, 17, 19, 28, 30]. The purpose of using fractional order derivative instead of integer order derivative is to have a better fit to the real data in these application areas and to overcome the limitations of the integer order derivatives [19, 32]. The real life models can be better explained with memory which has an important effects on fractional derivatives. Since the fractional order derivatives contain memory, it is especially preferred in biological models [36]. While there are many mathematically acceptable and common definitions for fractional derivative such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard and Riesz [29, 31, 32], two types of them, namely Riemann-Liouville and Caputo, are mostly used in problem formulations in the papers cited above.
Recent years, there have been many studies on fractional chaotic and hyperchaotic systems. For instance, the effects of fractional dynamics in chaotic systems are studied in [13], chaotic behaviours are found in the fractional-order Chen system in [20] where the authors found that the lowest order in the system is 2.1. At a later stage, it is found in [23] that chaos existed in the fractional-order Chen system for orders as low as 0.3. The chaotic behaviours of the fractional order Liu system, which connects both the Lorenz and Chen systems, and also represents transition from one to another, is investigated in [22]. Master slave synchronization of Liu system is also considered in [22]. The basic properties of dynamical behaviours in the fractional-order Rossler system, which have been analysed by means of Lyapunov exponents and bifurcation diagrams, are studied in [45]. The fractional-order complex Lorenz system is proposed and dynamic behaviours of a fractional-order chaotic system in complex space are investigated in [24]. In addition to these, high-dimensional (four to six dimensional) hyperchaotic Lorenz systems are also studied with fractional-order perspective in last decade [11, 27, 38]. However, a novel 7D continuous hyperchaotic Lorenz-like system, which is introduced by Yang et al. [44], has not studied with fractional order perspective in the literature, yet. For this reason, we introduce fractional order version of this new 7D continuous hyperchaotic Lorenz-like system in this study.
The main purpose of this paper is to investigate the hyperchaotic behaviour of the proposed fractional-order 7D hyperchaotic Lorenz-like system, which may provide potential applications in circuit implement and secure communication. The existence of chaos and hyperchaos

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are verified with using the sign of Lyapunov exponents. Besides, hyperchaotic attractors are obtained with in a certain range of specified three control parameters. Moreover, the stability analysis and numerical experiments of the fractional order system are investigated. Our stability analysis produces the lowest existence condition order of hyperchaotic behaviour for the fractional-order system when the parameters are fixed.

This study is organized as follows: in Section 2, preliminaries of the paper are introduced. Section 3 provides 7D hyperchaotic Lorenz-like system and presents analytical investigation of its steady states. The new fractional order 7D system is introduced in Section 4. The stability analysis and a comparative numerical investigations of Yang's model and the new fractional order 7D hyperchaotic system are also given in this section. Finally, concluding remarks of the paper are given in Section 5.

2. Preliminaries

In this section, we give some definitions and properties which are further used in this paper. In the following, it is aimed to analyse and investigate systems via Caputo fractional derivative due to its convenience for initial conditions of the differential equations. In this sense, the main focus of this paper is to study on this type of derivative.

Definition 2.1. *[32] The fractional integral of order $\gamma > 0$ for a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds,$$

where and elsewhere Γ denotes the Gamma function.

Definition 2.2. *[4] If $f(t) \in \mathbb{C}^n$, $t, \gamma \in \mathbb{R}$ then the Caputo fractional derivative with fractional order γ is defined as,*

$$^C D_t^{\gamma} f(t) = I^{n-\gamma} D^n f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds,$$

where $n-1 \leq \gamma < n \in \mathbb{N}$ and D^n is the usual differential operator of integer order n. Note that the value of the Caputo fractional derivative of the function g at point t involves all the values of $g^n(s)$ for $s \in [0,t]$.
Clearly $^C D_t^{\gamma} f(t)$ tends to $f'(t)$ as $\gamma \to 1$.

Definition 2.3. *[21] The constant z^* is an equilibrium point of the Caputo fractional dynamic system that is presented as below,*

$$^C D_t^{\gamma} z(t) = f(t, z(t)), \quad \gamma \in (0,1), \quad \textit{if and only if} \quad f(t, z^*) = 0.$$

Theorem 2.4. *[26] The autonomous system*

$$^C D_t^{\gamma} \mathbf{x} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \quad \textit{with} \quad 0 < \gamma < 1, \quad \mathbf{x} \in \mathbb{R}^n \quad \textit{and} \quad A \in \mathbb{R}^{n \times n}, \tag{2.1}$$

is asymptotically stable if and only if the following inequality,

$$|arg(\lambda(A))| > \frac{\gamma\pi}{2},$$

is satisfied for all $\lambda \in \sigma(A)$. Here, $\sigma(A)$ shows the spectrum of A. Also, system (2.1) is stable if and only if

$$|arg(\lambda(A))| \geq \frac{\gamma\pi}{2}, \quad \forall \lambda \in \sigma(A),$$

with these critical eigenvalues satisfying $|arg(\lambda(A))| = \dfrac{\gamma\pi}{2}$ and having geometric multiplicity of one. Here, the geometric multiplicity is the dimension of the subspace of vectors υ for which $A\upsilon = \lambda\upsilon$.

For simplicity, the symbol $^C D_t^{\gamma}$ is denoted as D_t^{γ} from now on.

3. Model Formulation

Yang et al. [44] presented a new seven-dimensional (7D) hyperchaotic Lorenz-Like system with five positive Lyapunov exponents by coupling 1D linear equation to the 6D hyperchaotic equations system as follows:

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + r y_6, \\
\dot{y}_2 &= c y_1 - y_2 - y_1 y_3 + y_5, \\
\dot{y}_3 &= -b y_3 + y_1 y_2, \\
\dot{y}_4 &= d y_4 - y_1 y_3, \\
\dot{y}_5 &= -k y_2 + y_6, \\
\dot{y}_6 &= q_1 y_1 + q_2 y_2, \\
\dot{y}_7 &= g y_7 + n y_4,
\end{aligned} \tag{3.1}$$

where a, b and c are constant parameters, n is the coupling parameter, r, d, k, q_1, q_2 and g are the six control parameters. Note that a, b, d, k and g are nonzero. Here, y_i and $\dot{y}_i$, $(i = 1, ..., 7)$, represent the state variables and their derivatives, respectively. All of the control parameters in the system generate chaotic and hyperchaotic behaviours, and also bifurcations can occur.

Yang et al. proved that system (3.1) has a unique unstable equilibrium point at $\bar{y}_i = 0$, $(i = 1,2,...,7)$, and five positive Lyapunov exponents in seven Lyapunov exponents as,

$$\lambda_{LE} = (1.000, 0.4128, 0.2255, 0.1360, 0.0880, 0.0000, -12.5289),$$

while the parameters are choosen as

$$(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1).$$

Furthermore, the eigenvalues of the equilibrium at $\bar{y}_i = 0$ with the same parameters are obtained as

$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7) = (-22.6230, -2.6667, 1, 2, 11.4755, 0.0737+0.3850i, 0.0737-0.3850i),$$

and this equilibrium point has a two-dimensional stable and a five-dimensional unstable manifolds.
Regardless of these determined parameters, system (3.1) has a unique equilibrium point at $\bar{y}_i = 0$, while the following conditions are satisfied,

- $bd\left[a + rk + a\frac{q_2}{q_1}\right] \geq 0$ and $q_1q_2 \neq 0$, or
- $q_1q_2 = 0$.

