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A study on Matrix Domain of Riesz-Euler Totient Matrix in the Space of p-Absolutely Summable Sequences

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Abstract
In this study, a special lower triangular matrix derived by combining Riesz matrix and Euler totient matrix is used to construct new Banach spaces. $\alpha-,\beta-,\gamma-$duals of the resulting spaces are obtained and some matrix operators are characterized. Finally by the aid of measure of non-compactness, the conditions for which a matrix operator on these spaces is compact are determined.

Keywords: Compact operators, Hausdorff measure of non-compactness, Matrix mappings, Sequence space, $\alpha-,\beta-,\gamma-$duals.
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1. Introduction and background

A sequence space is a vector subspace of the space ω of all sequences with real entries. Well known classical sequence spaces are ℓ_p (the space of p-absolutely summable sequences, $1 \leq p < \infty$), ℓ_∞ (the space of bounded sequences), c_0 (the space of null sequences), c (the space of convergent sequences). On the other hand, bs, cs_0 and cs are the most frequently encountered spaces consisting of sequences generating bounded, null and convergent series, respectively. Further ψ is the space of all finite sequences. A Banach sequence space having continuous coordinates is called a BK space. Examples of BK spaces are c_0 and c endowed with the supremum norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, where $\mathbb{N} = \{1,2,3,...\}$.

By virtue of the fact that the matrix mappings between BK-spaces are continuous, the theory of matrix mappings plays an important role in the study of sequence spaces. Let X and Y be two sequence spaces, $\mathscr{A} = (a_{nk})$ be an infinite matrix with real entries and $\mathscr{A}_n$ indicate the n^{th} row of $\mathscr{A}$. If each term of the sequence $\mathscr{A}x = \{(\mathscr{A}x)_n\} = \{\sum_{k=1}^{\infty} a_{nk}x_k\}$ is convergent, this sequence is called $\mathscr{A}$-transform of $x = (x_n)$. Further, if $\mathscr{A}x \in \mathsf{Y}$ for every sequence $x \in \mathsf{X}$, then the matrix $\mathscr{A}$ defines a matrix mapping from X into Y. (X,Y) represents the collection of all matrices defined from X into Y. Additionally, $B(\mathsf{X},\mathsf{Y})$ is the set of all bounded (continuous) linear operators from X to Y. A matrix $\mathscr{A} = (a_{nk})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nk} = 0$ for $k > n$.

The matrix domain $\mathsf{X}_\mathscr{A}$ of the matrix $\mathscr{A}$ in the space X is defined by

$$\mathsf{X}_\mathscr{A} = \{x \in \omega : \mathscr{A}x \in \mathsf{X}\}.$$

Since this space is also a sequnce space, the matrix domain has a crucial role to construct new sequence spaces. Moreover given any triangle $\mathscr{A}$ and a BK-space X, the sequence space $\mathsf{X}_{\mathscr{A}}$ gives a new BK-space equipped with the norm $\|x\|_{\mathsf{X}_{\mathscr{A}}} = \|\mathscr{A}x\|_{\mathsf{X}}$. Several authors applied this technique to construct new Banach spaces with the help of special triangles. For relevant literature, the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] can be referred.

The spaces

$$\mathsf{X}^{\alpha} = \left\{ t = (t_k) \in \omega : \sum_{k=1}^{\infty} |t_k x_k| < \infty \text{ for all } x = (x_k) \in \mathsf{X} \right\},$$

$$\mathsf{X}^{\beta} = \left\{ t = (t_k) \in \omega : \sum_{k=1}^{\infty} t_k x_k \text{ converges for all } x = (x_k) \in \mathsf{X} \right\},$$

$$\mathsf{X}^{\gamma} = \left\{ t = (t_k) \in \omega : \sup_n \left| \sum_{k=1}^{n} t_k x_k \right| < \infty \text{ for all } x = (x_k) \in \mathsf{X} \right\},$$

are called the α-, β-, γ-duals of a sequence space X, respectively.

Let $(\mathsf{X}, \|.\|_{\mathsf{X}})$ be a normed space and $B_{\mathsf{X}} = \{x \in \omega : \|x\|_{\mathsf{X}} = 1\}$. Given any BK-space $\mathsf{X} \supset \psi$ and $t = (t_n) \in \omega$,

$$\|t\|_{\mathsf{X}}^{*} = \sup_{x \in B_{\mathsf{X}}} \left| \sum_k t_k x_k \right|$$

implies that $t \in \mathsf{X}^{\beta}$.

