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A Study on Chen-like Inequalities for Half Lightlike Submanifolds of a Lorentzian Manifold Endowed with Semi-Symmetric Metric Connection

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ABSTRACT: In this paper, Chen-like inequalities of a half lightlike submanifolds of a real space form $\tilde{N}(c)$ with constant sectional curvature c, equipped with semi-symmetric metric connection are established and some important characterization theorems for such submanifolds are proved using these inequalities.

Keywords: Chen inequality; Half lightlike submanifold; Lorentzian manifold; Semi-symmetric metric connection.

1. INTRODUCTION

Lightlike submanifolds theory is an important field of study as it is the mathematical modeling of black holes. The lightlike submanifolds theory was first studied by Kupeli [1] and Duggal-Bejancu [2]. Later many geometers have worked in this area [3-5].

The connection on the manifold is a very important concept geometrically, as it enables us to operate algebraically on the manifold. One of the different connections that can be defined on manifolds is the semi-symmetric metric connection. A semi-symmetric metric connection defined by Hayden in [6] was studied by Yano in [7]. The first studies on Riemannian, semi-Riemannian and Lorentzian manifolds with semi-symmetric metric connection belong to Z. Nakao [8], Duggal and Sharma [9] and Konar and Biswas [10], respectively. Yaşar, Çöken and Yücesan introduced lightlike hypersurfaces of a semi-Riemannian manifold with a semi-symmetric metric connection in [11]. Later, Akyol, Vanlı and Fernandez studied curvature properties of a semi-symmetric metric connection on S manifolds in [12].

The Chen inequalities established by Chen in [13] are inequalities that extrinsic and intrinsic curvatures of the manifold and are very useful in characterizing the manifold. Then, various non-degenerate manifolds have been studied by establishing Chen inequalities on them [14-26]. However, Chen inequalities in lightlike geometry were first studied by Gülbahar, Kılıç and Keleş in [27] and [28]. Then Poyraz, Doğan and Yaşar introduced inequalities on the lightlike hypersurface of a Lorentzian manifold with a semi-symmetric metric connection in [29].

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Finally, some inequalities of screen conformal half lightlike submanifolds were derived by Gülbahar and Kılıç in [30].

In this paper, Chen-like inequalities of a half lightlike submanifolds of a real space form $\tilde{N}(c)$ with constant sectional curvature c, equipped with semi-symmetric metric connection are established and some important characterization theorems for such submanifolds are proved using these inequalities.

2. MATERIAL AND METHODS

Let $\tilde{\nabla}$ be a connection on a semi-Riemannian manifold $(\tilde{N},\tilde{g})$. $\tilde{\nabla}$ is called a semi-symmetric metric connection if it is metric, i.e., $(\tilde{\nabla},\tilde{g})=0$ and for $\forall\tilde{U},\tilde{V}\in\Gamma(T\tilde{N})$, its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(\tilde{U},\tilde{V})=\tilde{\pi}(\tilde{V})\tilde{U}-\tilde{\pi}(\tilde{U})\tilde{V}, \tag{2.1}$$

where $\tilde{P}$ is a vector field on $\tilde{N}$, which called the torsion vector field and $\tilde{\pi}$ is a 1-form defined by

$$\tilde{g}(\tilde{P},\tilde{U})=\tilde{\pi}(\tilde{U}).$$

Now, suppose that the semi-Riemannian manifold $\tilde{N}$ admits a semi-symmetric metric connection which is given by

$$\tilde{\nabla}_{\tilde{U}}\tilde{V}=\overset{\circ}{\tilde{\nabla}}_{\tilde{U}}\tilde{V}+\tilde{\pi}(\tilde{V})\tilde{U}-\tilde{g}(\tilde{U},\tilde{V})\tilde{P} \tag{2.2}$$

for arbitrary vector fields $\tilde{U}$ and $\tilde{V}$ of $\tilde{N}$, where $\overset{\circ}{\tilde{\nabla}}$ denotes the Levi-Civita connection concerning the semi-Riemannian metric $\tilde{g}$ [7].