Otherwise, system (3.1) has three equilibria for $bd\left[a + rk + a\frac{q_2}{q_1}\right] < 0$ and $q_1q_2 \neq 0$. The first one is origin $O(0,0,0,0,0,0,0)$, and the others are

$$E_{\pm}\left(\pm y_0,\ \mp\frac{q_1}{q_2}y_0,\ -\frac{q_1}{q_2 b}y_0^2,\ \mp\frac{q_1}{q_2 bd}y_0^3,\ \mp\left(c + \frac{q_1}{q_2} + \frac{q_1}{q_2 b}y_0^2\right)y_0,\ \mp\frac{q_1 k}{q_2}y_0,\ \mp\frac{q_1 n}{q_2 bdg}y_0^3\right),$$

where $y_0 = \sqrt{-bd(a + rk + a\frac{q_2}{q_1})}$.

4. Fractional Order 7D System

The aim of this study is to introduce the fractional version of system (3.1) and compare its dynamic results. System (3.1) can be transformed to the fractional order version as follows:

$$\begin{aligned}
D_t^{\gamma}y_1(t) &= a(y_2 - y_1) + y_4 + ry_6,\\
D_t^{\gamma}y_2(t) &= cy_1 - y_2 - y_1y_3 + y_5,\\
D_t^{\gamma}y_3(t) &= -by_3 + y_1y_2,\\
D_t^{\gamma}y_4(t) &= dy_4 - y_1y_3,\\
D_t^{\gamma}y_5(t) &= -ky_2 + y_6,\\
D_t^{\gamma}y_6(t) &= q_1y_1 + q_2y_2,\\
D_t^{\gamma}y_7(t) &= gy_7 + ny_4,
\end{aligned} \tag{4.1}$$

where γ decribes the arbitrary derivative order of the state variables y_i, $(i = 1,2,...,7)$. It is obvious that system (4.1) is exactly same with the dynamic system (3.1) while $\gamma = 1$.
By using the parameters $(a,b,c,d,k,q_1,q_2,g,n,r)$ as $(10,8/3,28,2,9.9,1,2,1,1,1)$, the calculation of $bd\left[a + rk + a\frac{q_2}{q_1}\right]$ is equal to 132.8, and this positive value shows that system (4.1) has a unique equilibrium point with these parameters which is $O(0,0,0,0,0,0,0)$. Therefore, the Jacobian matrix of system (4.1) with this equilibria is obtained as follow:

$$J = \begin{bmatrix} -10 & 10 & 0 & 1 & 0 & 1 & 0\\ 28 & -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -8/3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 0 & 0 & 0\\ 0 & -99/10 & 0 & 0 & 0 & 1 & 0\\ 1 & 2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Corresponding to this Jacobian, eigenvalues are obtained as $\lambda_1 = -2.6667$, $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = -22.623$, $\lambda_5 = 0.0737 + 0.385i$, $\lambda_6 = 0.0737 + -0.385i$, $\lambda_7 = 11.4755$.
The lowest fractional order to see chaotic attractor for the fractional order system (4.1) can be determined by using the fractional stability Theorem 2.4. According to this theorem, the chaotic attractor exists with the instability of the equilibrium points. Therefore, the lowest fractional order γ can be calculated with the following inequility,

$$\gamma > \frac{2}{\pi} min_i |arg(\lambda_i)|.$$

If the determined parameters are used for the corresponding eigenvalues, the lowest order γ is obtained as $\gamma > \frac{2}{\pi} min_i |arg(\lambda_i)| \approx 0.8796$ where $min_i |arg(\lambda_i)|$ is calculated as 1.3817. Thus, chaotic behaviour is not seen for system (4.1) while $\gamma < 0.8796$, which is verified by numerical implemantations in this study.
Now we calculate the fractional derivative order γ in range of $(0.6, 1)$ at which the fractional system (4.1) has hyperchaotic solutions. In our numerical implementations, Adams-Bashforth-Moulton method [9] is used within MATLAB to estimate the solution of system (4.1) [12]. Moreover, Benettin-Wolf algorithm [2, 39] is utilized to derive the Lyapunov exponents LE_j, $j = 1,2,\ldots,7$ of system (4.1) for different values of the derivative order γ.
The projections of the hyperchaotic attractor of system (4.1) with respect to fixing parameters $(a,b,c,d,k,q_1,q_2,g,n,r)$ and varying the fractional-order are demonstrated in Figures 4.1, 4.2 and 4.3.