Lemma 1.1. *[16, Theorem 1.29] $\ell_1^{\beta} = \ell_\infty$ and $\ell_p^{\beta} = \ell_q$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The equality $\|t\|_{\ell_p}^{*} = \|t\|_{\ell_p^{\beta}}$ holds for all $t \in \ell_p^{\beta}$, where $1 \leq p < \infty$.*

Lemma 1.2. *[16, Theorem 1.23 (a)] Given any BK-spaces X, Y and $\mathscr{A} \in (\mathsf{X}, \mathsf{Y})$, there exists a linear operator $\mathscr{L}_{\mathscr{A}} \in B(\mathsf{X}, \mathsf{Y})$ such that $\mathscr{L}_{\mathscr{A}}(x) = \mathscr{A}x$ for all $x \in \mathsf{X}$.*

Lemma 1.3. *[16] Let $\mathsf{X} \supset \psi$ be a BK-space and $\mathsf{Y} \in \{c_0, c, \ell_\infty\}$. If $\mathscr{A} \in (\mathsf{X}, \mathsf{Y})$, then*

$$\|\mathscr{L}_{\mathscr{A}}\| = \|\mathscr{A}\|_{(\mathsf{X},\mathsf{Y})} = \sup_{n \in \mathbb{N}} \|\mathscr{A}_n\|_{\mathsf{X}}^{*} < \infty.$$

Let $\mathscr{Q}$ be a bounded set in a metric space X and $B(x, \delta)$ be the open ball. The value

$$\chi(\mathscr{Q}) = \inf\{\varepsilon > 0 : \mathscr{Q} \subset \cup_{i=1}^{n} B(x_i, \delta_i), x_i \in \mathsf{X}, \delta_i < \varepsilon, n \in \mathbb{N}\}$$

is called the Hausdorff measure of noncompactness of $\mathscr{Q}$.

To compute the Hausdorff measure of noncompactness of a set in ℓ_p for $1 \leq p < \infty$, the following result is essential.

Theorem 1.4. *[17] Let $\mathscr{Q}$ be a bounded subset in ℓ_p for $1 \leq p < \infty$ and $P_r : \ell_p \to \ell_p$ be the operator defined by $P_r(x) = (x_0, x_1, x_2, ..., x_r, 0, 0, ...)$ for all $x = (x_k) \in \ell_p$ and each $r \in \mathbb{N}$. Then, we have*

$$\chi(\mathscr{Q}) = \lim_r \left(\sup_{x \in \mathscr{Q}} \|(I - P_r)(x)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

A linear operator $\mathscr{L} : \mathsf{X} \to \mathsf{Y}$ is a compact operator if the domain of $\mathscr{L}$ is all of X and for every bounded sequence $x = (x_n)$ in X, the sequence $(\mathscr{L}(x_n))$ has a convergent subsequence in Y. The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness. The Hausdorff measure of noncompactness of an operator $\mathscr{L} \in B(\mathsf{X}, \mathsf{Y})$, $\|\mathscr{L}\|_\chi = \chi(\mathscr{L}(B_{\mathsf{X}})) = 0$ if and only if $\mathscr{L}$ is compact.

In the theory of sequence spaces, the Hausdorff measure of noncompactness of a linear operator plays a role to characterize the compactness of an operator between BK spaces. For the relevant literature, see [18, 19, 20, 21, 22, 23, 24].

The Euler totient matrix $\Phi = (\phi_{nk})$ is defined as in [25]

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n, \end{cases}$$

where φ is the Euler totient function. In the recent time, by using this matrix, many new sequence and series spaces are defined and studied in the papers [26, 27, 28, 29, 30, 31, 32, 33].

For $p \in \mathbb{N}$ with $p \neq 1$, $\varphi(p)$ gives the number of positive integers less than p which are coprime with p and $\varphi(1) = 1$. Also, the equality

$$p = \sum_{k|p} \varphi(k)$$

holds for every $p \in \mathbb{N}$. For $p \in \mathbb{N}$ with $p \neq 1$, the Möbius function μ is defined as

$$\mu(p) = \begin{cases} (-1)^r & \text{if } p = p_1 p_2 ... p_r, \text{ where } p_1, p_2, ..., p_r \text{ are} \\ & \text{non-equivalent prime numbers} \\ 0 & \text{if } \tilde{p}^2 \mid p \text{ for some prime number } \tilde{p} \end{cases}$$

and $\mu(1) = 1$. The equality

$$\sum_{k|p} \mu(k) = 0 \tag{1.1}$$

holds except for $p = 1$.

The Riesz matrix $E = (e_{nk})$ is defined as

$$e_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad \text{if } 0 \leq k \leq n \\ 0 & , \quad \text{if } k > n, \end{cases}$$

where (q_k) is a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. By using these matrix, the authors of [34] introduced the Riesz sequence spaces of non-absolute type.

The main purpose of this study is to construct new *BK* spaces $\ell_p(R_\Phi)$ for $1 \leq p < \infty$. The matrix R_Φ is obtained by combining Euler totient matrix and Riesz matrix. After studying certain properties of the resulting spaces, α-, β- and γ-duals are computed. Finally some matrix mappings from the spaces $\ell_p(R_\Phi)$ to the classical spaces are characterized and compact operators are studied.

2. The sequence space $\ell_p(R_\Phi)$

In the present section, we introduce the sequence space $\ell_p(R_\Phi)$ by using the matrix R_Φ, where $1 \leq p < \infty$. Also, we present some theorems which give inclusion relations concerning this space.

