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SAKARYA ÜNİVERSİTESİ

A New Approach to $k-$Jacobsthal Lucas Sequences

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Abstract

In this study, Catalan transformation $CS_{k,n}$ of $k-$Jacobsthal-Lucas sequences $S_{k,n}$ is defined. In addition, the transformation of $CS_{k,n}$ is written as the product of the Catalan matrix C which is the lower triangular matrix and the S_k matrix of type $n\ x\ 1$, and the Hankel transformations of some $k-$Jacobsthal-Lucas numbers is found.

Keywords: $k-$Pell sequences, $k-$Lucas sequence, $k-$Fibonacci sequence, Catalan Transform, Hankel Transform

1. INTRODUCTION

For any integer n, it is called a generalized Fibonacci-type sequence in the following form $G(n+1)=aG(n)+bG(n-1)$, $G(0)=m, G(1)=t$,where m,t,a and b are any complex numbers. There is an extensive work in the literature concerning Fibonacci-type sequences and their applications in modern science (see e.g.[1-8]). The known Jacobstahal-Lucas numbers have some applications in many branches of mathematics such as group theory, calculus, applied mathematics, linear algebra, etc. [9-12]. There exist generalizations of the Jacobsthal-Lucas numbers. This study is an extension of the papers [13-15].

In this paper, we put in for Catalan transform to the $k-$Jacobsthal-Lucas sequence and present application of the Catalan transform of the $k-$Jacobsthal-Lucas sequence. In section 2, we introduce some fundamental definitions of $k-$Jacobsthal-Lucas sequences and some basic theorems. In Theorem 2.1, we obtain Binet's formula of $k-$Jacobsthal-Lucas sequences and in Theorem 2.2, we give the relationship between positive and negative terms of $k-$Jacobsthal-Lucas numbers. In Theorem 2.3, we get Cassini identity for this sequence. In section 3, Catalan transform of $k-$Jacobsthal-Lucas sequence is given. Hankel transform of Catalan transformation of $k-$Jacobsthal-Lucas sequence is obtained in section 4.

2. $k-$JACOBSTHAL-LUCAS SEQUENCES

Let k be any positive real number. Then the k-Jacobsthal-Lucas sequences is defined

$S_{k,n+1} = S_{k,n} + 2k.S_{k,n-1}$ for $n \geq 1$ with the initial values $S_{k,0} = 2$ and $S_{k,1} = 1$.

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When k=1, the known Jacobsthal-Lucas sequences is obtained. Characteristic equation of the sequence is

$r^2 - r - 2k = 0.$

Its characteristic roots are

$r_1 = \frac{1+\sqrt{1+8k}}{2}$

and

$r_2 = \frac{1-\sqrt{1+8k}}{2}$

Characteristic roots verify the properties

$r_1 + r_2 = 1, \quad r_1.r_2 = -2k$

$r_1 - r_2 = \sqrt{1+8k}$

Binet's formula for $S_{k,n}$ is

$S_{k,n} = r_1^n + r_2^n$.

k−Jacobsthal-Lucas sequences as numbered;

$S_{k,n+1} = S_{k,n} + 2k.S_{k,n-1}$

$S_{k,0} = 2$

$S_{k,1} = 1$

$S_{k,2} = S_{k,1} + 2k.S_{k,0} = 4k + 1$

$S_{k,3} = S_{k,2} + 2k.S_{k,1}$

$= 4k+1+2k.1$

$= 6k + 1.$

$S_{k,4} = S_{k,3} + 2k.S_{k,2}$

$= 6k+1+2k.(4k+1)$

$= 8k^2 + 8k + 1.$

$S_{k,5} = S_{k,4} + 2k.S_{k,3}$

$= 8k^2 + 8k + 1 + 2k.(6k+1)$

$= 20k^2 + 10k + 1.$

$S_{k,6} = S_{k,5} + 2k.S_{k,4}$

$= (20k^2 + 10k + 1) + 2k.(8k^2 + 8k + 1)$

$= 16k^3 + 36k^2 + 12k + 1.$

$S_{k,7} = S_{k,6} + 2kS_{k,5}$

$= 16k^3 + 36k^2 + 12k + 1 + 2k.(20k^2 + 10k + 1)$

$= 56k^3 + 56k^2 + 14k + 1.$

Theorem 2.1. Binet's formula of k−Jacobsthal-Lucas sequences are obtained from the relations.