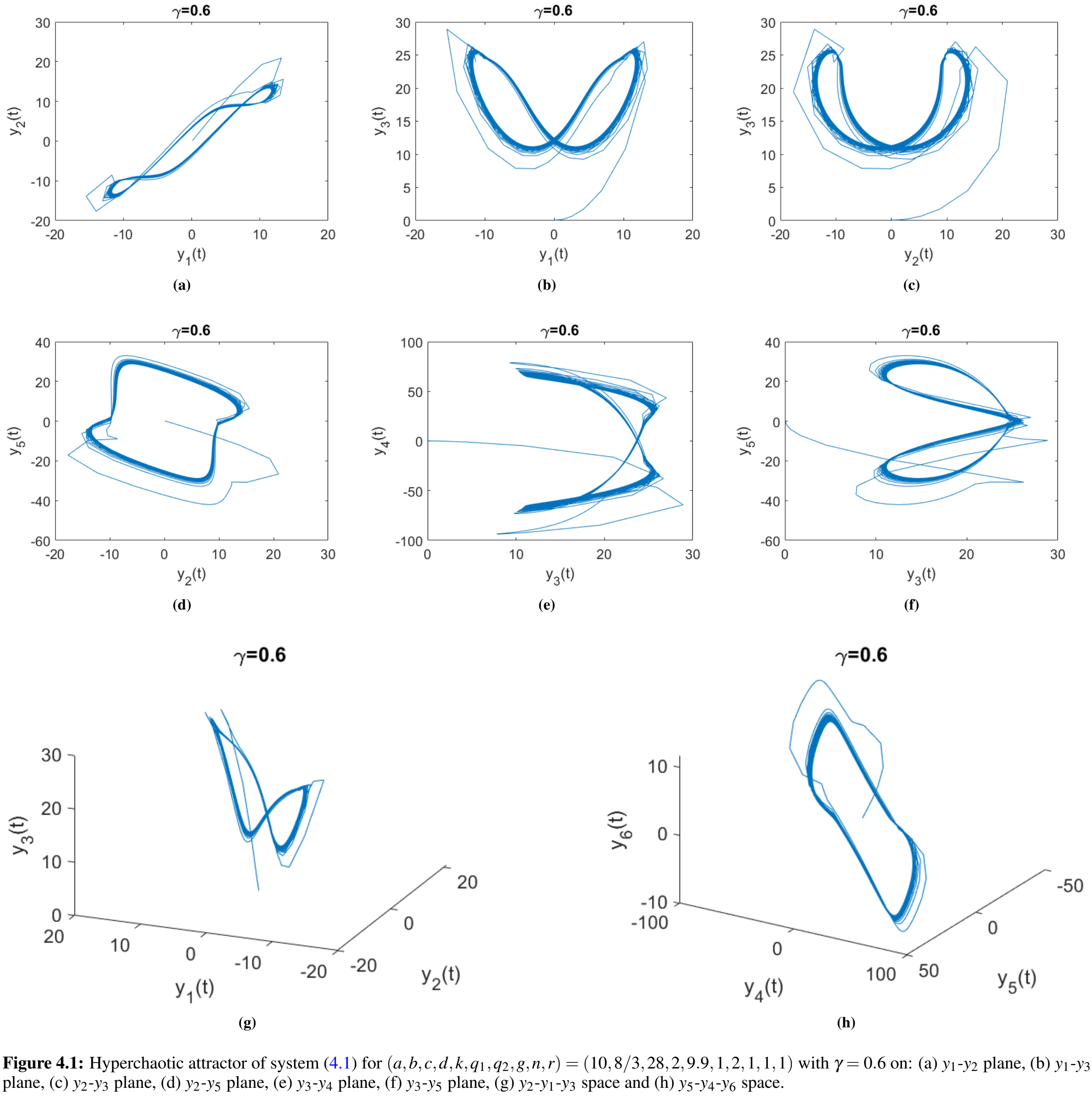


Figure 4.1: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$ with $\gamma = 0.6$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

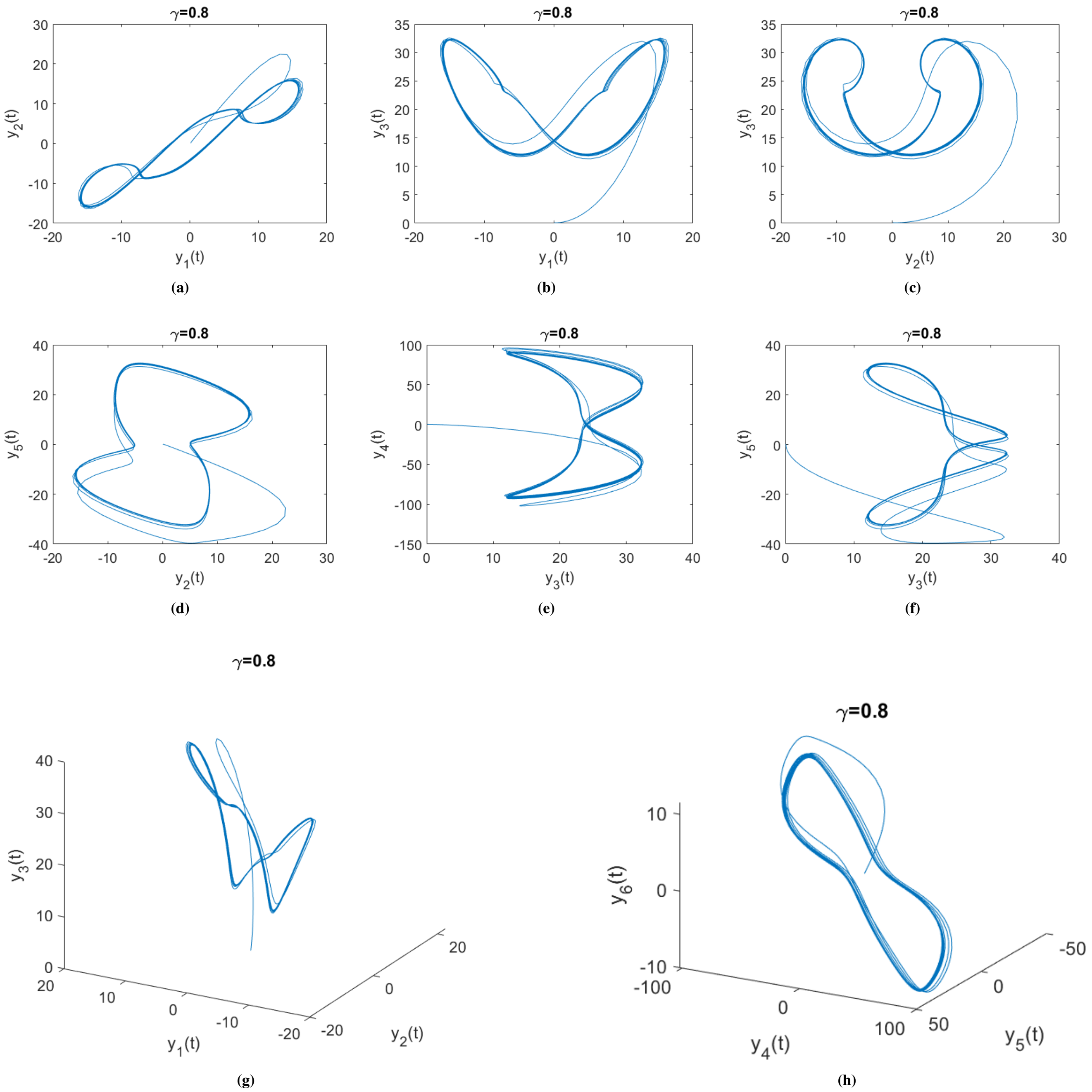


Figure 4.2: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$ with $\gamma = 0.8$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