The matrix $R_\Phi = (r_{nk})$ is defined as

$$r_{nk} = \begin{cases} \frac{q_k \varphi(k)}{Q_n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n, \end{cases}$$

where $Q_n = q_1 + q_2 + ... + q_n$. We call this matrix as *Riesz Euler Totient matrix operator*.

The inverse $R_\Phi^{-1} = (r_{nk}^{-1})$ of the matrix R_Φ is computed as

$$r_{nk}^{-1} = \begin{cases} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} & , \quad \text{if } k \mid n \\ 0 & , \quad \text{if } k \nmid n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the sequence space $\ell_p(R_\Phi)$ by

$$\ell_p(R_\Phi) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k \right|^p < \infty \right\} \quad (1 \leq p < \infty).$$

Unless otherwise stated, $y = (y_n)$ will be the R_Φ-transform of a sequence $x = (x_n)$, that is, $y_n = (R_\Phi x)_n = \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k$ for all $n \in \mathbb{N}$.

Theorem 2.1. *The space $\ell_p(R_\Phi)$ is a Banach space with the norm given by $\|x\|_{\ell_p(R_\Phi)} = \left(\sum_n \left| \frac{1}{Q_n} \sum_{k|n} q_k \varphi(k) x_k \right|^p \right)^{1/p}$, where $1 \leq p < \infty$.*

Proof. We omit the proof which is straightforward. □

Corollary 2.2. *The space $\ell_p(R_\Phi)$ is a BK-space, where $1 \leq p < \infty$.*

Theorem 2.3. *The space $\ell_p(R_\Phi)$ is linearly isomorphic to ℓ_p, where $1 \leq p < \infty$.*

Proof. Let f be a mapping defined from $\ell_p(R_\Phi)$ to ℓ_p such that $f(x) = R_\Phi x$ for all $x \in \ell_p(R_\Phi)$. It is clear that f is linear. Also it is injective since the kernel of f consists of only zero. To prove that f is surjective, consider the sequence $x = (x_n)$ whose terms are

$$x_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} y_k$$

for all $n \in \mathbb{N}$, where $y = (y_k)$ is any sequence in ℓ_p. It follows from (1.1) that

$$\begin{aligned}(R_\Phi x)_n &= \frac{1}{Q_n}\sum_{k|n} q_k\varphi(k)x_k = \frac{1}{Q_n}\sum_{k|n} q_k\varphi(k)\sum_{j|k}\frac{\mu(\frac{k}{j})}{\varphi(k)}\frac{Q_j}{q_k}y_j \\ &= \frac{1}{Q_n}\sum_{k|n}\sum_{j|k}\mu(\frac{k}{j})Q_j y_j = \frac{1}{Q_n}\sum_{k|n}\left(\sum_{j|k}\mu(j)\right)Q_{\frac{n}{k}}y_{\frac{n}{k}} = \frac{1}{Q_n}\mu(1)Q_n y_n = y_n\end{aligned}$$

and so $x = (x_n) \in \ell_p(R_\Phi)$. f preserves norms since the equality $\|x\|_{\ell_p(R_\Phi)} = \|f(x)\|_{\ell_p}$ holds. □

Remark 2.4. *The space $\ell_2(R_\Phi)$ is an inner product space with the inner product defined as $\langle x, \tilde{x}\rangle_{\ell_2(R_\Phi)} = \langle R_\Phi x, R_\Phi \tilde{x}\rangle_{\ell_2}$, where $\langle .,.\rangle_{\ell_2}$ is the inner product on ℓ_2 which induces $\|.\|_{\ell_2}$.*

Theorem 2.5. *The space $\ell_p(R_\Phi)$ is not an inner product space for $p \neq 2$.*

Proof. Consider the sequences $x = (x_n)$ and $\tilde{x} = (\tilde{x}_n)$, where

$$x_n = \begin{cases} \frac{\mu(n)}{\varphi(n)}\frac{Q_1}{q_n} + \frac{\mu(\frac{n}{2})}{\varphi(n)}\frac{Q_2}{q_n} & , \quad \text{if } n \text{ is even} \\ \frac{\mu(n)}{\varphi(n)}\frac{Q_1}{q_n} & , \quad \text{if } n \text{ is odd} \end{cases}$$

and

$$\tilde{x}_n = \begin{cases} \frac{\mu(n)}{\varphi(n)}\frac{Q_1}{q_n} - \frac{\mu(\frac{n}{2})}{\varphi(n)}\frac{Q_2}{q_n} & , \quad \text{if } n \text{ is even} \\ \frac{\mu(n)}{\varphi(n)}\frac{Q_1}{q_n} & , \quad \text{if } n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{N}$. Then, we have $R_\Phi x = (1,1,0,...,0,...) \in \ell_p$ and $R_\Phi \tilde{x} = (1,-1,0,...,0,...) \in \ell_p$. Hence, one can easily observe that

$$\|x+\tilde{x}\|_{\ell_p(R_\Phi)} + \|x-\tilde{x}\|_{\ell_p(R_\Phi)} \neq 2(\|x\|_{\ell_p(R_\Phi)} + \|\tilde{x}\|_{\ell_p(R_\Phi)}).$$