Let $(\tilde{N},\tilde{g})$ be a $(m+3)-$dimensional semi-Riemannian manifold of the index $q\geq 1$ and N be a lightlike submanifold of codimension 2 of $\tilde{N}$. Then the radical distribution $Rad(TN)=TN\cap TN^{\perp}$ of N is a vector subbundle of the tangent bundle TN and the normal bundle $TN^{\perp}$ of rank 1 or 2. If $rank(Rad(TN))=1$, then N is called half lightlike submanifold of $\tilde{N}$. Then there exist complementary non-degenerate distributions $S(TN)$ and $S(TN^{\perp})$ of $Rad(TN)$ in TN and $TN^{\perp}$, which are called the screen and the screen transversal distribution on N respectively. Thus we have

$$TN=Rad(TN)\perp S(TN),\quad TN^{\perp}=Rad(TN)\perp S(TN^{\perp}). \tag{2.3}$$

Consider the orthogonal complementary distribution $S(TN)^{\perp}$ to $S(TN)$ in $T\tilde{N}$. Then ξ and Z belong to $\Gamma(S(TN)^{\perp})$. Thus we obtain

$$S(TN)^{\perp}=S(TN^{\perp})\perp S(TN^{\perp})^{\perp}, \tag{2.4}$$

where $S(TN^{\perp})^{\perp}$ is the orthogonal complementary to $S(TN^{\perp})$ in $S(TN)^{\perp}$. For any null section

$\xi \in Rad(TN)$ on a coordinate neighborhood $U \subset N$, there exists a uniquely determined null vector field $L \in \Gamma(ltr(TN))$ satisfying

$$\tilde{g}(L,\xi)=1, \quad \tilde{g}(L,L)=\tilde{g}(L,U)=\tilde{g}(L,Z)=0, \ \forall U \in \Gamma(TN). \tag{2.5}$$

We call L, $ltr(TN)$ and $tr(TN)=S(TN^{\perp}) \perp ltr(TN)$ the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of N concerning $S(TN)$, respectively. Hence we have

$$\begin{aligned} T\tilde{N} &= TN \oplus tr(TN) \\ &= \{Rad(TN) \oplus ltr(TN)\} \perp S(TN) \perp S(TN^{\perp}). \end{aligned} \tag{2.6}$$

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{N}$. Using (2.6) we define the projection morphism $Q : \Gamma(TN) \to \Gamma(S(TN))$. Hence we derive

$$\tilde{\nabla}_U V = \nabla_U V + B(U,V)L + D(U,V)Z, \tag{2.7}$$
$$\tilde{\nabla}_U F = -A_F U + \nabla^t_U F, \tag{2.8}$$
$$\tilde{\nabla}_U L = -A_L U + \tau(U)L + \rho(U)Z, \tag{2.9}$$
$$\tilde{\nabla}_U Z = -A_Z U + \psi(U)L, \tag{2.10}$$
$$\nabla_U QV = \nabla^*_U QV + C(U,QV)\xi, \tag{2.11}$$
$$\nabla_U \xi = -A^*_\xi U - \tau(U)\xi, \tag{2.12}$$

for any $U,V \in \Gamma(TN)$, $\xi \in \Gamma(Rad(TN))$, $F \in \Gamma(tr(TN))$, $L \in \Gamma(ltr(TN))$ and $Z \in \Gamma(S(TN^{\perp}))$. Then ∇ and ∇^* are called induced linear connections on TN and $S(TN)$ respectively, B and D are called the local second fundamental forms of N, C is called the local second fundamental form on $S(TN)$. A_L, A^*_ξ and A_Z are called linear operators on TN. Also τ, ρ and ψ are called $1-$forms on TN.

The induced connection ∇ of N is not metric and satisfies

$$(\nabla_U g)(V,W) = B(U,V)\eta(W) + B(U,W)\eta(V), \tag{2.13}$$

for any $U,V,W \in \Gamma(TN)$, where η is a $1-$form defined by

$$\eta(U) = \tilde{g}(U,L), \ \forall U \in \Gamma(TN). \tag{2.14}$$

But the connection ∇^* is metric. Using (2.1) and (2.13), we show that

$$T(U,V) = \pi(V)U - \pi(U)V \tag{2.15}$$

and B and D are symmetric, where T is the torsion tensor concerning ∇. From (2.13) and (2.15), we show that the induced connection ∇ of N is a semi-symmetric non-metric connection of N. From the facts $B(U,V)=\tilde{g}(\tilde{\nabla}_U V,\xi)$ and $D(U,V)=\tilde{g}(\tilde{\nabla}_U V,Z)$, we know that B and D are independent of the choice of $S(TN)$ and satisfy

$$B(U,\xi)=0,\ D(U,\xi)=-\varepsilon\psi(U),\ \forall U\in\Gamma(TN). \tag{2.16}$$