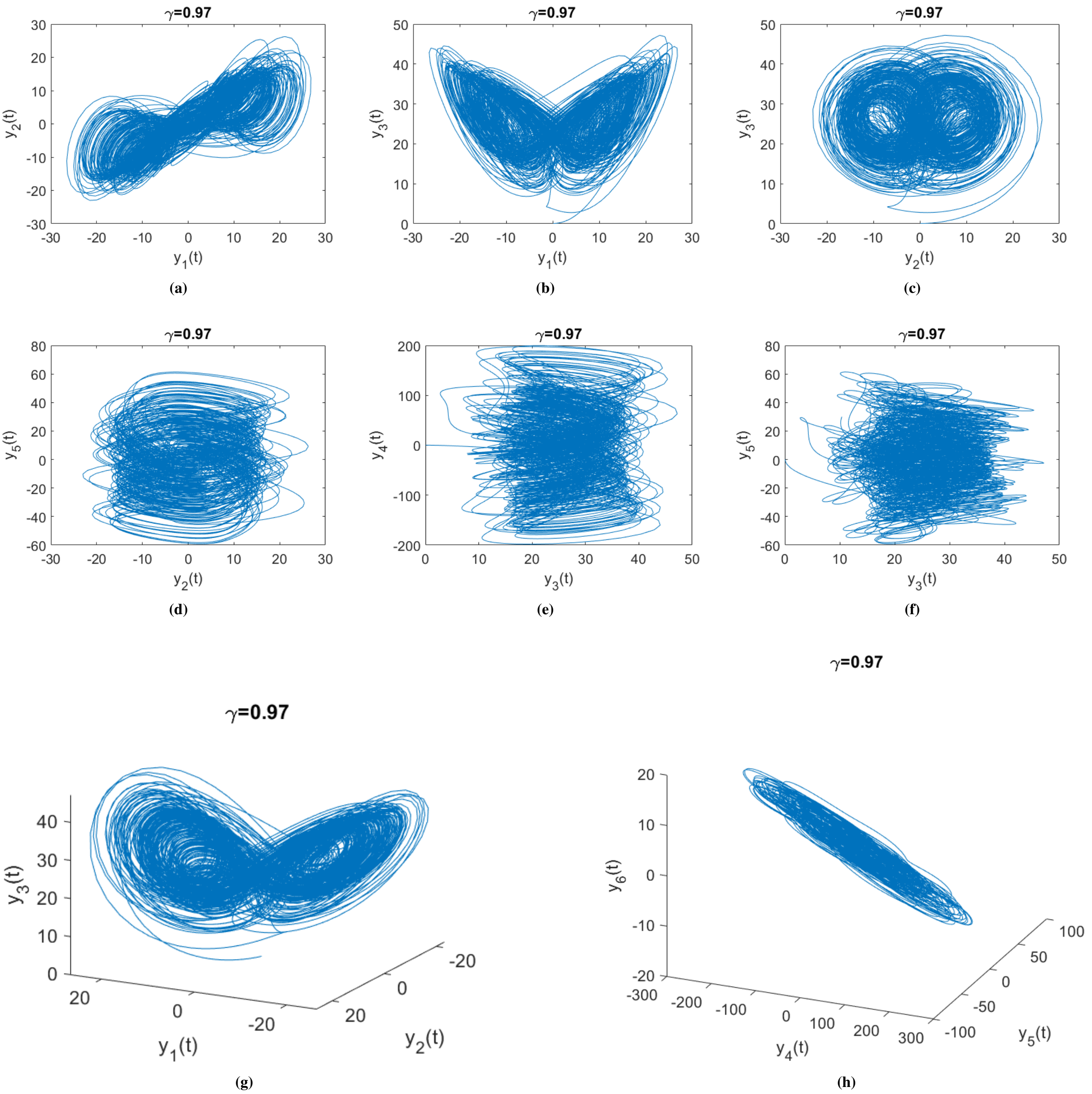


Figure 4.3: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$ with $\gamma = 0.97$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

The Lyapunov exponents of system (4.1) with respect to parameters $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$ and initial values of the state variables as $(0.1,0.1,0.1,0.1,0.1,0.1,0.1)$ are shown in Figure 4.4, where the fractional-order γ is varying in range between $(0.6,1)$.

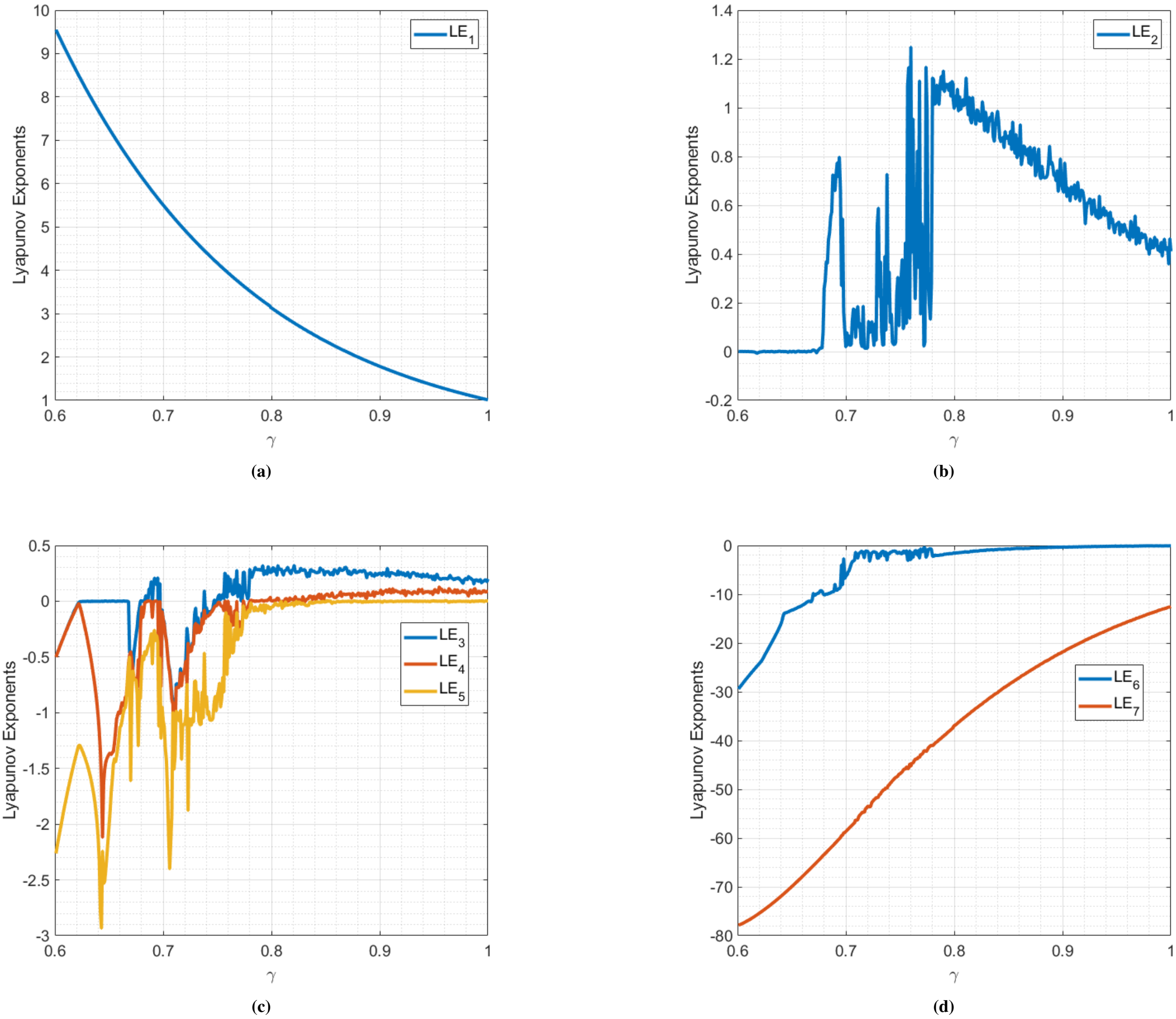


Figure 4.4: Lyapunov exponents of system (4.1) with parameters $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$, all initial values of state variables as 0.1 and γ in $(0.6,1)$. a-b) Lyapunov exponents LE_1 and LE_2 are positive in $(0.6,1)$, c) Lyapunov exponents LE_3, LE_4 and LE_5 show changeable behaviour in $(0.6,1)$, d) Lyapunov exponents LE_6 and LE_7 are negative in $(0.6,1)$.

According to numerical results, the fractional system (4.1) has no any periodic, quasi-periodic or chaotic solutions, and the solutions do not approach to fixed point since the value of LE_1 and LE_2 are positive for every point of γ in range between $(0.6,1)$. The results show that the fractional system (4.1) has only hyperchaotic solutions while the value of γ is in range between $(0.6,1)$. The range of γ orders for the fractional system (4.1) and their dynamic behaviours are given in Table 1.