□

Theorem 2.6. *The inclusion $\ell_p(R_\Phi) \subset \ell_q(R_\Phi)$ strictly holds for $1 \leq p < q < \infty$.*

Proof. It is clear that the inclusion $\ell_p(R_\Phi) \subset \ell_q(R_\Phi)$ holds since $\ell_p \subset \ell_q$ for $1 \leq p < q < \infty$. Also, $\ell_p \subset \ell_q$ is strict and so there exists a sequence $z = (z_n)$ in $\ell_q \backslash \ell_p$. By defining a sequence $x = (x_n)$ as

$$x_n = \sum_{k|n} \frac{\mu(\frac{n}{k})}{\varphi(n)} \frac{Q_k}{q_n} z_k$$

for all $n \in \mathbb{N}$, we conclude that $x \in \ell_q(R_\Phi)\backslash\ell_p(R_\Phi)$. Hence, the desired inclusion is strict. □

Before presenting the next result, we define the sequence space $\ell_\infty(R_\Phi)$ by

$$\ell_\infty(R_\Phi) = \{x \in \omega : R_\Phi x \in \ell_\infty\}.$$

Theorem 2.7. *The inclusion $\ell_p(R_\Phi) \subset \ell_\infty(R_\Phi)$ strictly holds for $1 \leq p < \infty$.*

Proof. The inclusion is obvious since $\ell_p \subset \ell_\infty$ holds for $1 \leq p < \infty$. Let $x = (x_n)$ be a sequence such that $x_n = \sum_{k|n}(-1)^k \frac{\mu(\frac{n}{k})}{\varphi(n)}\frac{Q_k}{q_n}$ for all $n \in \mathbb{N}$. We obtain that $R_\Phi x = \left(\frac{1}{Q_n}\sum_{k|n} q_k\varphi(k)\sum_{j|k}(-1)^j\frac{\mu(\frac{k}{j})}{\varphi(k)}\frac{Q_j}{q_k}\right) = ((-1)^n) \in \ell_\infty\backslash\ell_p$ which implies that $x \in \ell_\infty(R_\Phi)\backslash\ell_p(R_\Phi)$ for $1 \leq p < \infty$. □

3. The α-, β- and γ-duals of the space $\ell_p(R_\Phi)$

In this section, we determine the α-, β- and γ-duals of the sequence space $\ell_p(R_\Phi)$, where $1 \leq p < \infty$. The following lemmas are required to prove our main results in this section. Here and in what follows $\mathscr{K}$ denotes the family of all finite subsets of $\mathbb{N}$.

Lemma 3.1. *[35] The following statements hold:*
$\mathscr{A} = (a_{nk}) \in (\ell_p, \ell_1)$ *if and only if*

$$\sup_{F \in \mathscr{K}} \sum_k \left| \sum_{n \in F} a_{nk} \right|^q < \infty \tag{3.1}$$

holds, where $1 < p < \infty$.
$\mathscr{A} = (a_{nk}) \in (\ell_\infty, \ell_1)$ *if and only if* (3.1) *holds with* $q = 1$.
$\mathscr{A} = (a_{nk}) \in (\ell_1, \ell_1)$ *if and only if*

$$\sup_k \sum_n |a_{nk}| < \infty \tag{3.2}$$

holds.
$\mathscr{A} = (a_{nk}) \in (\ell_p, c)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} \text{ exists for each } k \in \mathbb{N} \tag{3.3}$$

and

$$\sup_n \sum_k |a_{nk}|^q < \infty \tag{3.4}$$

holds, where $1 < p < \infty$.
$\mathscr{A} = (a_{nk}) \in (\ell_\infty, c)$ *if and only if* (3.3) *and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = \sum_k \left| \lim_{n\to\infty} a_{nk} \right|$$

hold.
$\mathscr{A} = (a_{nk}) \in (\ell_1, c)$ *if and only if* (3.3) *and*

$$\sup_{n,k} |a_{nk}| < \infty \tag{3.5}$$

hold.
$\mathscr{A} = (a_{nk}) \in (\ell_p, c_0)$ *if and only if*

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N} \tag{3.6}$$

and (3.4) *holds, where* $1 < p < \infty$.
$\mathscr{A} = (a_{nk}) \in (\ell_\infty, c_0)$ *if and only if* (3.6) *and*

$$\lim_{n\to\infty} \sum_k |a_{nk}| = 0$$

hold.
$\mathscr{A} = (a_{nk}) \in (\ell_1, c_0)$ *if and only if* (3.5) *and* (3.6) *hold.*
$\mathscr{A} = (a_{nk}) \in (\ell_p, \ell_\infty)$ *if and only if* (3.4) *holds, where* $1 < p < \infty$.
$\mathscr{A} = (a_{nk}) \in (\ell_\infty, \ell_\infty)$ *if and only if* (3.4) *holds with* $q = 1$.
$\mathscr{A} = (a_{nk}) \in (\ell_1, \ell_\infty)$ *if and only if* (3.5) *holds.*

In the following theorem, we determine the α-duals of the spaces $\ell_p(R_\Phi)$ $(1 < p < \infty)$ and $\ell_1(R_\Phi)$.