Therefore one obtains

$$B(U,V)=g(A^{*}_{\xi}U,V),\quad g(A^{*}_{\xi}U,L)=0, \tag{2.17}$$

$$C(U,QV)=g(A_{L}U,QV),\quad g(A_{L}U,L)=0, \tag{2.18}$$

$$D(U,QV)=g(A_{Z}U,QV),\quad g(A_{Z}U,L)=\rho(U), \tag{2.19}$$

$$D(U,V)=g(A_{Z}U,V)-\psi(U)\eta(V),\quad \forall U,V\in\Gamma(TN). \tag{2.20}$$

By (2.17) and (2.18), A^{*}_{ξ} and A_{L} are $\Gamma(S(TN))-$valued shape operators related to B and D, respectively and $A^{*}_{\xi}\xi=0.$

Using (2.7), (2.12) and (2.16), one derives

$$\tilde{\nabla}_{U}\xi=-A^{*}_{\xi}U-\tau(U)\xi-\varepsilon\psi(U)Z, \tag{2.21}$$

for any $U\in\Gamma(TN)$.

Definition 1. *A half lightlike submanifold* (N,g) *of a semi-Riemannian manifold* $(\tilde{N},\tilde{g})$ *is said to be irrotational [31] if* $\tilde{\nabla}_{U}\xi\in\Gamma(TN)$ *for any* $U\in\Gamma(TN)$. *From (2.16) and (2.21), the definition of irrotational is equivalent to the condition* $\psi(U)=0$, *that is,* $D(U,\xi)=0$ *for any* $U\in\Gamma(TN)$.

Definition 2. *A half lightlike submanifold* (N,g) *of a semi-Riemannian manifold* $(\tilde{N},\tilde{g})$ *is called umbilical* N, *if there is a smooth vector field* $H\in\Gamma(tr(TN))$ *on any coordinate neighborhood* U *such that*

$$h(U,V)=Hg(U,V) \tag{2.22}$$

for any $U,V\in\Gamma(TN)$, *where*

$$h(U,V)=B(U,V)L+D(U,V)Z \tag{2.23}$$

is the global second fundamental form tensor of N. *In the case* $H=0$ *on* U, *we say that* N *is totally geodesic [32].*

It is easy to see that N is totally umbilical iff, on each coordinate neighborhood U, there exist smooth vector functions λ and δ such that

$$B(U,V)=\lambda g(U,V),\quad D(U,V)=\delta g(U,V), \tag{2.24}$$

for any $U,V\in\Gamma(TN)$.

Definition 3. *We say that the screen distribution* $S(TN)$ *of* N *is totally umbilical [32] in* N *if there is a smooth function* γ *on any coordinate neighborhood* $U\subset N$ *such that*

$$D(U,QV)=\gamma g(U,V), \tag{2.25}$$

for any $U,V \in \Gamma(TN)$. *If* $\gamma = 0$ *on* U, *then we say that* $S(TN)$ *is totally geodesic in* N.

Furthermore, $(N, g, S(TN))$ is called minimal if it is irrotational and

$$trace_{S(TN)} h = 0, \tag{2.26}$$

where $trace_{S(TN)}$ denotes the trace restricted to $S(TN)$ concerning the degenerate metric g [33].

Let $(N, g, S(TN))$ be a $(m+1)-$dimensional half-lightlike submanifold and $\{e_1,...,e_m\}$ be an orthonormal basis of $\Gamma(S(TN))$. Let us consider

$$\mu_1 = \frac{1}{m}\sum_{j=1}^{m} B(e_j, e_j), \; \mu_2 = \frac{1}{m}\sum_{j=1}^{m} D(e_j, e_j). \tag{2.27}$$

Then it is clear from (2.26) and (2.27) that ∇ is minimal if and only if $\mu_1 = \mu_2 = 0$.

A lightlike hypersurface (N, g) of a semi-Riemannian manifold $(\tilde{N}, \tilde{g})$ is called *screen locally conformal* if the shape operators A_L and A^*_ξ of N and $S(TN)$, respectively, are related by

$$A_L = \phi A^*_\xi, \tag{2.28}$$

i.e.,

$$C(U, PV) = \phi B(U,V), \quad \forall U,V \in \Gamma(TN), \tag{2.29}$$

where ϕ is a non-vanishing smooth function on a neighborhood U in N. In particular, if ϕ is a non-zero constant, N is called screen homothetic [34].