Table 1: Lyapunov exponents of the fractional system (4.1) while the parameters and initial values of the state variables are chosen as $(a,b,c,d,k,q_1,q_2,g,n,r) = (10,8/3,28,2,9.9,1,2,1,1,1)$ and $y_i(0) = 0.1$, $(i = 1,\cdots,7)$, respectively.

γ	LE_1	LE_2	LE_3	LE_4	LE_5	LE_6	LE_7	Dynamics
0.96	1.2632	0.4838	0.2046	0.0817	0.0064	-0.1283	-15.5549	hyperchaotic of order 5
0.925	1.5493	0.6265	0.2428	0.0784	0.0049	-0.2549	-19.0214	hyperchaotic of order 5
0.83	2.6475	0.9483	0.2820	0.0182	-0.0306	-0.8861	-31.7437	hyperchaotic of order 4
0.75	4.1673	0.2464	0.0073	-0.0309	-0.8467	-1.5318	-46.8075	hyperchaotic of order 3
0.70	5.5072	0.0191	-0.1573	-0.1616	-1.0719	-6.2933	-58.6417	hyperchaotic of order 2
0.601	9.5475	0.0008	-0.4973	-0.5002	-2.2624	-29.4041	-77.8684	hyperchaotic of order 2

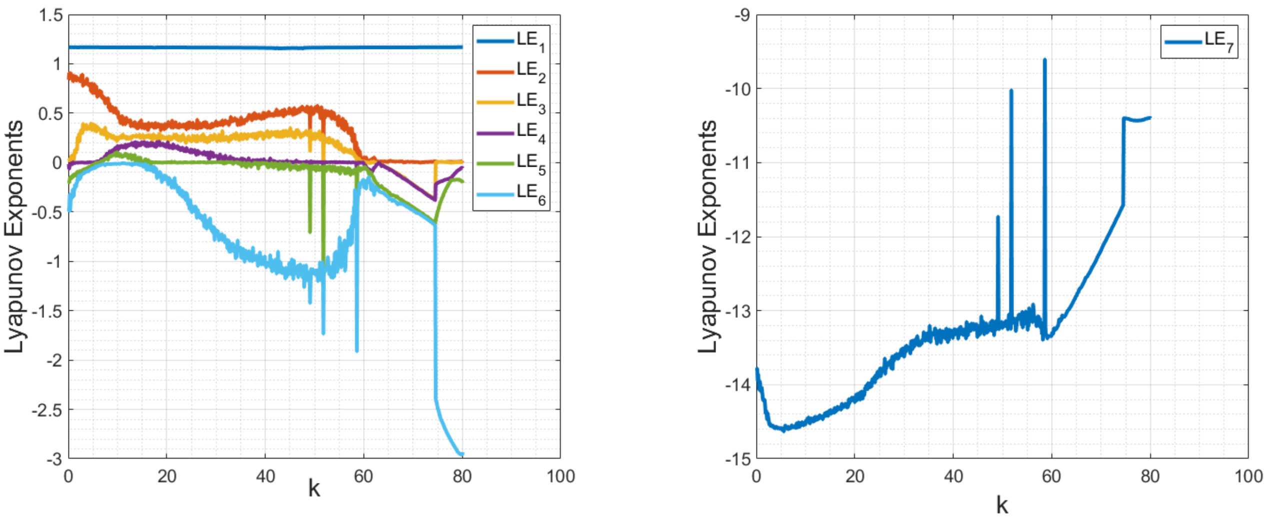
From this table, it can be said that the Lyapunov exponents of system (4.1) with different values of the fractional order are depicted, and this system is dissipative due to the sum of all LE values for each γ are negative.

After this point, the behaviour of system (4.1) is analysed by varying the control parameters k, d and q_2, respectively. The range of these parameter values are chosen same as in Yang's paper [44]. The initial values of state variables are chosen as $y_i(0) = 0.1$, $(i = 1, \cdots, 7)$, and the fractional order γ is 0.97.

- While the control parameters k is considered in range between [0, 80] as in Yang's paper [44], and the other parameters are fixed: System (4.1) has only hyperchaotic attractors. The Lyapunov exponents LE_1 is always positive and LE_2 is negative for only a few values of k in range between [68.3, 71.6] where they are very close to zero and for this reason, they can be ignored. Out of these values, system has always hyperchaotic attractors in range [0, 80]. It has hyperchaotic attractors with two (for $k \in [51.8, 80]$), three (for $k \in [0, 6]$ and $k \in [47.7, 80]$), four (for $k \in [2.1, 59.9]$) and five (for $k \in [6.1, 40.2]$) positive Lyapunov exponents, respectively. The numerical results of Lyapunov exponents for k in range between [0, 80] are written in Table 2 and presented in Figure 4.5, while the other parameters are fixed as $(a,b,c,d,q_1,q_2,g,n,r) = (10, 8/3, 28, 2, 1, 2, 1, 1, 1)$, and the initial values of state variables and fractional order are chosen as $y_i(0) = 0.1$ and $\gamma = 0.97$, respectively.

Table 2: Lyapunov exponents of the fractional system (4.1) while $\gamma = 0.97$ and the parameters and initial values of the state variables are $(a,b,c,d,q_1,q_2,g,n,r) = (10, 8/3, 28, 2, 1, 2, 1, 1, 1)$, $k \in [0, 80]$ and $y_i(0) = 0.1$, $(i = 1, \cdots, 7)$, respectively.

k	LE_1	LE_2	LE_3	LE_4	LE_5	LE_6	LE_7	Dynamics
2	1.1668	0.8334	0.2198	-0.0027	-0.1090	-0.1703	-14.3459	hyperchaotic of order 3 in [0, 6]
4	1.1671	0.7924	0.3494	0.0054	-0.0462	-0.0934	-14.5719	hyperchaotic of order 4 in [2.1, 59.9]
30	1.1647	0.4176	0.2584	0.0704	0.0050	-0.6818	-13.6251	hyperchaotic of order 5 in [6.1, 40.2]
45	1.1592	0.5083	0.3123	0.0069	-0.0545	-1.1110	-13.2134	hyperchaotic of order 4 in [2.1, 59.9]
73	1.1671	0.0051	-0.3250	-0.3332	-0.5538	-0.5785	-11.7782	hyperchaotic of order 2 in [51.8, 80]
75	1.1673	0.0073	0.0083	-0.2076	-0.4964	-2.4852	-10.4016	hyperchaotic of order 3 in [60, 80]



(a) **(b)**

Figure 4.5: Lyapunov exponents of the fractional system (4.1) with $(a,b,c,d,q_1,q_2,g,n,r) = (10, 8/3, 28, 2, 1, 2, 1, 1, 1)$, $k \in [0, 80]$ and $\gamma = 0.97$.

The following Figures 4.6, 4.7 and 4.8 show the behaviour of fractional system (4.1) while γ and k are fixed as 0.97 and 65, respectively. The other parameters and initial values of the state variables are same as above.