$S_{k,n} = r_1{}^n + r_2{}^n$

Proof.

The solutions of the characteristic equation are

$r^2 - r - 2k = 0,$

$r_1 = \frac{1+\sqrt{1+8k}}{2}$ and $r_2 = \frac{1-\sqrt{1+8k}}{2}$.

$S_{k,n} = c.r_1{}^n + d.r_2{}^n$

for $n = 0$, it is $S_{k,0} = 2$ and for $n=1$, it is $S_{k,1} = 1$. Thus $c = 1$ and $d = 1$ are obtained. So, the proof is completed.

Theorem 2.2. For the k−Jacobsthal-Lucas numbers, the following identity holds for:

$$S_{k,n} = (-2k)^n.S_{k,-n} .$$

Proof. By virtue of Binet's formula, we find that

$S_{k,-n} = r_1{}^{-n} + r_2{}^{-n}$

$= \frac{1}{r_1{}^n} + \frac{1}{r_2{}^n}$

$= \frac{r_1{}^n + r_2{}^n}{(r_1.r_2)^n} \quad ((r_1.r_2)^n = (-2k)^n)$

$= \frac{S_{k,n}}{(-2k)^n}$.

$S_{k,n} = (-2k)^n.S_{k,-n}.$

Theorem 2.3. (Cassini identity) For the $k-$Jacobsthal-Lucas numbers, the following equailty holds:

$$S_{k,n+1}.S_{k,n-1}-S_{k,n}^2=(-2k)^{n-1}.(8k+1)$$

Proof. By using the Binet's formula, we have $S_{k,n+1}.S_{k,n-1}-S_{k,n}^2=$

$(a^{n+1}+b^{n+1}).(a^{n-1}+b^{n-1})-(a^n+b^n)^2$

$=a^{2n}+a^{n+1}.b^{n-1}+a^{n-1}.b^{n+1}+b^{2n}-a^{2n}-2a^nb^n-b^{2n}$

$=(ab)^n\frac{a}{b}+(ab)^n\frac{b}{a}-2.(ab)^n$

$=(ab)^n[\frac{a}{b}+\frac{b}{a}-2]$

$=(-2k)^n(\frac{4k+1}{-2k}-2)$

$=(-2k)^{n-1}.(8k+1).$

3. CATALAN NUMBERS

For n ≥0, the n^{th} Catalan number [13] is defined as follows

$C_n=\frac{1}{n+1}$(2n,n) or $C_n=\frac{(2n)!}{(n+1)!.n!}$

Its generating function is given by

C(x) = $\frac{1-\sqrt{1-4x}}{2x}$.

The first Catalan numbers are {1,1,2,5,14,132,429,1430,4862, …}.

3.1. Catalan Transform of the $k-$Jacobsthal-Lucas sequences

We define the Catalan transform of the $k-$Jacobsthal-Lucas sequences { $S_{k,n}$} as

$CS_{k,n}=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}S_{k,i},\ n\geq 1$ with

$CS_{k,0}=0.$

We can give the first few of Catalan transform of the first $k-$Jacobsthal-Lucas numbers. These are the polynomials in k:

$$CS_{k,1}=\sum_{i=0}^{1}\frac{i}{2-i}\binom{2-i}{1-i}S_{k,i}$$

$=1.S_{k,1}=1,$

$$CS_{k,2}=\sum_{i=0}^{2}\frac{i}{4-i}\binom{4-i}{2-i}S_{k,i}$$

$=\frac{1}{3}\binom{3}{1}S_{k,1}+\binom{2}{0}S_{k,2}$

$=4k+2.$

$$CS_{k,3}=\sum_{i=0}^{3}\frac{i}{6-i}\binom{6-i}{3-i}S_{k,i}$$

$=\frac{1}{5}\binom{5}{2}S_{k,1}+\frac{2}{4}\binom{4}{1}S_{k,2}+\frac{3}{3}\binom{3}{0}S_{k,3}$

