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Formal İspatın Mevcut Olmadığı Bir Durumda İspat İmajı Var Olabilir Mi?: Başarısız Bir İspat Girişiminin Analizi

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Özet – Matematiğin temel bileşenlerinden olan ispat ve ispatlamayı farklı perspektiflerden ele alan pek çok teorik çerçeve sunulmuştur. Bunlardan biri olan ispat imajı, Kidron ve Dreyfus'un (2014) iki profesyonel matematikçinin ispat süreci üzerinde yaptığı analizler sayesinde ortaya çıkmıştır. Yazarlar, ispat imajını bileşenleri bağlamında tanımlamış ve bunun formal ispat ile ilişkisini vurgulamıştır. Diğer yandan ispat imajının, formal ispatın ortaya çıkmadığı durumlarda da ortaya çıkabileceği belirtilmesine rağmen böyle bir örnek sunulmamıştır. Teorik çerçevedeki bu boşluk araştırmanın motivasyon kaynağı olarak benimsenmiştir. Çalışma sürecinde çok aşamalı örnekleme yaklaşımı tercih edilmiş ve öncelikle 120 öğretmen adayından cebir ile ilgili iki teoremi ispatlamaları istenmiştir. Daha sonra her iki teoremi de doğru olarak ispatlayabilen 3 katılımcı ile etkinlik temelli mülakatlar gerçekleştirilmiştir. Toplanan veriler üzerinde mikro-analitik analizler yapılmış ve alt bileşenler arasındaki ilişki tartışılmıştır. Ayrıca "aydınlanma" kavramının rolü yorumlanmış ve hislerin etkisi detaylandırılmıştır. Bu sayede katılımcılardan birinin formal ispata ulaşamamasına rağmen ispat imajına sahip olduğu belirlenmiş ve bu çalışmada buna dair verilere yer verilmiştir.

Anahtar kelimeler: ispat imajı, ispat, cebir, matematik eğitimi

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Geniş Özet

Giriş

Kidron ve Dreyfus (2014) tarafından literatüre kazandırılan ispat imajı, bireyin ispata ilişkin zihninde taşıdığı yapının tamamı olarak düşünülebilir. Yazarlar iki matematikçinin ispat süreçleri üzerinde RBC (Recognizing, Building With and Construction) modeliyle yaptıkları analizler sonucunda bu kavrama ulaşmışlardır. RBC, üç gözlenebilir (*epistemik*) eylemle açıklanabilmektedir (Hershkowitz, Schwarz ve Dreyfus, 2001). Sürecin ilk basamağı "Tanıma" (Recognizing), bir problem çözümünde önceden var olan/bilinen yapının fark edilmesini içermektedir (Türnüklü ve Özcan, 2014). İkinci basamağı oluşturan "Kullanma" (Building with), fark edilen bu bilgiden yararlanmayı ifade etmektedir (Schwarz, Dreyfus, Hadas ve Hershkowitz, 2004). Son basamak olan "Oluşturma" (Construction) ise mevcut bilgi yapılarının kısmi değişimle yeniden yapılandırılmasıdır (Bikner–Ahsbahs, 2004). Dahası, bunun öngörülemeyen bir şans sayesinde gerçekleşmediği "Aha! Deneyimi'' (Liljedahl, 2004) ve Aydınlanma (Rota, 1997) gibi mekanizmaların sürece eşlik ettiği söylenebilir. Tüm bu bileşenleri dikkate alan Kidron ve Dreyfus (2014) ispat imajı bileşenlerini aşağıda yer alan Tablo 1'deki gibi ifade etmişlerdir:

Tablo 1 İspat İmajının Bileşenleri

Bilişsel Boyut	Duyuşsal Boyut
B_1. *Kişisel Olma:* İmajın, bireyin kişisel çıkarımlarından/deneyimlerinden iz taşıması ve bunlardan beslenerek gelişim göstermesidir.	İmajın ikinci bileşeni "duyuşsal düzey" olarak kabul edilen sezgisel iknadır. Bu bileşen "kesinliğin duyuşsal hissi" olarak ifade edilebilir. Sezgi, temelinde öznel bir kesinliğin bulunduğu özel bir biliş türü olarak tanımlanabilir (Fischbein, 1999). Hisler ise Clore (1992) tarafından ifade edildiği gibi "bedensel, bilişsel veya duyuşsal durumlardan kullanılabilir bir geribildirim sağlayan tüm içsel işaretler" olarak değerlendirilebilir.
B_2. *Mantıksal Bağlar İçerme:* İspat sürecinde tanınan ve seçilen matematiksel yapılar arasında sezgisel iplikleri dokumaya benzer biçimde anlamlı bağlantılar kurulmasıdır (Kidron & Dreyfus, 2014). İspat imajı açısından, seçilen kurulan kişiye özgü bu bağlantıların formel olma zorunluluğu bulunmamaktadır.	
B_3. *Dinamizm:* Davydov'un (1990) soyutlamaya ilişkin görüşlerine paralel olarak ispat imajının gelişmemiş formdan gelişmiş forma, basitten karmaşığa doğru gelişim göstermesidir.	
B_4. *Bir Oluşuma Sebebiyet Verme (Bütünlük):* Matematiksel durumun tek bir imgesidir (Kidron & Dreyfus, 2014). Diğer bir ifadeyle imajın "oluşuma" olanak sağlayacak şekilde bütüncül gelişim göstermesidir.	

Blum ve Kirsch (1991) formel ispatı, kabul edilebilir mantıksal çıkarımların yer aldığı ispatlar şeklinde tanımlamaktadır. Buna paralel olarak Kidron ve Dreyfus (2014) da ispat imajının formel ispatla birlikte bulunduğu aktiviteler üzerine odaklanmışlardır. İspat imajının ortaya çıktığı ancak formel ispatın ortaya çıkmadığı durumlar olabileceğini belirtmelerine rağmen bu kısmı derinleştirmemişlerdir. Bahsedilen boşluk, araştırmanın motivasyon kaynağı olarak benimsenmiş ve araştırma problemi "İspat imajının ortaya çıktığı ve formel ispatın ortaya çıkmadığı bir ispat sürecinin bileşenleri nasıl şekillenir?" şeklinde belirlenmiştir. Bu inceleme, formel ispatla ispat imajı arasındaki ilişkiyi betimlemeye olanak sağlaması açısından önemli görülmektedir.

Yöntem

Gözlem, görüşme ve yazılı verilerden faydalanılan bu araştırmada nitel analiz yöntemlerinden örnek olay çalışması kullanılmıştır. Çalışma grubu, ilköğretim matematik öğretmenliği 3. sınıf öğrencilerinden oluşmaktadır. Örnekleme işlemi iki aşamada tamamlanmıştır. Birinci aşamada, kolay ulaşılabilir durum örneklemesi tercih edilmiştir. Bu kapsamda bir devlet üniversitesinde soyut cebir dersini almakta olan 120 öğretmen adayına 2 sorudan oluşan bir form uygulanmıştır. İkinci aşamada, görüşme yapılacak bireyler ölçüt örnekleme yöntemiyle seçilmiştir. Bu bağlamda iki soruya doğru cevap veren ve gönüllü olan 3 öğretmen adayıyla mülakat gerçekleştirilmiş ve onlardan başka bir teoremi sesli düşünerek ispatlamaları istenmiştir. Görüşme video ile kayıt altına alınmıştır. Elde edilen veriler içerik analizi ile analiz edilmiştir. Analizler sonucunda katılımcılardan sadece birinin formel ispata ulaşamadığı halde ispat imajı oluşturabildiği belirlenmiştir. "Büşra" olarak isimlendirilen bu katılımcıya dair veriler bulgularda paylaşılmıştır.