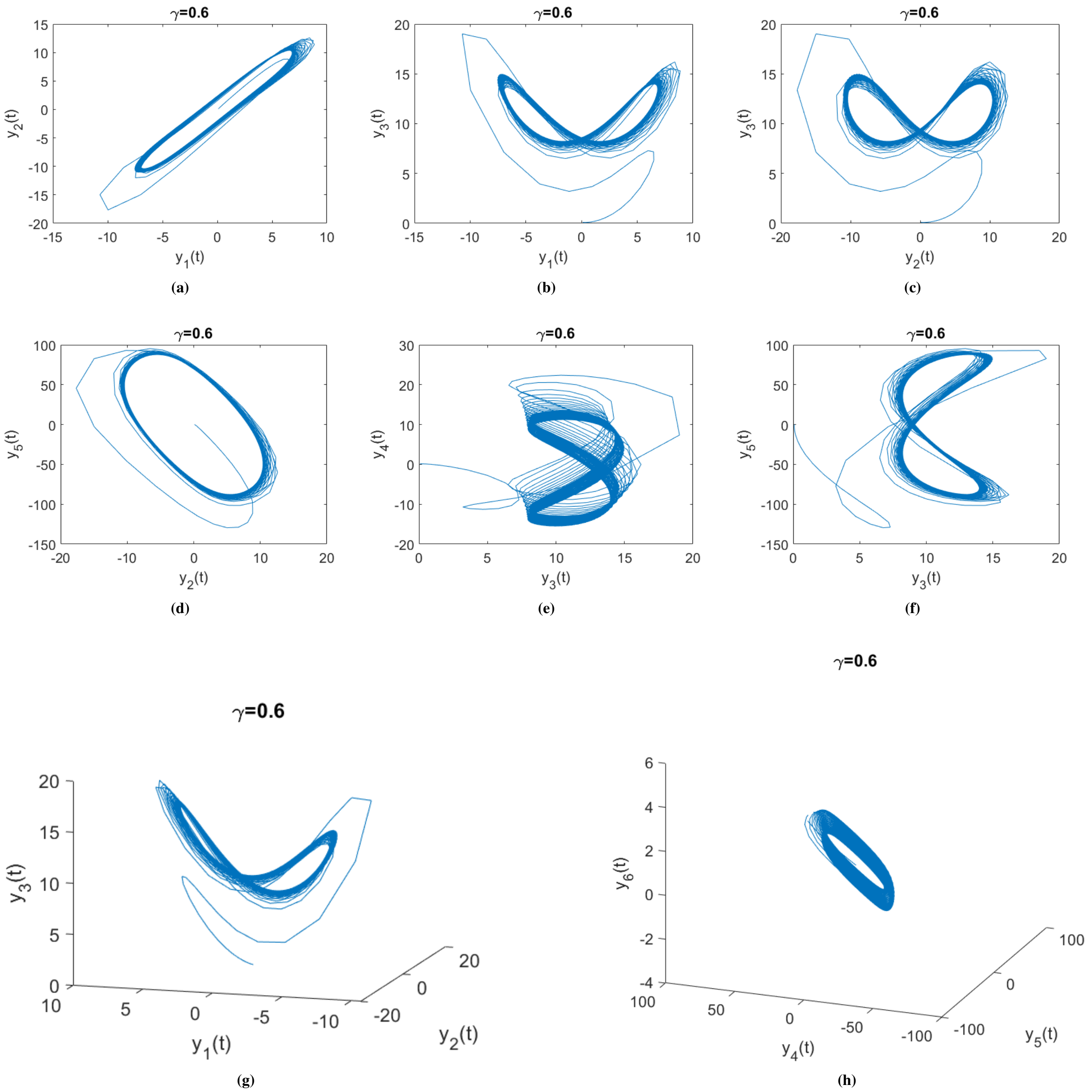


Figure 4.6: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,q_1,q_2,g,n,r) = (10,8/3,28,2,1,2,1,1,1)$ with $k = 65$ and $\gamma = 0.6$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

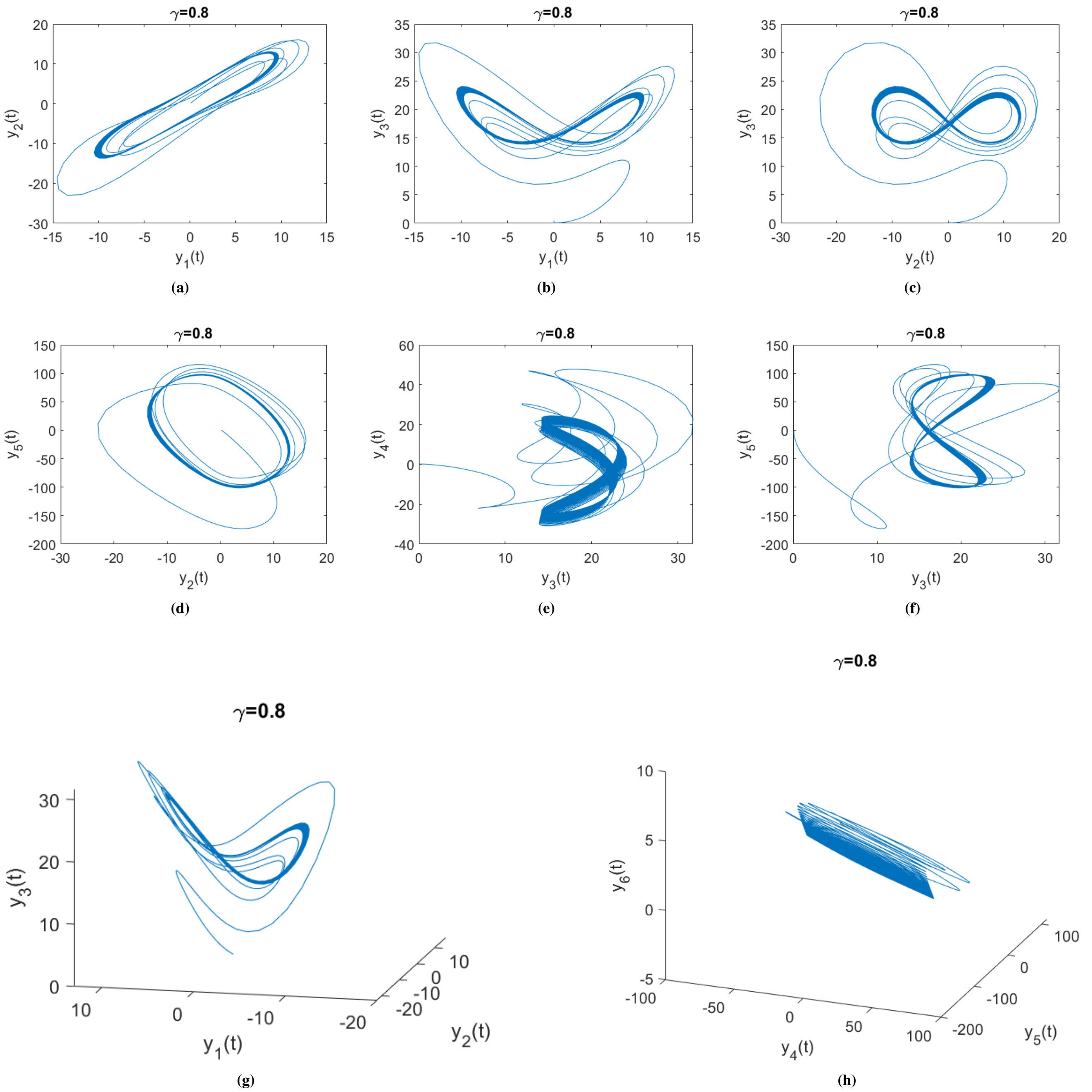


Figure 4.7: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,q_1,q_2,g,n,r) = (10,8/3,28,2,1,2,1,1,1)$ with $k = 65$ and $\gamma = 0.8$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