We denote by $\tilde{R}$, R and R^* the curvature tensors of the semi-symmetric metric connection $\tilde{\nabla}$ on $\tilde{N}$, the induced connection ∇ on N and the induced connection ∇^* on $S(TN)$, respectively. Using the Gauss-Weingarten equations (2.7)-(2.12) for N and $S(TN)$, we obtain the Gauss-Codazzi equations for N and $S(TN)$:

$$\begin{aligned} \tilde{g}(\tilde{R}(U,V)W, QT) = g(R(U,V)W, QT) \\ +B(U,W)C(V,QT) - B(V,W)C(U,QT) \\ +D(U,W)D(V,QT) - D(V,W)D(U,QT), \end{aligned} \tag{2.30}$$

$$\begin{aligned} \tilde{g}(\tilde{R}(U,V)W, \xi) = (\nabla_U B)(V,W) - (\nabla_V B)(U,W) \\ +[\tau(U) - \pi(U)]B(V,W) - [\tau(V) - \pi(V)]B(U,W) \\ +\psi(U)D(V,W) - \psi(V)D(U,W), \end{aligned} \tag{2.31}$$

$$\begin{aligned} \tilde{g}(\tilde{R}(U,V)W, L) = g(R(U,V)W, L) \\ +\rho(V)D(U,W) - \rho(U)D(V,W), \end{aligned} \tag{2.32}$$

$$\begin{aligned}\tilde{g}(\tilde{R}(U,V)\xi,L) &= g(A^*_\xi U, A_L V) - g(A^*_\xi V, A_L U) \\ &\quad -2d\tau(U,V) + \rho(U)\psi(V) - \rho(V)\psi(U),\end{aligned} \tag{2.33}$$

$$\begin{aligned}g(R(U,V)QW,QT) &= g(R^*(U,V)W,QT) + B(V,QT)C(U,QW) \\ &\quad -B(U,QT)C(V,QW),\end{aligned} \tag{2.34}$$

$$\begin{aligned}\tilde{g}(R(U,V)QW,L) &= (\nabla_U C)(V,QW) - (\nabla_V C)(U,QW) \\ &\quad +[\tau(V)+\pi(V)]C(U,QW) - [\tau(U)+\pi(U)]C(V,QW),\end{aligned} \tag{2.35}$$

for any $U,V,W,T \in \Gamma(TN)$ [35].

Now let us choose a $2-$dimensional non-degenerate plane section

$$\Pi = Span\{U,V\}, \tag{2.36}$$

in T_pN , $p \in N$. Then the sectional curvature at p is expressed by [36]

$$K(\Pi) = \frac{g(R(U,V)V,U)}{g(U,U)g(V,V) - g(U,V)^2}. \tag{2.37}$$

Let $p \in N$ and ξ be a null vector of T_pN. A plane Π of T_pN is said to be null plane if it contains ξ and e_i such that $g(\xi, e_i) = 0$ and $g(e_i, e_i) = \varepsilon_i = \pm 1$. The null sectional curvature of Π is defined by

$$K_i^{null} = \frac{g(R_p(e_i,\xi)\xi, e_i)}{g_p(e_i, e_i)}.$$

The Ricci tensor $\overline{Ric}$ of $\tilde{N}$ and the induced Ricci type tensor $R^{(0,2)}$ of N are given by

$$\begin{aligned}\overline{Ric}(U,V) &= trace\{W \to \tilde{R}(W,U)V\}, \ \ \forall U,V \in \Gamma(T\tilde{N}), \\ R^{(0,2)}(U,V) &= trace\{W \to R(W,U)V\}, \ \ \forall U,V \in \Gamma(TN),\end{aligned} \tag{2.38}$$

where

$$R^{(0,2)}(U,V) = \sum_{i=1}^{m} \varepsilon_i g(R(e_i,U)V,e_i) + \tilde{g}(R(\xi,U)V,L) \tag{2.39}$$

for the quasi-orthonormal frame $\{e_1,...,e_m,\xi\}$ of T_pN . From the equations (2.30)-(2.33), it can be shown that the Ricci type tensor doesn't need to be symmetric as the sectional curvature map. This tensor is called Ricci tensor if it is symmetric.