Theorem 3.2. *The α-duals of the spaces $\ell_p(R_\Phi)$ $(1 < p < \infty)$ and $\ell_1(R_\Phi)$ are as follows:*

$$(\ell_p(R_\Phi))^\alpha = \left\{ t = (t_n) \in \omega : \sup_{F \in \mathscr{K}} \sum_k \left| \sum_{n \in F, k|n} \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n \right|^q < \infty \right\},$$

and

$$(\ell_1(R_\Phi))^\alpha = \left\{ t = (t_n) \in \omega : \sup_k \sum_{n \in \mathbb{N}, k|n} \left| \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n \right| < \infty \right\}.$$

Proof. Consider the matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{\mu(\frac{n}{k})}{\varphi(k)} \frac{Q_k}{q_n} t_n & , \quad k \mid n \\ 0 & , \quad k \nmid n \end{cases}$$

for any sequence $t = (t_n) \in \omega$. Hence, given any $x = (x_n) \in \ell_p(R_\Phi)$ for $1 \leq p < \infty$, we have $t_n x_n = (Cy)_n$ for all $n \in \mathbb{N}$. This implies that $tx \in \ell_1$ with $x \in \ell_p(R_\Phi)$ if and only if $Cy \in \ell_1$ with $y \in \ell_p$. It follows that $t \in (\ell_p(R_\Phi))^\alpha$ if and only if $C \in (\ell_p, \ell_1)$ which completes the proof in view of Lemma 3.1. □

Theorem 3.3. *Let us define the following sets:*

$$A_1 = \left\{ t = (t_k) \in \omega : \lim_{n \to \infty} \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$A_2 = \left\{ t = (t_k) \in \omega : \sup_n \sum_k \left| \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \right|^q < \infty \right\},$$

and

$$A_3 = \left\{ t = (t_k) \in \omega : \sup_{n,k} \left| \sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j \right| < \infty \right\}.$$

The β and γ-duals of the spaces $\ell_p(R_\Phi)$ $(1 < p < \infty)$ and $\ell_1(R_\Phi)$ are as follows:
$(\ell_p(R_\Phi))^\beta = A_1 \cap A_2$ *and* $(\ell_1(R_\Phi))^\beta = A_1 \cap A_3$,
$(\ell_p(R_\Phi))^\gamma = A_2$ *and* $(\ell_1(R_\Phi))^\gamma = A_3$.

Proof. Let $t = (t_k) \in \omega$ and $B = (b_{nk})$ be an infinite matrix with terms

$$b_{nk} = \begin{cases} \sum_{j=k,k|j}^{n} t_j \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} & , \quad \text{if} \quad 1 \leq k \leq n \\ 0 & , \quad \text{if} \quad k > n. \end{cases}$$

Hence it follows that

$$\sum_{k=1}^{n} t_k x_k = \sum_{k=1}^{n} t_k \left(\sum_{j|k} \frac{\mu(\frac{k}{j})}{\varphi(k)} \frac{Q_j}{q_k} y_j \right) = \sum_{k=1}^{n} \left(\sum_{j=k,k|j}^{n} t_j \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} \right) y_k = (By)_n$$

for any $x = (x_n) \in \ell_p(R_\Phi)$. This equality yields that $tx \in cs$ for $x \in \ell_p(R_\Phi)$ if and only if $By \in c$ for $y \in \ell_p$. That is, $t \in (\ell_p(R_\Phi))^\beta$ if and only if $B \in (\ell_p, c)$ for $1 \leq p < \infty$. Hence, by Lemma 3.1, it is concluded that $(\ell_p(R_\Phi))^\beta = A_1 \cap A_2$ and $(\ell_1(R_\Phi))^\beta = A_1 \cap A_3$.

This equality also yields that $tx \in bs$ for $x \in \ell_p(R_\Phi)$ if and only if $By \in \ell_\infty$ for $y \in \ell_p$. That is, $t \in (\ell_p(R_\Phi))^\gamma$ if and only if $B \in (\ell_p, \ell_\infty)$ for $1 \leq p < \infty$. Hence, by Lemma 3.1, it is concluded that $(\ell_p(R_\Phi))^\gamma = A_2$ and $(\ell_1(R_\Phi))^\gamma = A_3$.