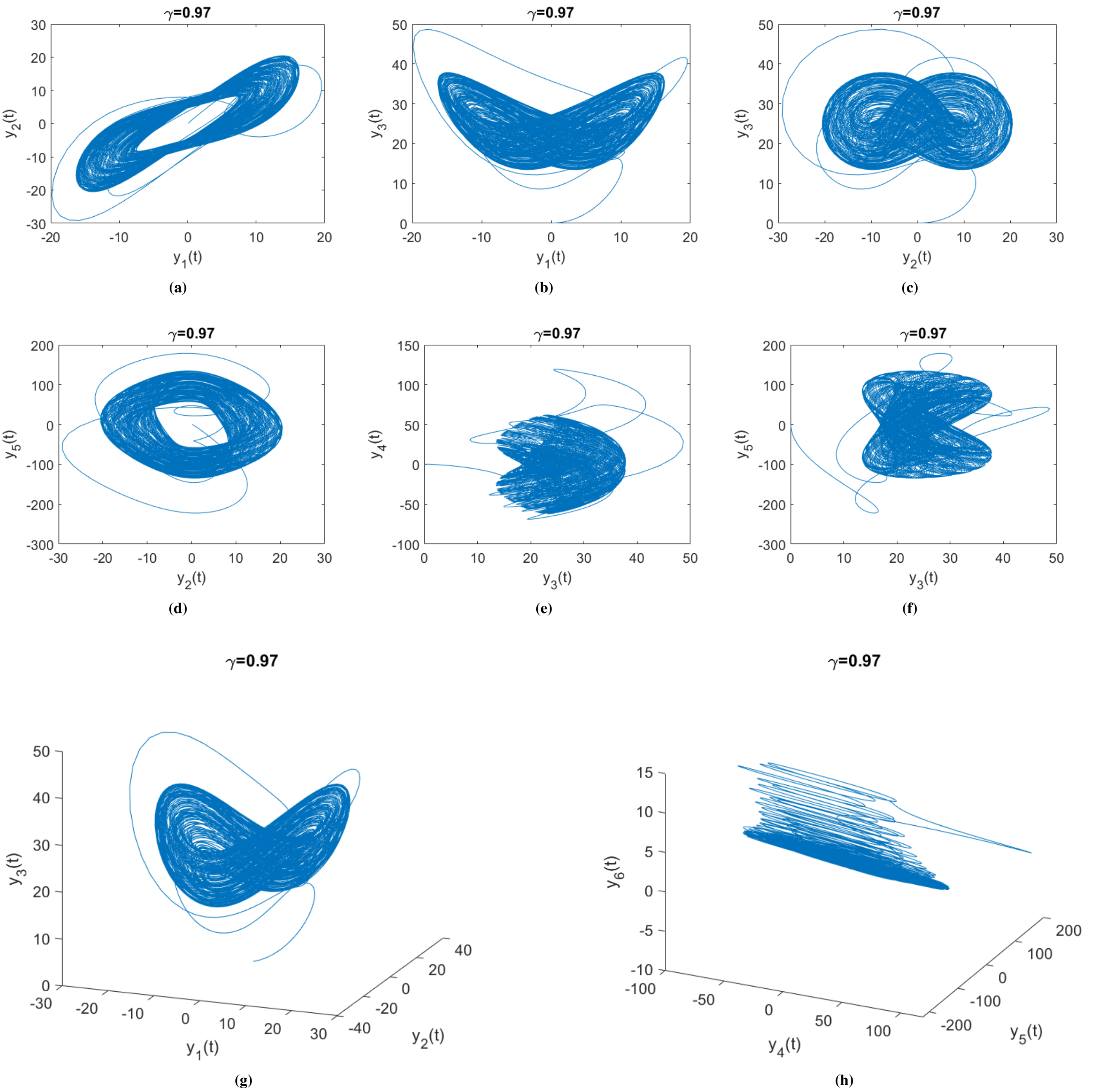
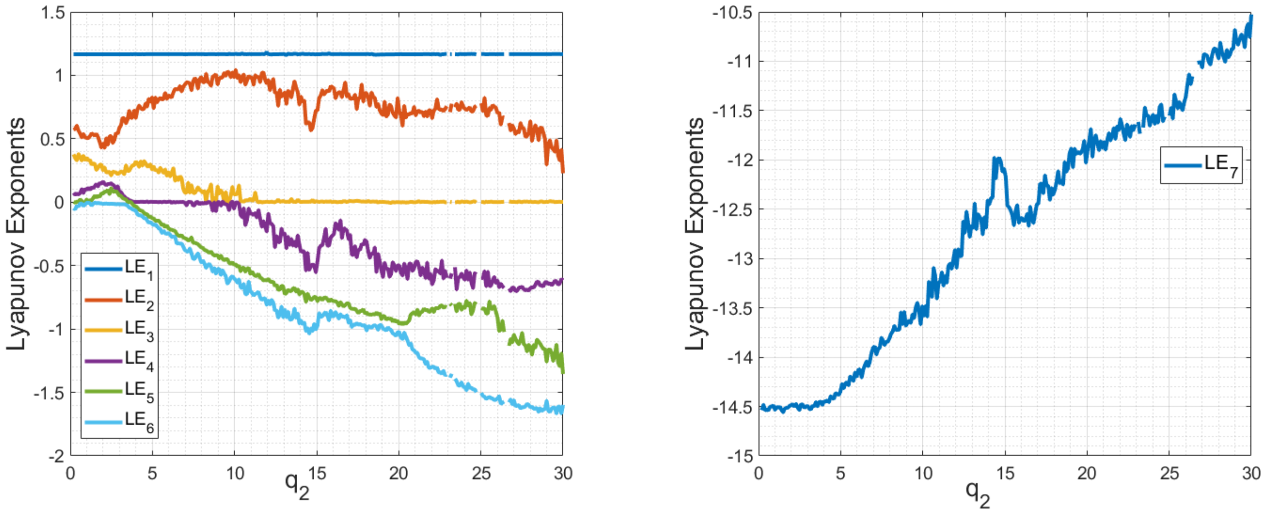


Figure 4.8: Hyperchaotic attractor of system (4.1) for $(a,b,c,d,q_1,q_2,g,n,r) = (10,8/3,28,2,1,2,1,1,1)$ with $k = 65$ and $\gamma = 0.97$ on: (a) y_1-y_2 plane, (b) y_1-y_3 plane, (c) y_2-y_3 plane, (d) y_2-y_5 plane, (e) y_3-y_4 plane, (f) y_3-y_5 plane, (g) y_2-y_1-y_3 space and (h) y_5-y_4-y_6 space.