Bulgular

Bu bölümde Büşra'nın ispat süreci öncelikle özetlenerek doğrudan aktarılmıştır. Daha sonra ise ispat, bir akış diyagramı ile ayrıntılı olarak görselleştirilmiş ve ispatın bilişsel ve duyuşsal boyutları için ispat imajı bağlamında analizler gerçekleştirilmiştir. Bu analizler sonucunda ulaşılan sonuçlar aşağıda özetlenmiştir:

B_1. Kişisel Olma: İspat sürecinin tamamı dışarıdan herhangi bir müdahale olmaksızın Büşra'nın tercihleri ve kendi eylemleri doğrultusunda şekillendiğinden kişisel bir anlayışı içerisinde barındırdığı yorumu yapılabilir.

B_2. Mantıksal Bağlar İçerme: Büşra'nın farklı noktada anlayışını derinleştirmek için ön bilgileri arasından eşitlik, küme, merkezleştirici, gibi matematiksel yapıları seçtiği (R-) ve çoğunlukla

mantıksal gerekçelendirmeye dayanan bağlantılar kurarak bunları kullandığı (B-) yorumu yapılabilir. Bağlantıların bir bölümünün matematiksel açıdan doğru olmasına karşın önemli bir bölümünün de doğru olmayan gerekçelendirmelere dayandığı belirlenmiştir. Bu aşamada Büşra iki kümenin de grup yapısında olduğunu bilmesine karşın bunlar arasındaki ilişkiyi "grup olma özellikleri" açısından tekrar incelemiştir. Dahası, yeterli sorgulama gerçekleştirmediğinden bu eksikliği fark edememiş ve dolayısı ile mevcut bilgi yapısı içerisinde bir dengesizlik yaşamamıştır. Bu nedenle işlemlerin doğru olduğu kanısı ile formel ispata ulaştığını belirtmiştir.

B_3. Dinamizm: Seçilen (R-) ve Kullanılan (B-) yapılar arasındaki mantıksal gerekçelendirme sayesinde ispatın gelişmemiş bir formdan gelişmiş bir forma doğru gelişim gösterdiği belirlenmiştir.

B_4. Bütünlük: Büşra'nın farklı yaklaşımlar benimsemiş olmasına rağmen tek ve bütün bir imaja sahip olduğu söylenebilir. Bu karakteristik özellik Büşra'nın tüm süreci zihninde taşımasına olanak sağlamıştır. Bu sayede Büşra, gerekli noktalarda deneyimlerinin sonuçlarını da gözden geçirerek ispatına yön vermiştir.

İç görü anları ve aydınlanma deneyimleri: Büşra'nın üç farklı iç görü (Aha! Deneyimi) yaşadığı belirlenmiştir. Bunlardan ilk ikisi, onun M(a) ve M(a^{-1}) kümelerini ayrık olarak ele almasından kaynaklı güçlüğü aşmasını sağlayacak yöntemler ortaya koymasını mümkün kılmıştır. Ardından ispatı sonuçlandırmasında önemli bir adım olan üçüncü bir iç görü daha gerçekleşmiş ve "eşitlik, merkezleştirici ve alt grup" yapılarını birlikte düşünerek "$M(a) < M(a^{-1}) \wedge M(a^{-1}) < M(a) \Rightarrow M(a) = M(a^{-1})$" ölçütünü oluşturmuştur. Bu ölçüt, formel olmayan bağlantılar içermesine karşın bu onun bilgi yapısı içerisinde bir tutarsızlık ortaya çıkarmamıştır. Dolayısıyla, Büşra'nın teoremde yer alan kavramlar arasındaki ilişkilere kendi anlayışı bağlamında anlam verebildiği ve mevcut bilgi yapısı içerisinde aydınlandığı söylenebilir.

His Boyutu: Büşra'nın süreci his deneyimleri açısından incelendiğinde sürecin ilk aşamalarından son aşamalarına kadar çeşitli noktalarda aşina olma hissi, bilme hissi, doğru iz üstünde olma hissi ve kesinlik hissi gibi bu sürece yön veren çeşitli hisleri deneyimlediği ve eylemlerine bu hisleri doğrultusunda bir yön verdiği söylenebilir. Dahası, pek çok noktada doğru iz üstünde olma hissinin varlığından da söz edilebilir. Büşra'nın çelişkiye düştüğü anlarda bu hissin kısmen kesintiye uğradığı belirlenmiştir. Fakat bunun uzun sürmediği ve onun kavramsal ilişki ağını farklı yapılar/yöntemler sayesinde zenginleştirdiği ve bunu yaparken de kendisini cesaretlendirdiği görülmüştür. Dolayısıyla Büşra'nın sürecin sonunda sezgisini kesin olarak doğruladığını düşündüğünden ispatın geneline ilişkin bir tamamlanmışlık hissine ulaştığı yorumu yapılabilir.

Tartışma ve Sonuç

İspat imajının bilişsel boyutunu oluşturan bileşenler arasında, kişisel anlayış bileşeni ile şekillenen hiyerarşik bir ilişkinin varlığından söylenebilir. Buna göre, ispatlama sürecini gerçekleştiren birey sürecin herhangi bir aşamasında belirli bir amacı gerçekleştirmek için kişisel anlayışı doğrultusunda girişimde bulunarak belirli matematiksel yapıları seçer ve bunlar arasında sezgisel ya da formel olabilecek ilişkiler kurar. Eğer kurulan bu ilişkiler, mevcut aşamaya kadar benimsenmiş olan gerekçelendirme ağı ile tutarlılık gösteriyor ise bu aşama sürecin önceki aşamaları ile birleşir ve Davydov (1990) tarafından ifade edildiği anlamı ile basit bir formdan daha gelişmiş bir forma doğru gelişim mümkün olabilir. Ayrıca bu dinamik gelişim sayesinde ispatın söz konusu aşaması ile önceki aşamaları arasında bir bütünlük ortaya çıkabilir ve birey önceki adımlarda ulaştığı sonuçlardan hareketle yeni bir girişimde daha bulunarak sürece yön verir. İspatlama eylemine eşlik eden imajın oluşumunun, beraberinde getirdiği ilişki ağı sayesinde bireye iddianın neden doğru olduğu ile ilgili içsel bir görüş sağladığı yorumu da yapılabilir. Dahası, Büşra'nın olayında gözlendiği gibi ispat imajını biçimlendiren bir bireyin ulaştığı kesinlik hissi ile birlikte enformel yaklaşımının da ötesine geçerek daha formel bir bakış elde etme ihtiyacı duyduğu söylenebilir. Bu noktada sağlanan içsel motivasyon, bireyi tanımlar gibi matematikçiler tarafından kabul edilen formel yapıları kullanmaya teşvik etmektedir. Bu noktada Kidron & Dreyfus (2014) tarafından da ifade edildiği gibi söz konusu formel bilgi yapılarının, daha zayıf yapıları destekleyerek bireyin daha gerekçelendirilmiş bir çerçeveye ulaşmasını mümkün kıldığı yorumu yapılabilir. Diğer yandan kurulan matematiksel bağlantıların formel bilgi tutarlılığı ise başta hazırbulunuşluk olmak üzere pek çok faktör ile yakından ilişkilidir. Bu çalışmada olduğu gibi yeterli matematiksel olgunluk düzeyinde olmayan ve bir matematiksel durumu başka bir duruma dönüştürmede (Simon, 1996) yeterli gerekçelendirmeyi sağlayamayan bireylerin ispat imajını inşa ettikleri durumlarda dahi formel ispatı oluşturamayabilecekleri söylenebilir.