□

4. Some matrix transformations related to the sequence space $\ell_p(R_\Phi)$

In this section, we give the characterization of the classes $(\ell_p(R_\Phi), \mathsf{Y})$, where $1 \leq p < \infty$ and $\mathsf{Y} \in \{\ell_\infty, c, c_0, \ell_1\}$. Throughout this section, we write $d(n,k) = \sum_{j=0}^{n} d_{jk}$ for an infinite matrix $D = (d_{nk})$ and all $n,k \in \mathbb{N}$.

Theorem 4.1. *Let $1 \leq p < \infty$ and Y be any sequence space. Then, we have $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), \mathsf{Y})$ if and only if*

$$D^{(n)} = \left(d_{mk}^{(n)}\right) \in (\ell_p, c) \; for \; each \; n \in \mathbb{N},$$

$$D = (d_{nk}) \in (\ell_p, \mathsf{Y}),$$

where $d_{mk}^{(n)} = \begin{cases} 0 & , \quad k > m \\ \sum_{j=k,k|j}^{m} a_{nj} \frac{\mu(\frac{j}{k})}{\varphi(k)} \frac{Q_k}{q_j} & , \quad 0 \leq k \leq m \end{cases}$ *and* $d_{nk} = \sum_{j=k,k|j}^{\infty} a_{nj} \frac{\mu(\frac{j}{k})}{\varphi(k)} \frac{Q_k}{q_j}$ *for all $k,m,n \in \mathbb{N}$.*

Proof. We omit the proof since it follows with the same technique in [6, Theorem 4.1]. □

The following results are obtained by combining Theorem 4.1 with Lemma 3.1.

Theorem 4.2.

(a) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), \ell_\infty)$ if and only if

$$\lim_{m\to\infty} d_{mk}^{(n)} \; exists \; for \; each \; n,k \in \mathbb{N}, \tag{4.1}$$

$$\sup_{m,k} \left| d_{mk}^{(n)} \right| < \infty \; for \; each \; n \in \mathbb{N} \tag{4.2}$$

and (3.5) holds with d_{nk} instead of a_{nk}.

(b) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), c)$ if and only if (4.1) and (4.2) hold, and (3.3) and (3.5) also hold with d_{nk} instead of a_{nk}.

(c) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), c_0)$ if and only if (4.1) and (4.2) hold, and (3.5) and (3.6) also hold with d_{nk} instead of a_{nk}.

(d) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), \ell_1)$ if and only if (4.1) and (4.2) hold, and (3.2) also holds with d_{nk} instead of a_{nk}.

Theorem 4.3. *Let $1 < p < \infty$.*

(a) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), \ell_\infty)$ if and only if (4.1) and

$$\sup_m \sum_{k=0}^{m} \left| d_{mk}^{(n)} \right|^q < \infty \; for \; each \; n \in \mathbb{N} \tag{4.3}$$

hold, and (3.4) also holds with d_{nk} instead of a_{nk}.

(b) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), c)$ if and only if (4.1) and (4.3) hold, and (3.3) and (3.4) also hold with d_{nk} instead of a_{nk}.

(c) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), c_0)$ if and only if (4.1) and (4.3) hold, and (3.6) and (3.4) also hold with d_{nk} instead of a_{nk}.

(d) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), \ell_1)$ if and only if (4.1) and (4.3) hold, and (3.1) also holds with d_{nk} instead of a_{nk}.

The following results are derived by using Theorems 4.2-4.3.

Corollary 4.4. *The following statements hold:*

(a) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), bs)$ if and only if (4.1), (4.2) hold and (3.5) holds with $d(n,k)$ instead of a_{nk}.

(b) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), cs)$ if and only if (4.1), (4.2) hold and (3.3),(3.5) hold with $d(n,k)$ instead of a_{nk}.

(c) $\mathscr{A} = (a_{nk}) \in (\ell_1(R_\Phi), cs_0)$ if and only if (4.1), (4.2) hold and (3.5),(3.6) hold with $d(n,k)$ instead of a_{nk}.

Corollary 4.5. *Let $1 < p < \infty$. Then, the following statements hold:*

(a) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), bs)$ if and only if (4.1), (4.3) hold and (3.4) holds with $d(n,k)$ instead of a_{nk}.

(b) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), cs)$ if and only if (4.1), (4.3) hold and (3.3),(3.4) hold with $d(n,k)$ instead of a_{nk}.

(c) $\mathscr{A} = (a_{nk}) \in (\ell_p(R_\Phi), cs_0)$ if and only if (4.1), (4.3) hold and (3.4),(3.6) hold with $d(n,k)$ instead of a_{nk}.

5. Compact operators on the space $\ell_p(R_\Phi)$

Let the matrix $\tilde{\mathscr{A}} = (\tilde{a}_{nk})$ defined by an infinite matrix $\mathscr{A} = (a_{nk})$ as

$$\tilde{a}_{nk} = \sum_{j=k,k|j}^{\infty} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} a_{nj}$$

for all $n,k \in \mathbb{N}$.