- While the control parameters q_2 is considered in range between $[0,30]$ as in Yang's paper [44], and the other parameters are fixed: System (4.1) has only hyperchaotic attractors. When $q_2 \in [0,3.7]$, system has five positive Lyapunov exponents and the number of positive exponents decrease while the value of q_2 increase. When q_2 is in range between [11.5, 30], there are two positive Lyapunov exponents. The details are given in Table 3 and the numerical results are illustrated in Figure 4.9.

Table 3: Lyapunov exponents of the fractional system (4.1) while $\gamma = 0.97$ and the parameters and initial values of the state variables are $(a,b,c,d,k,q_1,g,n,r) = (10,8/3,28,2,9.9,1,1,1,1)$, $q_2 \in [0,30]$ and $y_i(0) = 0.1$, $(i = 1,\cdots,7)$, respectively.

q_2	LE_1	LE_2	LE_3	LE_4	LE_5	LE_6	LE_7	Dynamics
1	1.1663	0.5066	0.3516	0.0991	0.0223	-0.0056	-14.5343	hyperchaotic of order 5 in [0, 3.7]
3.6	1.1667	0.7218	0.2489	0.0147	-0.0007	-0.0427	-14.4982	hyperchaotic of order 4 in [0, 9.9]
4.8	1.1668	0.8142	0.2808	-0.0003	-0.1194	-0.1626	-14.3715	hyperchaotic of order 3 in [4.8, 30]
11.5	1.1674	0.9916	-0.0035	-0.0614	-0.5651	-0.7061	-13.2462	hyperchaotic of order 2 in [11.5, 30]



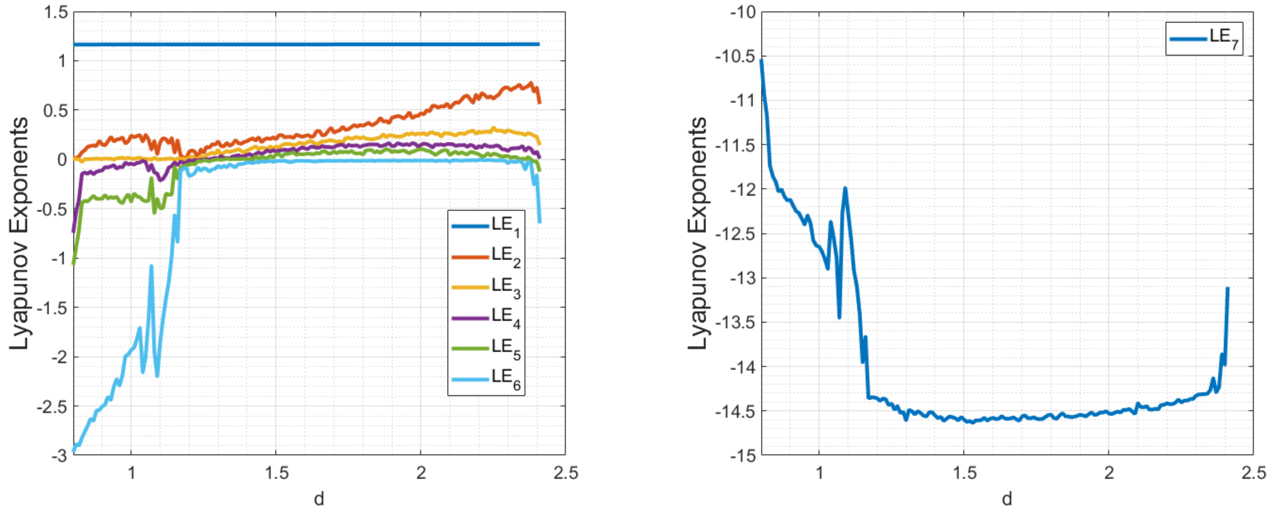
(a) **(b)**

Figure 4.9: Lyapunov exponents of the fractional system (4.1) with $(a,b,c,d,k,q_1,g,n,r) = (10,8/3,28,2,9.9,1,1,1,1)$, $q_2 \in [0,30]$ and $\gamma = 0.97$.

- While the control parameters d is considered in range between $[0.8,2.4]$ as in Yang's paper [44], and the other parameters are fixed: System has only hyperchaotic attractors due to the value of γ is chosen as 0.97. If it is chosen as 1, there also exists chaos orbits when d is around $(0.94,1.07)$ (see [44]). When $d \in [0.8,1.15]$, the system has only two positive Lyapunov exponents, and this number is increasing while the value of d is increasing. The details are given in Table 4, and the numerical results are illustrated in Figure 4.10.

Table 4: Lyapunov exponents of the fractional system (4.1) while $\gamma = 0.97$ and the parameters and initial values of the state variables are $(a,b,c,k,q_1,q_2,g,n,r) = (10,8/3,28,9.9,1,2,1,1,1)$, $d \in [0.8,2.4]$ and $y_i(0) = 0.1$, $(i = 1,\cdots,7)$, respectively.

d	LE_1	LE_2	LE_3	LE_4	LE_5	LE_6	LE_7	Dynamics
0.83	1.1645	0.0726	-0.0269	-0.1472	-0.4401	-2.8235	-11.7376	hyperchaotic of order 2 in [0.83, 1.15]
1.16	1.1651	0.1948	0.0078	-0.0598	-0.1975	-0.8395	-13.6618	hyperchaotic of order 3 in [0.8, 1.27]
1.28	1.1652	0.1539	0.0698	0.0098	-0.0089	-0.1011	-14.5231	hyperchaotic of order 4 in [1.28, 1.31]
1.32	1.1652	0.1322	0.0686	0.0133	0.0003	-0.0642	-14.5048	hyperchaotic of order 5 in [1.32, 2.37]
2.38	1.1669	0.7777	0.2549	0.1061	-0.0006	-0.0469	-14.2300	hyperchaotic of order 4 in [2.38, 2.41]



(a) (b)

Figure 4.10: Lyapunov exponents of the fractional system (4.1) with $(a,b,c,k,q_1,q_2,g,n,r) = (10,8/3,28,9.9,1,2,1,1,1)$, $d \in [0.8,2.4]$ and $\gamma = 0.97$.

5. Conclusion

In this paper, the fractional-order version of the new 7D Lorenz-like system is proposed and the dynamics of such a system is investigated in detail with numerical simulations. Since there is no appreciable result for order less than 0.6 in fractional system (4), the numerical results are obtained for order in range between 0.6 to 1. It is obtained that the newly proposed fractional-order 7D Lorenz-like system shows quite a variety of dynamic behaviour, including chaotic and hyperchaotic motions, which are verified based on Lyapunov exponents and phase portraits. Moreover, the numerical analysis of Lyapunov exponents for the fractional order system (4) verified the existence of hyperchaos for less than five orders. When the parameters are fixed, the lowest existence condition order of chaotic behaviour for the fractional-order system (4) is obtained approximately as 0.8796 by using fractional order stability theory. The fractional stability and numerical investigations of fractional system (4) verified that limited number of parameter values exist for chaotic phenomena where the majority of them exist for hyperchaotic behaviour.

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