One defines scalar curvature τ by

$$\tau(p) = \sum_{i,j=1}^{m} K_{ij} + \sum_{i=1}^{m} K_i^{null} + K_{iL}, \tag{2.40}$$

where $K_{iL} = \tilde{g}(R(\xi,e_i)e_i,L)$ for $i \in \{1,...,m\}$.

3. CHEN LIKE INEQUALITIES FOR HALF-LIGHTLIKE SUBMANIFOLDS

Let N be a $(m+1)-$dimensional half lightlike submanifold of a $(m+3)-$dimensional of a Lorentzian manifold $\tilde{N}$ with a semi-symmetric metric connection and $\{e_1,....,e_m,\xi\}$ be a basis of $\Gamma(TN)$ where $\{e_1,....,e_m\}$ be an orthonormal basis of $\Gamma(S(TN))$. For $k \leq m$, we establish $\pi_{k,\xi} = sp\{e_1,....,e_k,\xi\}$ is a $(k+1)-$dimensional degenerate plane section and $\pi_k = sp\{e_1,....,e_k\}$ is $k-$dimensional non-degenerate plane section. The $k-$degenerate Ricci curvature and the $k-$ Ricci curvature are defined by

$$Ric_{\pi_{k,\xi}}(U) = R^{(0,2)}(U,U) = \sum_{j=1}^{k} g(R(e_j,U)U,e_j) + \widehat{g}(R(\xi,U)U,L), \tag{3.1}$$

$$Ric_{\pi_k}(U) = R^{(0,2)}(U,U) = \sum_{j=1}^{k} g(R(e_j,U)U,e_j), \tag{3.2}$$

respectively for a unit vector $U \in \Gamma(TN)$. Also, $k-$degenerate scalar curvature and $k-$scalar curvature are at $p \in N$ are given by

$$\tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{null} + K_{iL}, \tag{3.3}$$

$$\tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij}, \tag{3.4}$$

respectively. For $k = m$, $\pi_m = sp\{e_1,....,e_m\} = \Gamma(S(TN))$, we have the screen Ricci curvature and the screen scalar curvature given by

$$Ric_{S(TN)}(e_1) = Ric_{\pi_m}(e_1) = \sum_{j=1}^{m} K_{1j} = K_{12} + ... + K_{1m}, \tag{3.5}$$

and

$$\tau_{S(TN)} = \sum_{i,j=1}^{m} K_{ij}, \tag{3.6}$$

respectively.

Let $\tilde{N}(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$. The curvature tensor $\mathring{\tilde{R}}$ concerning the Levi-Civita connection $\mathring{\tilde{\nabla}}$ on $\tilde{N}(c)$ is expressed by

$$\tilde{g}(\mathring{\tilde{R}}(U,V)W,QT) = c\{g(U,T)g(V,W) - g(V,T)g(U,W)\}. \tag{3.7}$$

Using (2.2), we have the relation between the curvature tensor $\mathring{\tilde{R}}$ concerning the Levi-Civita connection $\mathring{\tilde{\nabla}}$ and the curvature tensor $\tilde{R}$ concerning the semi-symmetric metric connection $\tilde{\nabla}$ given by

$$\tilde{g}(\tilde{R}(U,V)W,QT)=\tilde{g}(\mathring{\tilde{R}}(U,V)W,QT)-\alpha(V,W)g(U,T)+\alpha(U,W)g(V,T)$$
$$-\alpha(U,T)g(V,W)+\alpha(V,T)g(U,W), \tag{3.8}$$

for any vector fields $U,V,W,T\in\Gamma(TN)$, where α is a $(0,2)$ tensor field defined by

$$\alpha(U,V)=(\mathring{\tilde{\nabla}}_U\,\pi)V-\pi(U)\pi(V)+\frac{1}{2}\pi(Q)g(U,V) \tag{3.9}$$

[20].

From (2.30), (3.6), (3.7) and (3.8), we can write

$$\tau_{S(TN)}(p)=m(m-1)c-2(m-1)\lambda+\sum_{i,j=1}^{m}B_{ii}C_{jj}-B_{ij}C_{ji}+\sum_{i,j=1}^{m}D_{ii}D_{jj}-D_{ij}D_{ji}, \tag{3.10}$$

where λ is the trace of α and $B_{ij}=B(e_i,e_j)$, $C_{ij}=C(e_i,e_j)$, $D_{ij}=D(e_i,e_j)$ for $i,j\in\{1,...,m\}$.