$=\frac{1}{5}.10.1+\frac{2}{4}.4.(4k+1)+\frac{3}{3}.1.(6k+1)$

$=14k+5.$

$$CS_{k,4}=\sum_{i=0}^{4}\frac{i}{8-i}\binom{8-i}{4-i}S_{k,i}$$

$=\frac{1}{7}\binom{7}{3}S_{k,1}+\frac{2}{6}\binom{6}{2}S_{k,2}+\frac{3}{5}\binom{5}{3}S_{k,3}+\frac{4}{4}\binom{4}{0}S_{k,4}$

$=\frac{1}{7}.35.1+\frac{2}{6}.15.(4k+1)+\frac{3}{5}.5.(6k+1)+\frac{4}{4}.1.(8k^2+8k+1)$

$=8k^2+46k+14.$

$$CS_{k,5}=\sum_{i=0}^{5}\frac{i}{10-i}\binom{10-i}{5-i}S_{k,i}$$

$=\frac{1}{9}\binom{9}{4}S_{k,1}+\frac{2}{8}\binom{8}{3}S_{k,2}+\frac{3}{7}\binom{7}{2}S_{k,3}+\frac{4}{6}\binom{6}{1}S_{k,4}+\frac{5}{5}\binom{5}{0}S_{k,5}$

$=\frac{1}{9}.126.1+\frac{2}{8}.56.(4k+1)+\frac{3}{7}.21.(6k+1)+\frac{4}{6}.6.(8k^2+8k+1)+\frac{5}{5}.1.(20k^2+10k+1)$

$=52k^2+152k+42.$

$$CS_{k,6} = \sum_{i=0}^{6} \frac{i}{12-i}\binom{12-i}{6-i} S_{k,i}$$

$$= \frac{1}{11}\binom{11}{5} S_{k,1} + \frac{2}{10}\binom{10}{4} S_{k,2} + \frac{3}{9}\binom{9}{3} S_{k,3} + \frac{4}{8}\binom{8}{2} S_{k,4} + \frac{5}{7}\binom{7}{1} S_{k,5} + \frac{6}{6}\binom{6}{0} S_{k,6}$$

$$= \frac{1}{11}.462.1 + \frac{2}{10}.210.(4k+1) + \frac{3}{9}.84.(6k+1) + \frac{4}{8}.28.(8k^2+8k+1) + \frac{5}{7}.7.(20k^2+10k+1) + \frac{6}{6}.1.(16k^3+56k^2+14k+1)$$

$$= 16k^3+268k^2+512k+132.$$

We can show $\{S_{k,n}\}$ as the n x 1 matrix S_k and the product of the lower triangular matrix C as

$$\begin{bmatrix} CS_{k,1} \\ CS_{k,2} \\ CS_{k,3} \\ CS_{k,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} S_{k,1} \\ S_{k,2} \\ S_{k,3} \\ S_{k,4} \\ \vdots \end{bmatrix}$$

So, we have

$$\begin{bmatrix} 1 \\ 4k+2 \\ 14k+5 \\ 8k^2+46k+14 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \cdots \\ 1 & 1 & & & \cdots \\ 2 & 2 & 1 & & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 4k+1 \\ 6k+1 \\ 8k^2+8k+1 \\ \vdots \end{bmatrix}.$$

4. HANKEL DETERMINANT OF THE CATALAN k−JACOBSTHAL-LUCAS SEQUENCES

The Hankel matrix H of the integer sequence $A = \{a_0, a_1, a_2 \dots\}$ is the infinite matrix

$$H_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

with elements $h_{i,j} = a_{i+j-1}$. The Hankel matrix H_n of order n of A is the upper-left nxn submatrix of H, and h_n, the Hankel determinant of order n of A, is the determinant of the corresponding Hankel matrix of order n, h_n = det (H_n) [14,15].