For a sequence $t = (t_k) \in \omega$, define a sequence $\tilde{t} = (\tilde{t}_k)$ as $\tilde{t}_k = \sum_{j=k,k|j}^{\infty} \frac{\mu(\frac{j}{k})}{\varphi(j)} \frac{Q_k}{q_j} t_j$ for all $k \in \mathbb{N}$.

Lemma 5.1. *Let $t = (t_k) \in (\ell_p(R_\Phi))^\beta$, where $1 \le p < \infty$. Then $\tilde{t} = (\tilde{t}_k) \in \ell_q$ and*

$$\sum_k t_k x_k = \sum_k \tilde{t}_k y_k$$

for all $x = (x_k) \in \ell_p(R_\Phi)$.

Lemma 5.2. *The following statements hold.*
(a) $\|t\|^*_{\ell_1(R_\Phi)} = \sup_k |\tilde{t}_k| < \infty$ for all $t = (t_k) \in (\ell_1(R_\Phi))^\beta$.
(b) $\|t\|^*_{\ell_p(R_\Phi)} = (\sum_k |\tilde{t}_k|^q)^{1/q} < \infty$ for all $t = (t_k) \in (\ell_p(R_\Phi))^\beta$, where $1 < p < \infty$.

Lemma 5.3. *Let X be any sequence space and $\mathscr{A} = (a_{nk})$ be an infinite matrix. If $\mathscr{A} \in (\ell_p(R_\Phi), \mathsf{X})$, then $\tilde{\mathscr{A}} \in (\ell_p, \mathsf{X})$ and $\mathscr{A}x = \tilde{\mathscr{A}}y$ for all $x \in \ell_p(R_\Phi)$, where $1 \le p < \infty$.*

Proof. It follows from Lemma 5.1. □

Lemma 5.4. *If $\mathscr{A} \in (\ell_1(R_\Phi), \ell_p)$, then we have*

$$\|\mathscr{L}_{\mathscr{A}}\| = \|\mathscr{A}\|_{(\ell_1(R_\Phi),\ell_p)} = \sup_k \left(\sum_n |\tilde{a}_{nk}|^p\right)^{1/p} < \infty,$$

where $1 \le p < \infty$.

Lemma 5.5. *[22, Theorem 3.7] Let $\mathsf{X} \supset \psi$ be a BK-space. Then, the following statements hold.*
(a) $\mathscr{A} \in (\mathsf{X}, \ell_\infty)$, then $0 \le \|\mathscr{L}_{\mathscr{A}}\|_\chi \le \limsup_n \|\mathscr{A}_n\|^*_{\mathsf{X}}$.
(b) $\mathscr{A} \in (\mathsf{X}, c_0)$, then $\|\mathscr{L}_{\mathscr{A}}\|_\chi = \limsup_n \|\mathscr{A}_n\|^*_{\mathsf{X}}$.
(c) If X has AK or $\mathsf{X} = \ell_\infty$ and $\mathscr{A} \in (\mathsf{X}, c)$, then

$$\frac{1}{2}\limsup_n \|\mathscr{A}_n - a\|^*_{\mathsf{X}} \le \|\mathscr{L}_{\mathscr{A}}\|_\chi \le \limsup_n \|\mathscr{A}_n - a\|^*_{\mathsf{X}},$$

where $a = (a_k)$ and $a_k = \lim_n a_{nk}$ for each $k \in \mathbb{N}$.

Lemma 5.6. *[22, Theorem 3.11] Let $\mathsf{X} \supset \psi$ be a BK-space. If $\mathscr{A} \in (\mathsf{X}, \ell_1)$, then*

$$\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\|\sum_{n \in N} \mathscr{A}_n\right\|^*_{\mathsf{X}}\right) \le \|\mathscr{L}_{\mathscr{A}}\|_\chi \le 4\lim_r \left(\sup_{N \in \mathscr{K}_r} \left\|\sum_{n \in N} \mathscr{A}_n\right\|^*_{\mathsf{X}}\right)$$

and $\mathscr{L}_{\mathscr{A}}$ is compact if and only if $\lim_r \left(\sup_{N \in \mathscr{K}_r} \|\sum_{n \in N} \mathscr{A}_n\|^*_{\mathsf{X}}\right) = 0$, where $\mathscr{K}_r$ is the subcollection of $\mathscr{K}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than r.

Theorem 5.7. *Let $1 < p < \infty$.*

1. *For $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$,*

$$0 \le \|\mathscr{L}_{\mathscr{A}}\|_\chi \le \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$$

holds.

2. *For* $\mathscr{A} \in (\ell_p(R_\Phi), c)$,

$$\frac{1}{2}\limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q\right)^{1/q} \leq \|\mathscr{L}_\mathscr{A}\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q\right)^{1/q}$$

holds, where $\tilde{a} = (\tilde{a}_k)$ *and* $\tilde{a}_k = \lim_n \tilde{a}_{nk}$ *for each* $k \in \mathbb{N}$.

3. *For* $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$,

$$\|\mathscr{L}_\mathscr{A}\|_\chi = \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$$

holds.