Let N be a screen homothetic half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$. From the Gauss-Codazzi equations and using (2.29) and (3.10) we have the following equations:

$$\tau_{S(TN)}(p)=m(m-1)c-2(m-1)\lambda+\phi m^2\mu_1^2+m^2\mu_2^2-\sum_{i,j=1}^{m}[\phi(B_{ij})^2+(D_{ij})^2]. \tag{3.11}$$

Theorem 4. *Let N be a $(m+1)-$dimensional screen homothetic half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\nabla}$.*

a) If $\phi>0$, then

$$\tau_{S(TN)}(p)\le m(m-1)c-2(m-1)\lambda+\phi m^2\mu_1^2+m^2\mu_2^2. \tag{3.12}$$

The equality case of (3.12) holds if and only if $S(TN)$ is totally geodesic and for $\forall U,V\in\Gamma(S(TN)),\quad D(U,V)=0$.

b) If $\phi<0$, then

$$\tau_{S(TN)}(p)\ge m(m-1)c-2(m-1)\lambda+\phi m^2\mu_1^2+m^2\mu_2^2-\sum_{i,j=1}^{m}(D_{ij})^2. \tag{3.13}$$

The equality case of (3.13) holds if and only if $S(TN)$ is totally geodesic.

Proof. The proof is obvious from (3.11).

The following corollary is obtained from the previous theorem.

Corollary 5. *Let N be a $(m+1)-$dimensional irrotational screen homothetic half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\tilde{\nabla}}$.*

a) If $\phi > 0$, then

$$\tau_{S(TN)}(p) \leq m(m-1)c - 2(m-1)\lambda + \phi m^2 \mu_1^2 + m^2 \mu_2^2. \tag{3.14}$$

The equality of (3.14) holds if and only if N is totally geodesic.

b) If $\phi < 0$, then

$$\tau_{S(TN)}(p) \geq m(m-1)c - 2(m-1)\lambda + \phi m^2 \mu_1^2 + m^2 \mu_2^2 - \sum_{i,j=1}^{m} (D_{ij})^2. \tag{3.15}$$

The equality of (3.15) holds if and only if $S(TN)$ is totally geodesic on N.

Furthermore, the second fundamental form B and the screen second fundamental form C provide

$$\sum_{i,j=1}^{m} B_{ij} C_{ji} = \frac{1}{2}\left\{ \sum_{i,j=1}^{m} (B_{ij} + C_{ji})^2 - \sum_{i,j=1}^{m} (B_{ij})^2 + (C_{ji})^2 \right\} \tag{3.16}$$

and

$$\sum_{i,j=1}^{m} B_{ii} C_{jj} = \frac{1}{2}\left\{ \left(\sum_{i,j=1}^{m} B_{ii} + C_{jj} \right)^2 - \left(\sum_{i=1}^{m} B_{ii} \right)^2 - \left(\sum_{j=1}^{m} C_{jj} \right)^2 \right\}. \tag{3.17}$$

Theorem 6. *Let N be a $(m+1)-$dimensional half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then*

$$\begin{aligned} \tau_{S(TN)}(p) \leq\ & m(m-1)c - 2(m-1)\lambda + m\mu_1 trace(\tilde{A}_L) \\ & + m^2 \mu_2^2 + \frac{1}{4} \sum_{i,j=1}^{m} (B_{ij} - C_{ji})^2. \end{aligned} \tag{3.18}$$

The equality case of (3.18) satisfies if and only if either $\phi = -1$ or B vanishes on N and C and D vanish on $S(TN)$.