In addition, by applying Hankel determinant return to the CS_k polynomials we obtain;

$HCS_1 = Det[1] = 1$

$$HCS_2 = \begin{vmatrix} 1 & 4k+2 \\ 4k+2 & 6k+1 \end{vmatrix}$$

$= 6k+1-(16k^2+16k+4)$

$= -16k^2-10k-3.$

$HCS_3=$

$$\begin{vmatrix} 1 & 4k+2 & 14k+5 \\ 4k+2 & 14k+5 & 8k^2+16k+14 \\ 14k+5 & 8k^2+16k+14 & 52k^2+152k+42 \end{vmatrix}$$

$=-384k^4-3232k^3+2024k^2+2282k-255.$

5. CONCLUSION

We introduced Catalan transformation of k−Jacobsthal-Lucas sequences and Hankel determinant of the Catalan k−Jacobsthal-Lucas sequences.

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REFERENCES

[1] A. F. Horadam, ‘’A generalized Fibonacci sequence’’. Amer. Math. Monthly, vol. 68, no. 5, pp. 455–459, 1961.

[2] N. Anuradha, ‘’Number of points on certain hyperelliptic curves defined over finite

fields'' Finite Fields and Their Applications, vol. 14, pp. 314-328, 2008.

[3] H. Bruhn, L. Gellert, and J. Günther, ''Jacobsthal numbers in generalized Petersen graphs'' Electronic Notes in Discrete Mathematics, vol. 49, pp. 465-472, 2015.

[4] E. Özkan, İ. Altun, A. Göçer, "On Relationship Among A New Family of k−Fibonacci, k−Lucas Numbers, Fibonacci And Lucas Numbers", Chiang Mai Journal of Scıence, vol. 44, pp. 1744-1750, 2017.

[5] E. Özkan, M. Taştan, A. Aydoğdu, ''3-Fibonacci Polynomials in The Family of Fibonacci Numbers'', Erzincan University Journal of the Institute of Science and Technology, vol. 12, no. 2, pp. 926-933, 2019.

[6] S. Falc´on and A. Plaza,2007. ''On the Fibonacci k−numbers'', Chaos Solitons Fractals, vol. 32, no. 5, pp. 1615–1624, 2007.

[7] C. Kizilates, "On the Quadra Lucas-Jacobsthal Numbers", Karaelmas Science and Engineering Journal, vol. 7, no. 2, pp. 619-621, 2017.

[8] C. Kizilates, N. Tuğlu, and B. Çekim, Binomial Transforms Of Quadrapell Sequences And Quadrapell Matrix Sequences, Journal of Arts and Scince, vol. 1, no. 38, pp. 69-80, 2017.

[9] T. Koshy,''Fibonacci and Lucas Numbers with Applications''. New York, NY: Wiley-Interscience Publication. 2001.

[10] M. Taştan, and E. Özkan, ''Catalan Transform of The k−Jacobsthal Sequence'', Electronic Journal of Mathematical Analysis and Applications, vol. 8, no. 2, pp. 70-74, 2020.

[11] A. Boussayoud, M. Kerada, M. and N. Harrouche, ''On the k -Lucas Numbers and Lucas Polynomials'', Turkish Journal of Analysis and Number Theory, vol. 5, no. 4, pp. 121-125, 2017.

[12] P. Barry, ''A Catalan transform and related transformations on integer sequences''. J.Integer Seq., vol. 8, no. 4, pp. 1-24, 2005.

[13] Falcon, S. (2013). ''Catalan Transform of the k−Fibonacci sequence''. Commun. Korean Math. Soc., vol. 28, no. 4, pp. 827–832, 2013.

[14] J. W. Layman, ''The Hankel transform and some of its properties''. J. Integer Seq., vol. 4, no. 1, pp. 1-11, 2001.

[15] P. M. Rajkovi´c, M. D. Petkovi´c,. and P. Barry, ''The Hankel transform of the sum of consecutive generalized Catalan numbers", Integral Transforms Spec. Funct., vol. 18, no. 4, pp. 285–296, 2007.