Can the Proof Image Exist in the Absence of the Formal Proof?: Analyses of an Unsuccessful Proving Attempt

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Abstract – As proof and proving are the key elements of mathematics, several frameworks evaluating this process have been presented. Proof image, being one of them, was introduced by Kidron and Dreyfus (2014) through analyses of two mathematicians' activities. Authors clarified it in the context of components, and emphasized its relation with formal proof. However, despite mentioning its possibility, they didn't present any case where proof image exists without the formal proof. This led us investigating dynamics of such cases. Multi-stage sampling was preferred, and 120 pre-service teachers were asked to prove two theorems about algebra firstly. Then, task-based interviews were conducted with 3 participants, who proved both theorems. Moment-by-moment analyses were conducted and sub-dimensions were discussed in detail. Additionally, role of "enlightenment" was reinterpreted and feeling dimension was elaborated. Consequently, it was identified that one participant had an image although she couldn't reach the formal proof, and her story was presented.

Key words: proof image, proving, algebra, mathematics education

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The data of this study were collected in the 2017/2018 academic year, and some of them were presented as oral presentations at the 42nd Annual Meeting of the International Group for the Psychology of Mathematics Education symposium held in Umeå/Sweden, between 3 and 8 July 2018.

Introduction

As proof and proving are the key elements of mathematics, difficulties experienced (eg., Moore, 1994; Almeida, 2000; Knapp, 2005; Harel & Sowder, 2007; Ko & Knuth, 2009; Weber & Alcock, 2009) in proving in mathematics education have caused the process to be discussed in detail, and several theoretical frames (eg., Balacheff, 1988; Harel & Sowder, 1998; Tall, 1998; Blum & Kirsh, 1991; Miyazaki, 2000) which are evaluating this process through different perspectives have been presented. One of these perspectives, the proof image was introduced by Kidron and Dreyfus (2014) by correlating different frameworks which explain the learning process of individuals. Because it approaches the proving process from the knowledge construction perspective and considers its both cognitive and affective dimensions, it can be said that the proof image provides a deep understanding of proof. However, is there always a possibility of transition from proof image to formal proof? Moreover, can the proof image exist in the absence of formal proof?

In this article, firstly, basic theoretical frameworks constituting the basis for the proof image were mentioned, and the sub-components of the image were discussed in detail. Then a proving process of a prospective mathematics teacher was presented in micro-analytic (moment-by-moment) level. This process was interpreted by using the proof image and findings of all characteristics were presented with specific examples. Based on these results, the discussion about the relation between the proof image and the formal proof was presented at the end of article.

Theoretical Framework

While the importance of proving in the studies of mathematics education has been underlined in this regard, studies have shown that undergraduates have difficulties with this activity. For example, Moore (1994) stated that students do not know mathematical definitions, or they cannot explain them. However, Antonini and Mariotti (2008) stated that methods of proof are not known sufficiently and argued that the known methods are often applied incorrectly. On the other hand, Knapp (2005) stated that undergraduates have difficulty in formal thinking and understanding formal mathematics. Another point in which students have difficulties is that they cannot decide where or how to start to prove. Although, on the basis of many theories which try to explain the proof process underlining the significance of intuitive understanding, lack of intuitive understanding is one of the difficulties that the students experience (Moore, 1994). However, intuitive structures are the key elements in every type of active understanding and productive thinking (Fischbein, 1982). Weber and Alcock (2004) stated that students use intuitive reasoning in addition to formal reasoning, defined this situation as different ways that students can produce a proof. In producing a syntactic proof, which is one of them, the individual tries to prove by using mathematical expressions in a logically acceptable way, that is, in a formal way. In producing a semantic proof, the individual uses informal and intuitive representations to lead the formal processes. Having looked at difficulties experienced in the proof process from a different perspective, Weber (2001) mentioned that although students know the concepts, definitions, theorems and can apply them, they could fail in proving. She explained that situation by strategic knowledge and categorized it. The strategic knowledge is "knowledge of the domain's proof techniques, knowledge of which theorems are important and when they will be useful, knowledge of when and when not to use 'syntactic' strategies". This strategic knowledge and attempt to prove in a semantic way through intuitive understanding can be seen as a reflection of concept image in the individual's mind. Because having a rich concept image is of great importance to be able to use a concept in a flexible way. On the other hand, having a rich concept image is possible by structuring the knowledge construction process correctly. With the aim of analysing this construction on a micro-analytic level, Hershkowitz, Schwarz and Dreyfus (2001) put forward RBC model.

RBC (Recognizing, Building- With and Construction)

According to RBC, which is a model presented in the framework of Abstraction in Context (AiC), construction of mathematical knowledge is seen as a vertical mathematization process. Vertical mathematization pointed out by Freudenthal (1991) means constructing a new knowledge by analyzing and correlating present mathematical knowledge in mathematical context (Treffers & Goffree, 1985). According to Hershkowitz, Schwarz and Dreyfus (2001) taking this view as a reference, the abstraction process can be explained by three epistemic actions. These actions are Recognizing (R-), Building- with (B-) and Construction (C-) respectively. Recognizing (R-) which forms the first step of the abstraction process is the realization of a construction already formed by students before in the process of problem solving (Türnüklü & Özcan, 2014). On the other hand, using preformed mathematical constructions to achieve a certain goal is explained through Building with- action. (Schwarz, Dreyfus, Hadas & Hershkowitz, 2004). Construction (C-), the last step of the process, refers to the reconstruction and regulation of mathematical constructions by partial changes. (Bikner-Ahsbahs, 2004). The reconstruction and regulation actions mentioned herein indicates a vertical mathematization. Thus, thanks to the Construction (C-) action, a new mathematical construction, which has not been accessible for the individual before, is put forward (Hershkowitz et al., 2001). However, it can be interpreted that this discovery has not been made by an unforeseen chance (Liljedahl, 2004) and some basic mechanisms which take place in the mathematical thinking process have important roles in forming and understanding new constructions. The "Aha! Experience" and "Enlightenment" are two of those mechanisms accompanying the knowledge construction process.

The "Aha! Experience" and Enlightenment

"Aha! Experience", which expressed by Liljedahl (2005), can be imagined as an electric spark that occurs suddenly and connects the various pre-existing knowledge constructs through appropriate selection. Thanks to this phenomenon, a new construct can be formed. Moreover, it can be said that the individual can better understand the experienced situation, and thus he or she is "enlightened" in the sense of Rota (1997). From the viewpoint of the proving process, this step can be interpreted as "providing an insight to the connections underlying the claim to be proved". By this means, for example, the role of a concept (or argument etc.) in the context of other mathematical constructs can be comprehended.

The concept of the "Aha! Experience" and the "Enlightenment" can be described as important stages in the knowledge construction (C-) process (Kidron & Dreyfus, 2010) and are explained in a mutual metaphor by the authors of this study as follows (Figure 1).

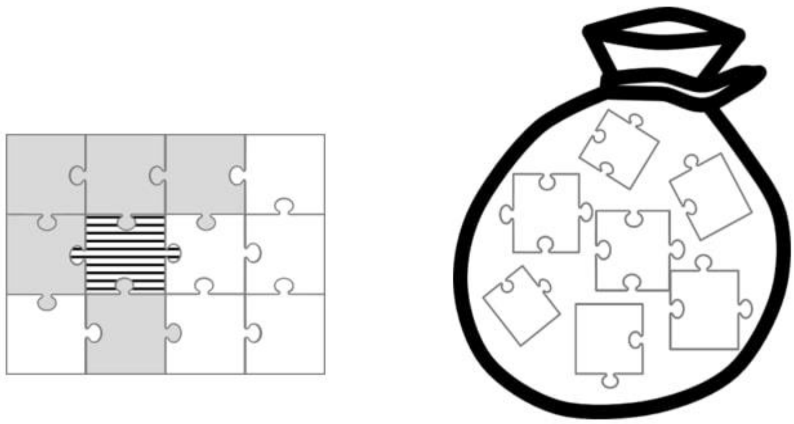

Figure 1 A metaphor explaining the relation between "Enlightenment" and "Aha! Moment"

While the marked part of the presented puzzle is determined by an appropriate approach, different features such as shape, size and color of the remaining shaded parts are taken into consideration. All of these features are considered as a whole and then the appropriate part is chosen from the pouch. At the moment in which all the features are evaluated simultaneously, clarification of which part in the pouch is appropriate for the shaded space in an undoubted way for the individual is interpreted in the context of "Aha! Experience" stated by Liljedahl (2005). On the other hand, creating a meaningful whole by this chosen part for the individual by being placed on the puzzle, and the increase of this meaningfulness level gradually have been evaluated in the context of the "Enlightenment" idea stated by Rota (1997).