Proof Using (3.10) and (3.16), we have

$$\begin{aligned}\tau_{S(TN)}(p) &= m(m-1)c - 2(m-1)\lambda + \sum_{i,j=1}^{m} B_{ii}C_{jj} \\ &\quad -\frac{1}{2}\left\{\sum_{i,j=1}^{m}(B_{ij}+C_{ji})^2 - \sum_{i,j=1}^{m}(B_{ij})^2 + (C_{ji})^2\right\} \\ &\quad + \sum_{i,j=1}^{m} D_{ii}D_{jj} - (D_{ij})^2.\end{aligned} \tag{3.19}$$

since

$$\frac{1}{2}((B_{ij})^2 + (C_{ji})^2) = \frac{1}{4}(B_{ij}+C_{ji})^2 + \frac{1}{4}(B_{ij}-C_{ji})^2, \tag{3.20}$$

from (3.19) and (3.20)

$$\begin{aligned}\tau_{S(TN)}(p) &= m(m-1)c - 2(m-1)\lambda + m\mu_1 trace(\tilde{A}_L) - \frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}+C_{ji})^2 \\ &\quad + \frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}-C_{ji})^2 + m^2\mu_2^2 - \sum_{i,j=1}^{m}(D_{ij})^2\end{aligned} \tag{3.21}$$

is obtained. Thus, we get equation (3.18). The equality case of (3.18) satisfies if and only if $B_{ij} = -C_{ij}$ and $D_{ij} = 0$, in the other word, either $\phi = -1$ or local second fundamental form B vanishes on TN and local second fundamental forms C and D vanish on $S(TN)$.

Theorem 7. *Let N be a $(m+1)-$dimensional half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then we have*

$$\begin{aligned}\tau_{S(TN)}(p) &\le m(m-1)c - 2(m-1)\lambda + \frac{1}{2}trace(\tilde{A}_L)^2 \\ &\quad + m^2\mu_2^2 + \frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}-C_{ji})^2\end{aligned} \tag{3.22}$$

where

$$\tilde{A}_L = \begin{pmatrix} B_{11}+C_{11} & B_{12}+C_{12} & \dots & B_{1m}+C_{1m} \\ B_{21}+C_{21} & B_{22}+C_{22} & \dots & B_{2m}+C_{2m} \\ & & & \\ B_{m1}+C_{m1} & B_{m2}+C_{m2} & \dots & B_{mm}+C_{mm} \end{pmatrix}. \tag{3.23}$$

The equality case of (3.23) holds for all $p \in N$ if and only if $\mu_1 = \mu_2 = trace(\tilde{A}_L) = 0$.

Proof. Using (3.17) and (3.21), we obtain

$$\begin{aligned}\tau_{S(TN)}(p) &= m(m-1)c - 2(m-1)\lambda \\ &+\frac{1}{2}\left\{\left(\sum_{i,j=1}^{m}(B_{ii}+C_{jj}\right)^2-\left(\sum_{i=1}^{m}B_{ii}\right)^2-\left(\sum_{j=1}^{m}C_{jj}\right)^2\right\} \\ &-\frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}+C_{ji})^2+\frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}-C_{ji})^2 \\ &+\sum_{i,j=1}^{m}D_{ii}D_{jj}-(D_{ij})^2.\end{aligned} \tag{3.24}$$

From equation (3.24), the inequality (3.22) is obtained and it is clear that the equality case of (3.22) holds if and only if $\mu_1=\mu_2=trace(\tilde{A}_L)$, $B_{ij}=-C_{ji}$ and $D_{ij}=0$.

The following corollaries is obtained from the previous theorem.

Corollary 8. *Let N be a $(m+1)-$dimensional irrotational half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then we have*

$$\begin{aligned}\tau_{S(TN)}(p) &\le m(m-1)c-2(m-1)\lambda+\frac{1}{2}trace(\tilde{A}_L)^2 \\ &+m^2\mu_2^2+\frac{1}{4}\sum_{i,j=1}^{m}(B_{ij}-C_{ji})^2,\end{aligned} \tag{3.25}$$

where $\tilde{A}_L$ is as defined in (3.23). The equality case of (3.25) holds for all $p\in N$ if and only if N is minimal.

Corollary 9. *Let N be a $(m+1)-$dimensional irrotational screen homothetic half lightlike submanifold of a $(m+3)-$dimensional Lorentzian space form $\tilde{N}(c)$ of constant sectional curvature c, endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then we derive*

$$\begin{aligned}\tau_{S(TN)}(p) &\le m(m-1)c-2(m-1)\lambda+\frac{(\phi+1)^2}{2}m^2\mu_1^2 \\ &+m^2\mu_2^2+\frac{(\phi-1)^2}{4}\sum_{i,j=1}^{m}(B_{ij})^2.\end{aligned} \tag{3.26}$$

The equality case of (3.26) holds for all $p\in N$ if and only if N is minimal. That is, $S(TN)$ is umbilical in N. The converse is trivial.

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