Proof Image

Taking conceptual framework presented above into account, Kidron and Dreyfus (2014), examined the interplay between intuitive and logical thinking in the proving process micro-analytically and they reached the concept of the proof image by using RBC (Recognizing – Building With - Constructing) model. They described this concept as *"total cognitive structure in the person's mind that is associated with her or his proof* (Kidron & Dreyfus, 2014, p. 305)" and introduced it by comparing it to different viewpoints related to proving such as intuition, conceptual insight, semantic proof production, and they created the proof image-formal proof analogy by using a double strand concept image-concept definition structure. In this context, the writers underlined that the individual has certainly a proof image if he or she, who has attempted to understand why a given claim is true, has two main components together. Main components and their subcomponents are in the Figure 2 below.

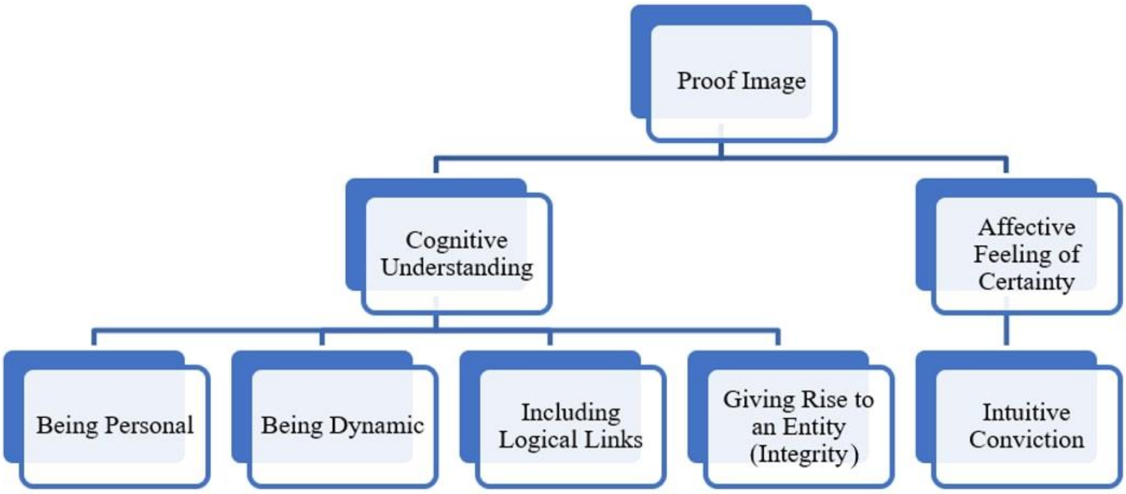

Figure 2 Components of the Proof Image

The first component is the cognitive understanding of why the proof is true. According to Kidron and Dreyfus (2014), the cognitive dimension of the proof image consists four characteristics. These are presented following Table 1:

Table 1. Cognitive Components of the Proof Image

C_1. Being personal: It means the image's having the traces of the individual's personal inferences and experiences, and making progress by being fed with them. According to Kidron and Dreyfus (2014), cognitive intuition and logic enriching the understanding of the individual on why each suggested claim is true in the proving process can be expressed in the context of personal understanding.

C_2. Including logical links: Mathematical proof requires a synthesis of various cognitive links to establish a new connection (Barnard & Tall, 1997). In this context, the meaningful links are established by the way like weaving the intuitive threads between previous mathematical constructions recognized and selected by the individual in the proving process (Kidron and Dreyfus, 2014).

C_3. Being dynamic: This characteristic which is inherent to the image's process of progress (Kidron and Dreyfus, 2014) can be expressed as image's further forms' including the former ones. In other words, in line with Davydov's view of abstraction, the progress of proof image is from backward form to developed form, from simple to complicate.

C_4. Giving rise to an entity: It means a single image of entire mathematical situation (Kidron and Dreyfus, 2014). In other words, the progress of the image in a single and entire way, which enables a construction. In present study, this characteristic was used by emphasizing the meaning of "integrity".

Kidron and Dreyfus (2014) defined the second component of the proof image as "the affective feeling of certainty, consists of the intuitive conviction". When defining the affective part of framework, they emphasized the following two concepts:

Intuition: According to the English version of the Oxford Dictionary (1989), intuition means "the ability to understand something instinctively, without the need for conscious reasoning". Fischbein (1999) defined intuition as a special cognition type on whose basis a personal feeling of certainty and generality take place, and stated that the intuition is explained together with many characteristics such as self-evident, immediacy etc. In addition to this, Fischbein (1987) defined intuitive knowledge as a knowledge type that is not based on experimental proofs or certain logical arguments.

Feelings: The context of the feeling concept is a subject of debate in the literature. Many researchers (e.g., Hannula, Evans, Philippou & Zan, 2004; Grootenboer & Marshman, 2016; Selden, McKee & Selden, 2010) described this concept as affective. However, some researchers claim that feelings cannot be restricted in a domain. One of them, Clore (1992) stated that "feelings" can be seen as both affective and cognitive and even physical. Because he considered the feeling as internal indications deducted from the bodily, cognitive and affective states, which provide usable feedback or information about these states. Affective feelings include fear, happiness, satisfaction, and many other feelings like those attached to the affective domain, while reactions for physical conditions such as hunger, ache and pain are bodily feelings (Clore, 1992). Cognitive feelings are also the feelings that the individual provides feedback for their own cognitive processes (ibid). Knowing, confusion, certainty, rightness, familiarity and completeness are some cognitive feelings. These three domains take place in different categories in regard to the way that they are felt; however, they are connected with each other and are interchangeable. That is to say, feelings in different categories can direct, launch or have an interaction with each other mutually. To illustrate, we can be happy to know something, be in dread of a situation that we feel confused about or be sad about being tired (Clore, Ortony & Foss, 1987, cited from Clore, 1992) or we can welcome an "Aha! Experience" with enthusiasm (Goldin, 2000). As Clore (1992) said, stating that some feelings aren't affective doesn't mean that they won't cause affective reactions.

As accepted by Kidron and Dreyfus (2014), in this present article, the intuition was discussed in the sense, which Fischbein (1982; 1987; 1999) used. In addition, the concept of feeling was adopted in the sense, which Clore (1992) used. Especially, affective and cognitive feelings have been taken into consideration due to their effect on justification in proving activities.

Motivation of Study

In their article, Kidron and Dreyfus (2014) presented two proving stories (the case of K and the case of L) both of which included a proof image and formal proof together. On the other hand, despite mentioning its possibility, they did not present any specific case, which includes a proof image and does not include formal proof. This gap, which is an incomplete part of the theoretical framework, led us to be sensitive about these type possibilities. For instance, Pala and Narlı (2020) examined the proof image of a student when the student had and did not have formal knowledge, and detailed the effect of formal knowledge on the proof image and on reaching formal proof. As a result, they emphasized the positive effect of formal knowledge on the proof image and on reaching formal proof, but also stated that the student had the proof image even when she did not have formal knowledge. Thus, such findings directed us to the following question for this study:

"How can a proving activity occur in which the proof image emerges but the formal proof does not?"

By the light of this question, in this present study, proof image, feelings, and epistemic actions were used to analyze an unsuccessful proving process of an undergraduate who attended abstract algebra course. Because of having a rich content in terms of proving activities, abstract algebra course is considered as a suitable context for revealing the difficulties in proving. In this way, it is thought that the relationship between the formal proof and the proof image can be elaborated and some of the reasons preventing the transition to the formal proof can be revealed. This analysis is of importance mainly because of its contributions to the educational practices for instructors' proof presentations, and the description of the process, which students experience within the scope of mathematics education in a multidimensional way. Furthermore, the data obtained from in-depth and multi-perspective analyses of the proving process can include suggestions for the educational practices for proof.

Method

Because a student's proving process was aimed to be analyzed without any intervention, the case study, which is one of the qualitative analysis methods, was used for this study. Case study is an in-depth description and analysis of a limited area (Merriam, 2009). Observation, interview, and written data were used to comprehend the student's proving process throughly.

Sampling

Participants of the study were pre-service middle school mathematics teachers attending a state university. Before the research, they took the Fundamentals of Mathematics, Abstract Mathematics, Linear Algebra I-II, and Analysis I-II courses previously. During this research, they were also taking Introduction to Algebra, Analysis III and Analytic Geometry I courses. The third writer was the instructor of Introduction to Algebra course. Because the sampling process completed in two stages, the multi-stage sampling was preferred in the study. In the first stage, the convenience-sampling method was preferred and 120 juniors taking Introduction to Algebra course were given a questionnaire including two proof questions. In the second stage, students whom would be interviewed were selected by the criterion sampling method. This method was preferred because it enables the selection of the individuals who had the determined qualification. In this stage, three students who answered both of the questions correctly and volunteered for the study were selected for task-based interview.

During the research and analysis process, attention was paid to the criteria that would ensure the validity and reliability of the study. The most basic and first of the strategies that increase credibility in a qualitative study is data triangulation (Merriam, 2009). In this study, data triangulation (interview and observation) was performed to ensure the credibility of the findings. Sufficient (rich and dense) descriptive data were provided in the findings to ensure transferability (Merriam, 2009). Peer evaluation, which is one of the strategies that can be used to ensure the reliability of qualitative research, is explained in the data analysis section.

Data Collection

The questionnaire, given to 120 students to determine the students with whom a task-based interview would be conducted, was an open-ended exam, which did not include any multiple-choice questions. The exam consisted of two questions about proving two theorems. Because the questions include advanced cognitive skills and required long answers, the number of questions was limited. Expert opinions on the validity of the questions were taken from two instructors working in the primary mathematics teaching department. The theorems in the questionnaire included proving tasks on a subgroup and centralizer concepts in abstract algebra. These concepts were preferred because they are the basis of many subjects in abstract algebra. The theorems were as follows:

- Theorem 1: Let (G,.) be a group and a ∈ G. In this case: Is the set of C (a) = {x ∈ G: a.x = x.a} a subgroup of the group G? Prove it.
- Theorem 2: Let (G,.) be a group and H<G. In this case: The set of C (H) = {x∈ G: ∀ h∈ H, h.x=x.h} is defined. What kind of a set is this? Explain it. Is the set of C(H) a subgroup of the group G? Prove it.

Before the practice exam, the students learned the subjects of operation, group, subgroup, and the center of a group in the introduction to the algebra course. Also, the statement "the center of a group is a sub-group" was proven in the course with the students. Therefore, the students were thought to have a sufficient background necessary to use statements such as "the center of group, subgroup requirements, properties of commutation and association, etc." to prove the two theorems above. In answering the questions, no limitation was placed on the undergraduates. After the preparation of the questionnaire, a key form was prepared to evaluate the items in the questionnaire. Taking into consideration that an individual can prove in different ways, all-different proving ways, which can be formed through the subjects taught within the scope of the course, were tried and added onto the key form. To be accepted as successful in demanded proving, an individual should complete the steps on the form. Answers taken from the questionnaire were evaluated by the first and second writers individually, and three students who received full marks from both of the researchers were selected. These students, whom we called Rabia, Büşra, and Merve, were given the necessary ethical information about the study. In the interview, they were asked to prove the theorem given below in order to observe how they proved a theorem, which they had never come across before.

- Theorem 3: Let G be a group and $a \in G$. If $C(a) = \{x \in G: a.x = x.a\}$, show that $C(a) = C(a^{-1})$

This interview was conducted by the first writer and recorded by two video cameras. One of these cameras was placed in such a way that it could record the paper of the student, and the other was placed in such a way that it could record the face of the student. When proving the given theorems, the students were asked to think aloud and to give explanations at critical points. Right after each proving activity, a semi-structured interview was carried out with participants about their proving process. This interview form, which was developed by the light of opinions of three mathematics education experts, included 13 questions about components of the proof image.

Data Analysis

Content analysis was used in the analysis of the data received from the task based activity and the interview. After the interview was transcribed, the data was analyzed according to the cognitive and affective components and their sub-components. The RBC (Recognizing-, Building With- and Construction-) model was used as an analytic tool in the evaluation of knowledge construction processes on proving.

In the data coding process, firstly, main indicators of each sub-component were identified by three authors. For example "clear expression of remembering any pre-knowledge" was one of the indicators of Recognizing (R-) epistemic action, "relating at least two concepts for any procedure" was one of the indicators of including logical links characteristics or "expressing thoughts without any hesitation" was one of the indicators of the feeling of rightness. Especially, camera records (focused to face) and statements in the interview were used for coding feelings. Having identified these indicators, the coding process was executed by the first and the second authors separately. The same coded variables of both authors were included in the study. For different situations, the third author's opinion was taken into account. The codes of the researchers were compared and the consistency between the coders was calculated using Miles and Huberman's (1994) formula (Reliability of the study = consensus / (disagreement + consensus) x100). As a result, the reliability percentage between the encoders was calculated as approximately 80%. After the coding process, analyzes were conducted. As a result of these analyzes, it was observed that only one of the participants had the proof image but did not reach the formal proof. In light of the motivation of the study, the proving process of this participant, whose name is Büşra, will be presented with findings related to her activity.

Findings

In this section, first of all, Büşra's proving activity is summarized and then interpreted in the context of the components of the proof image.

Summary of Büşra's Proving Process

After having read the theorem Büşra started to explain her thoughts to herself. She stated that there existed two centralizer sets in the theorem and she should show their equality. After having thought for a while, she said: "*I got what the question asked but I don't know how to point this out".* At this time she first questioned whether the commutative elements with an element of $a \in G$ could also be commutative with the inverse of this element. She linked the condition of equality of the sets to the condition of being commutative with exactly the same elements, and so started to create a framework within the context of commutative property on the equality of the sets. Then she decided to focus on the concept of a centralizer. She clearly expressed that the objects used to comprehend this concept existed in her mind, and she detailed her thoughts as follows using the concrete objects on her desk:

> For example *(she takes an eraser*) let this be the element of 'a'. If this eraser is commutative with these 4 pencils (*pointing out the pencils*), the inverse of the element of 'a' should be commutative only with these 4 pencils.

She stated that she had difficulty to express this relationship mathematically. At this point, she first focused on the approach of finding a contradiction. According to this approach, if any element in the centralizer sets were commutative with only one of the elements of $a \in G$ and $a^{-1} \in G$, it would certainly be commutative with the other. However, she gave up on proceeding in that way because she had a doubt that this approach could provide the essential conditions in terms of the formal proof. After having thought for a while, she excitingly expressed her thoughts:

> I wonder whether there exists an element a^{-1} in the set of C(a)? If I choose an element of a^{-1} from the set of C(a) and show that it is commutative with every element in the set of C(a), it will mean that this set is the centralizer of the element a^{-1}. Yes, I can do this now!

After making this explanation self-assuredly, she focused on the question of $a^{-1} \overset{?}{\in} C(a)$.

To answer this question, she focused on $a \overset{?}{\in} C(a^{-1})$ which she thought that it was easier to be answered and equivalent to other question. After having remained silent for a while, by stating that the set of $C(a^{-1})$ is a subgroup just like the set of *C(a)*, and so it should provide the group characteristics, she wrote "$a.a^{-1}=a^{-1}.a=e \Rightarrow a \in C(a^{-1})$". She couldn't continue with this expression and in order to overcome this difficulty, she expressed that she would define a new centralizer set and wrote "$C(a)=\{\forall x \in C(a^{-1}):\ a.x=x.a\}$". She stated that the sets needed to be equal according to the new definition. Because she had difficulty in expressing her thoughts and she repeated same concrete objects as follows:

> I said that (*she took the eraser*) let this be the element 'a'. Let them (pencils) be the centralizer set of the inverse of 'a'. If this one (*means the element 'a'*) is commutative with all these (*means the centralizer set of the inverse of the 'a'*), then the centralizer of the element 'a' is the same set.

Having made this explanation confidently, she stopped and didn't continue prove. After having thought for a while, she said that she made it much more confusing by defining the set of $C(a)=\{\forall x \in C(a^{-1}) | a.x = x.a\}$ in that way and then deleted it.

> If all elements of the set $C(a^{-1})$ is commutative with the element a, then it means that I'll show the set C(a) as a subgroup of the $C(a^{-1})$. If I show these sets are subgroups of each other, then it means I'll show the equality.

After having formed her criterion about the proof, Büşra, finally, was observed as being motivated. She produced the proof of "$C(a) < C(a^{-1})$" through a similar procedure of the proof of "$C(a) < G$" which she knew already. At the end of the following operations (presented in Figure 3), which she performed without hesitation, she arrived at the conclusion that this is a subgroup.

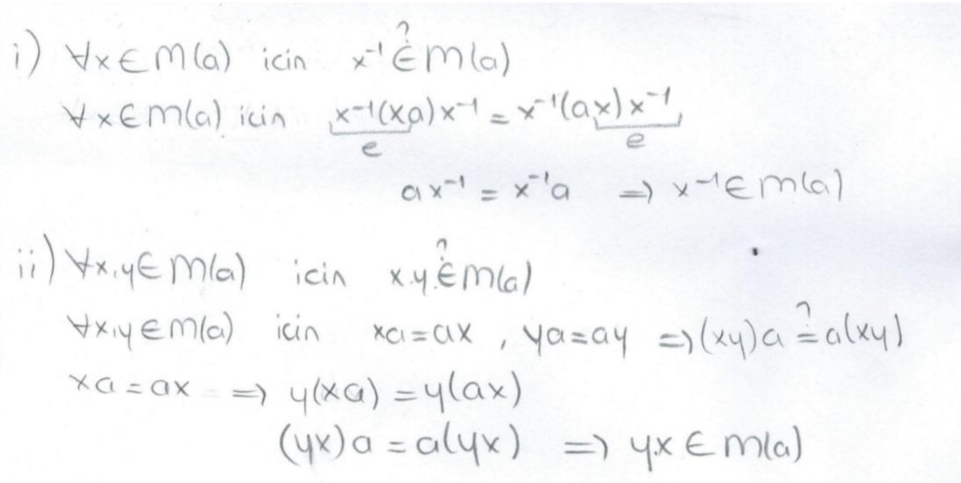

Figure 3 Büşra's approach for $C(a) < C(a^{-1})$

Following this step, she showed through a similar procedure that the expression $C(a^{-1}) < C(a)$, which is the inverse of the previous step, was valid, and then she stated that she completed her proof.

Figure 4 visualizes Büşra's proving approach (including mathematical connections, statements, actions, mimics and feelings) moment-by-moment. In addition, Table 2 elaborates her statements and actions step-by-step and shows her epistemic actions in each step.

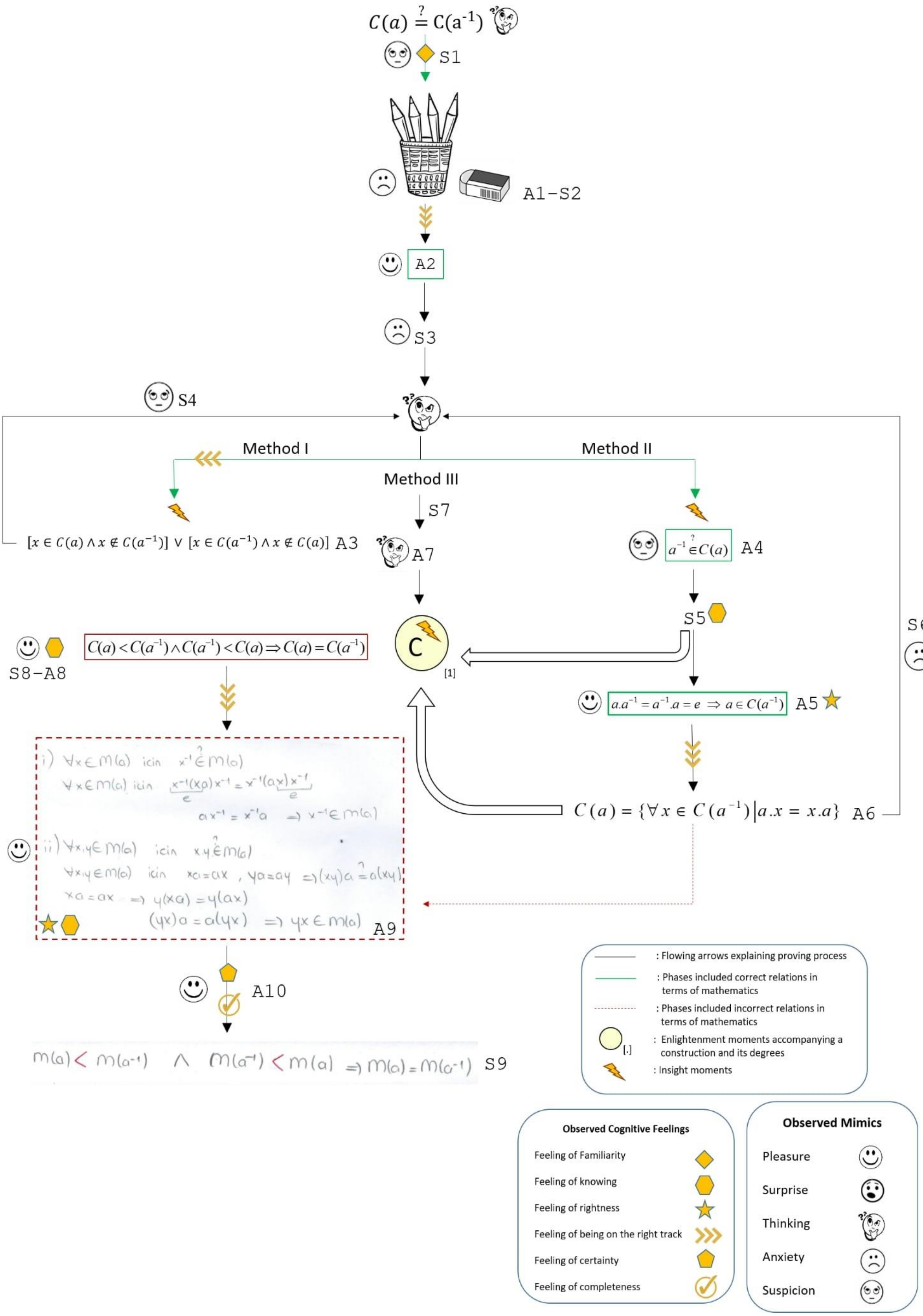

Figure 4 The Case of Büşra

Table 2. Micro-analytic analyses of Büşra's proving process

Statement (S) Action (A)	**Explanation**	**Observed Epistemic Actions**
S1	She stated that she has to demonstrate equality of centralizer sets. However, she also stated that she was not sure about this equality.	Action (R-)
A1-S2	She stated that the constructs, which she need for understand centralizer concept, exist in her mind. Afterwards, she interpreted the meaning of the equality of centralizer sets by using concrete objects (pens & eraser).	Action (R- & B-)
A2	She reached the conclusion that equal centralizers are the sets, which are commutative with same elements.	Action (R- & B-)
S3	She stated that she could deal centralizer sets separately and could not deal them together.	
A3	After for a while she suggested following approach: '*'Demonstrating that if any element in the centralizer sets were commutative with only one of the elements of* $a \in G$ *and* $a^{-1} \in G$*, it would certainly be commutative with the other*''	Action (R- & B-)
S4	Having stated she doubted the adequacy of this method, she abandoned it.	
A4	To understand better the relation between centralizer sets and their elements, she focused on a question.	
S5	«*If I select* a^{-1} *from C(a) and demonstrate that this element is commutative with all the elements of C(a), then I prove that C(a) set is also centralizer of the* a^{-1} *element.*»	Action (R- & B-)
A5	She answered the question asked in A4 action by using her pre-knowledge.	Action (R- & B-)
A6	She re-defined centralizer sets by using conclusion reached in A5 action.	Action (R- & B-)
S6	She stated that this approach made the relationship network more complicated and gave up using the definition she wrote in the last step.	

S7	She gave up her approach, which based on synthesizing two centralizer sets in single representation, and stated that she thought $C(a^{-1})$ as a simpler construct.	
A7	She thought about the properties of both centralizer sets and their elements.	
S8- A8	By considering S5 statement and A6 action, she formed following criterion to prove equality of the sets: *«If these two centralizer sets are sub-group of each other, then they must be equal.»*	Action (C-)
A9	She constructed the proof of $C(a) < C(a^{-1})$.	Action (R- & B-)
A10	She constructed the proof of $C(a^{-1}) < C(a)$ in a similar way.	Action (R- & B-)
S9	She stated that her proof was finished.	

Evaluation of Büşra's Proving Process

When cognitive and affective components taken into consideration, it could be said that Büşra had a proof image. The findings of the subcomponents will be exemplified below.

Cognitive Dimension

C_1. Personal Understanding: Because the proving process was shaped by Büşra's own preferences and her own actions, it can be interpreted that it includes a personal understanding which belongs to her. To illustrate, having considered using an approach based on finding a contradiction in the process, but then giving up on this approach with the impacts of her experiences and leaning towards a different approach can be interpreted as a reflection of her personal understanding giving direction to her proof. Personal understanding dimension expressed at this point can be said to provide an opportunity for presenting a flexible framework for herself. It can be expressed that this characteristic formed a branching construct rather than a linear proceeding.

On the other hand, it was observed that with the impact of Büşra's personal understanding, Büşra had difficulty in expressing her thoughts in the beginning stages and frequently hesitated in the communication process. To illustrate this, after having decided to define the elements of set C(a) with reference to the set $C(a^{-1})$, she remained silent for a while and then made the following explanation with a concerned face:

...I thought what I had at the beginning was logical for me but how can I express it?

It can be interpreted that at that moment the image was "silo thinking". On the other hand, with the impact of the dynamism provided in the next stages of the process, Büşra can be said to have managed to make contact with the outside, and thus she was able to express her thoughts in writing and orally in a more organized manner.

C_2. Including Logical Links: It can be interpreted that Büşra selected (R-) various mathematical constructs at several points and used them by establishing links most of which were based on logical justifications. To illustrate this, she focused on the relation between the centralizer sets and their elements to verify her initial intuition on the equality of the sets, and thus, she formed more formal framework between the sets by developing a mathematical definition which would correlate the elements of the set C(a) with the set of $C(a^{-1})$. Similarly, it can be interpreted that Büşra established logical links which enabled her to discuss different concepts such as equality, set, centralizer, group, and subgroup interactively at several points. Besides, it can be said that although some parts of those links which Büşra established in the process had correct justifications mathematically, the majority of the links were based on incorrect justifications.

On the other side, it can be said that the relation $C(a)<C(a^{-1})$ which Büşra expressed towards the last stages of the proving process was based on a mathematically inadequate justification process. Because she preferred to analyze the sets once more in terms of the characteristic of being a subgroup instead of analyzing their subgroup relations between them even though she knew that both sets were in a group structure. In this context, Büşra can be interpreted as having developed an inadequate understanding of the concept of the subgroup. On the other hand, because the subset relations between the sets weren't analyzed, a proof which could be valid for the proposition of $C(a)<G$ was put forward by Büşra. However, because she didn't question the operations adequately she put forward at the last stages of the proving process, she didn't realize this deficiency, and thus, any imbalance didn't emerge within her present knowledge construct. Therefore, Büşra stated that she produced the formal proof with the view that the operations she performed were correct.

C_3. Being Dynamic: Thanks to the logical justification networks among the mathematical constructs which were selected (R-) and used (B-), it can be stated that the proof made a dynamic progress from the backward form to the more developed form. Büşra illustrated this by managing to explain the equality, which was an abstract relation that she comprehended on merely concrete materials at the beginning of the process, within a mathematical form, thanks to the logical relation networks, which she established within the process. As in this example, thanks to the dynamism, it can be interpreted that the next stages within the proving process of

Büşra made progress in such a way that including the previous stages. Therefore, progressing from an intuitional characteristic to a formal framework was possible.

C_4. Giving Rise to an Entity: Although Büşra adopted different approaches methodologically and didn't proceed within the process linearly, she can be said to have a single and entire image in terms of her proof thanks to the emerged dynamism. In the most general sense, this characteristic allowed her to carry all steps of the process in her mind as a whole. By means of this integrity, Büşra can be said to have given direction to her proof by reviewing the results of previous experiences at the necessary points. For instance, when she foresaw that she would not be able to obtain the proof she wanted in the case of adopting the approach based on contradiction, she gave up this approach.

Insight Moments and Enlightenment Experiences: Thanks to the justified relation networks and the integrity based on this, Büşra can be said to have gone through insight (Aha! Experience) in the meaning by Liljedahl (2005) at three different points. First, two of these experiences enabled her to put forward proof approaches which would make it possible for her to overcome the difficulty resulting from dealing with the sets C(a) and C(a^{-1}) as separated constructs. In these moments, by taking into consideration of her dead-ends in proving, she produced new initiatives based on inferences of her previous experiences. The first method based on finding a contradiction and the second method grounded on expressing the sets under a single representation provided a beginning point for Büşra's proof approaches. However, these attempts could not provide enough sense making to her for continuing proving because of weakness and complexity. On the other side, following these experiences, the third insight moment occurred, which was a significant step in Büşra's proof. In this phase, she formed (C-) the following criterion confidently by binding the constructs of the equality, the centralizer and the subgroup, which comes into the forefront among the concepts:

$$C(a) < C(a^{-1}) \wedge C(a^{-1}) < C(a) \Rightarrow C(a) = C(a^{-1})$$

Although this criterion had inadequate links mathematically, this inadequacy was not questioned in detail because of the impact of Büşra's belief in her criterion. Because these inadequacies weren't realized, and any inconsistency didn't emerge within her knowledge construct. In other words, considering through Büşra's perspective, it could be said that there was no contradiction in terms of the relations hidden in the theorem. Thus, it can be said that Büşra could comprehend the relations among the concepts laying the basis of the theorem in the context of her understanding and she was enlightened in her present knowledge context.

Büşra's Feelings

When her proving process was analyzed, it can be said that she experienced various feelings such as the feeling of familiarity, the feeling of knowing, the feeling of being on the right track, and the feeling of certainty, which gave direction to the process. To illustrate this, after having read the theorem, she recognized (R-) the centralizer concept, and experienced the intense feeling of familiarity, as is seen in her explanations below:

> I am making assumptions right now, I mean the objects I used while comprehending the centralizer at the beginning are in my mind after all.

Along with this, she modelled the relations she mentioned within a concrete model and stated as follows that this model helped her comprehend the relations.

> Now, the centralizer isn't confusing for me any longer. What the centralizer is, is just a simple set, 'a' commutative one with 'a'...

Within this context, it can be said that the feeling of knowing of the concept of the centralizer is dominant in the explanations Büşra presented above. Besides, the existence of the feeling of being on the right track can be mentioned in some phases. The following explanation, while she was in search of a method to demonstrate the equality of the sets, can be interpreted as an example experience of this feeling.

> If I select the inverse of the element 'a' from the set of C(a) now and demonstrate that it is commutative with every element in C(a), I will have shown that this set is its centralizer... Okay, it made sense to me after all.

The moment that Büşra runs into a contradiction within the different methods she adopted, it can be said that the feeling of being on the right track was interrupted by the feeling of confusion at certain points as in the following example.

> ... I mean the thought I had at the beginning was logical for me but how can I express it? Because I am not sure about its correctness.

On the other side, it was observed that these interruptions didn't last long. Because, Büşra enriched the conceptual relation network at these points by different constructs and methods, and while doing this she strived to encourage herself at once with the explanations such as *"... I will do it now"* and *"...yes yes, I am doing it right now"*. Along with this, it can be said that Büşra experienced a feeling of certainty about "the centralizer sets were subgroups" as a result of actions that she had performed, thus she reached the feeling of rightness on her initial intuition. Therefore, it can be interpreted that although she didn't experience a feeling of rightness for some approaches, which they did not enable her to reach the point she wanted, she certainly had verified her intuition at the end of the process. In general, she had the feeling of completeness for the proof. On the other side, when her following statement considered, which she made when evaluating her proof in interview, it can be inferred that she was not adequately satisfied with the form of the proof despite having reached the feelings of rightness and completeness:

> ... now, I believe, okay, it might be long, I don't know, I may have gotten it very differently but I believe it is correct.

Discussion and Results

Based on the analysis of the presented case, some conclusions regarding the "proof image" and the "formal proof" are shared in this section.

Hierarchical Nature of Proof Image

When the proving process of the undergraduate in this study and two mathematicians presented by Kidron and Dreyfus (2014) have been evaluated together, it can be seen that a hierarchical relation is generally taking shape among the components of the cognitive dimension of the proof image. According to this, the individual realizing the proving process selects certain mathematical constructs by making an attempt on his/her own personal understanding to realize a certain purpose at any stage in the process, and establishes a relation among them which can be considered as intuitional or formal. If these established relations are consistent with the justification network adopted until the current stage, this stage is articulated with the previous stages of the process, and progress from a simple form to a more developed form in Davydov's (1990) sense could be possible. Furthermore, this dynamic development can lead to an entity between the present stage and the previous stages of the proof, and the individual gives direction to the process by making a new attempt considering the results of the previous steps (Pala, 2020).

Transition to Formal Proof In Terms of Sub-Components

It can be said that the construction of the image accompanying a proving activity provides an insight into the individual about why his or her claim is true thanks to the logical links. Along with this, an individual giving shape to a proof image can be said to be in need of obtaining more formal perspective by passing beyond the informal approach (Dreyfus & Kidron, 2014) with the certainty feeling which he/she reaches just as observed in the case Büşra. Moreover, the provided internal motivation encourages the individual to use formal constructs such as definitions (Weber, 2005). That is, the present formal knowledge constructs enable the individual to reach a more justified framework by contributing weaker constructs, just as expressed by Kidron and Dreyfus (2014). For example, after Büşra interpreted about the equality of the centralizer sets by the concrete model, which she formed through pencils, she sought to generalize her thoughts and formed a criterion for her judgement. This can be interpreted in the context of the transformational thought, which provides the transition from the image of proof to the formal proof. The importance of transformational thought in terms of constructing new mathematical situations has been also emphasized by Simon (1996) and has been closely related to the "dynamism" dimension of the proof image by Pala and Narlı (2020).

At this point, it can be interpreted that especially the justification dimension of the relation network in the proving process is an important element in both the dynamic progress of the proof and the individual's reaching the formal proof. Because it can be said that the relation networks established with insufficient justifications can be an obstacle against the meaningful progress (Duval, 2007), and as a result of this, it may not create an integrity. Moreover, dynamism dimension of the image includes "justification" within it, and thus consistent connections can be established between steps (Pala & Narlı, 2020). On the other side, it can be interpreted that the knowledge structure which can be synthesized (C-) in the circumstances in which the characteristic of the entity emerges, is strong within the context of its logical connections and accompanying epistemic actions; and plays an essential role in reaching the formal proof in this context. The dynamism in Büşra's image is parallel with Davydov's (1990) viewpoints on abstraction as expressed before. At this point, it can be said that the dynamism which was observed in Büşra's images enabled her to convey their intuitive approaches to a more formal mathematically framework. It can be said that the proving process of Büşra included the discovery of a new characteristic, which is a synthesis, by the intersection of the common characteristics of knowledge sets taking shape due to the gathering together of the different concepts. According to Kidron and Dreyfus (2014), when the image has the characteristic of giving rise to an entity, it includes whole mathematical circumstance within itself in full. It was clearly observed that the progress enabled by the dynamism occurred in the process had an important role in giving rise to the entity. It can be said that Büşra was able to carry consequences of previous stages continuously in her mind thanks to the entity they gave rise to and, thus experienced insightful moments due to their appropriate selections. Therefore, it can be interpreted that the entity characteristic of the proof image paves the way for the realization of important turning points which enables the revealing of a product in the proving process.

When the process had been analyzed in terms of the feeling dimension, it can be said that Büşra successively experienced some of the main cognitive feelings such as the feeling of familiarity, the feeling of knowing, the feeling of rightness, and the feeling of certainty. The experience of these feelings gave direction to her proving activity at various points, which in the end, allowed her to reach the feeling of "completeness" for the entire of the image. Moreover, it can be said that when the image is formed, cognitive and affective feelings reach in a specific step, give direction to the next step, and along with this, the product composed at the end of the process is reanalyzed with an integrated perspective in the context of the feelings of "completeness" and "satisfaction". It can be said that the occurrence of these feelings undertakes an important function in terms of termination of the proving activity (Selden & Selden, 2008).

Implications For the Further Studies

Kidron and Dreyfus (2014), made the following explanation while putting forward the analogy between the concept image and proof image:

> …while we do not have a specific example, we think that this might also occur in some cases where the proof image does not lead to a formal proof (Kidron & Dreyfus, 2014, p. 305)

Along with this, when the findings of this present study had been taken into consideration, it can be said that the foresight of Kidron and Dreyfus (2014) was verified by the case of Büşra. Because, despite the fact that she couldn't reach the formal proof, she had a proof image. On the other hand, the question of whether the individuals will be able to reach a formal proof even though they don't have any proof image is still pending.

This study was carried out with an undergraduate who can be considered as non-expert mathematicians unlike Kidron and Dreyfus (2014), and the analysis of proof image within the context of abstract algebra was managed by selecting the concept of the "centralizer". Along with this, understanding the nature of the proof image can be deepened by analysis of the proof image within the context of different age groups and different subject fields as suggested by Kidron and Dreyfus (2014).

Considering the findings of this study, it can be said that the relationships between mathematical concepts in the Abstract Algebra course should be taught by making formal justifications and by asking students for such justifications. It can be suggested that the logical connections that students establish between mathematical concepts should be frequently evaluated in the context of proving, and it should be ensured that they express these connections with formal justifications.

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