4. *For* $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$,

$$\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)} \leq \|\mathscr{L}_\mathscr{A}\|_\chi \leq 4\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)}$$

holds, where $\|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi),\ell_1)} = \sup_{N\in\mathscr{K}_r} \left(\sum_k |\sum_{n\in N} \tilde{a}_{nk}|^q\right)^{1/q}$ $(r \in \mathbb{N})$.

Proof.

1. Let $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$. Since the series $\sum_{k=1}^\infty a_{nk}x_k$ converges for each $n \in \mathbb{N}$, we have $\mathscr{A}_n \in (\ell_p(R_\Phi))^\beta$. From Lemma 5.2 (b), we write $\|\mathscr{A}_n\|^*_{\ell_p(R_\Phi)} = \|\tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n}\|_{\ell_q} = \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$ for each $n \in \mathbb{N}$. By using Lemma 5.5 (a), we conclude that

$$0 \leq \|\mathscr{L}_\mathscr{A}\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}.$$

2. Let $\mathscr{A} \in (\ell_p(R_\Phi), c)$. By Lemma 5.3, we have $\tilde{\mathscr{A}} \in (\ell_p, c)$. Hence, from Lemma 5.5 (c), we write

$$\frac{1}{2}\limsup_n \|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p} \leq \|\mathscr{L}_\mathscr{A}\|_\chi \leq \limsup_n \|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p},$$

where $\tilde{a} = (\tilde{a}_k)$ and $\tilde{a}_k = \lim_n \tilde{a}_{nk}$ for each $k \in \mathbb{N}$. Moreover, Lemma 1.1 implies that $\|\tilde{\mathscr{A}_n} - \tilde{a}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n} - \tilde{a}\|_{\ell_q} = \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q\right)^{1/q}$ for each $n \in \mathbb{N}$. This completes the proof.

3. Let $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$. Since we have $\|\mathscr{A}_n\|^*_{\ell_p(R_\Phi)} = \|\tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\tilde{\mathscr{A}_n}\|_{\ell_q} = \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}$ for each $n \in \mathbb{N}$, we conclude from Lemma 5.5 (b) that

$$\|\mathscr{L}_\mathscr{A}\|_\chi = \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q\right)^{1/q}.$$

4. Let $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$. By Lemma 5.3, we have $\tilde{\mathscr{A}} \in (\ell_p, \ell_1)$. It follows from Lemma 5.6 that

$$\lim_r \left(\sup_{N\in\mathscr{K}_r} \left\|\sum_{n\in N} \tilde{\mathscr{A}_n}\right\|^*_{\ell_p}\right) \leq \|\mathscr{L}_\mathscr{A}\|_\chi \leq 4\lim_r \left(\sup_{N\in\mathscr{K}_r} \left\|\sum_{n\in N} \tilde{\mathscr{A}_n}\right\|^*_{\ell_p}\right).$$

Moreover, Lemma 1.1 implies that $\|\sum_{n\in N} \tilde{\mathscr{A}_n}\|^*_{\ell_p} = \|\sum_{n\in N} \tilde{\mathscr{A}_n}\|_{\ell_q} = \left(\sum_k |\sum_{n\in N} \tilde{a}_{nk}|^q\right)^{1/q}$ which completes the proof.

□

Corollary 5.8. *Let* $1 < p < \infty$.

1. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), \ell_\infty)$ if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0.$$

2. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), c)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k|^q \right)^{1/q} = 0.$$

3. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), c_0)$ if and only if*

$$\lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0.$$

4. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_p(R_\Phi), \ell_1)$ if and only if*

$$\lim_r \|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi), \ell_1)} = 0,$$

where $\|\mathscr{A}\|^{(r)}_{(\ell_p(R_\Phi), \ell_1)} = \sup_{N \in \mathscr{K}_r} \left(\sum_k |\sum_{n \in N} \tilde{a}_{nk}|^q \right)^{1/q}$.

Theorem 5.9.

1. *For $\mathscr{A} \in (\ell_1(R_\Phi), \ell_\infty)$,*

$$0 \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

holds.

2. *For $\mathscr{A} \in (\ell_1(R_\Phi), c)$,*

$$\frac{1}{2} \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) \leq \|\mathscr{L}_{\mathscr{A}}\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right)$$

holds.

3. *For $\mathscr{A} \in (\ell_1(R_\Phi), c_0)$,*

$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

holds.

4. *For $\mathscr{A} \in (\ell_1(R_\Phi), \ell_1)$,*

$$\|\mathscr{L}_{\mathscr{A}}\|_\chi = \lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}| \right)$$

holds.

Proof. It follows with the same technique in Theorem 5.7. □

Corollary 5.10.

1. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), \ell_\infty)$ if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

2. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), c)$ if and only if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) = 0.$$

3. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), c_0)$ if and only if*

$$\lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

4. *$\mathscr{L}_{\mathscr{A}}$ is compact for $\mathscr{A} \in (\ell_1(R_\Phi), \ell_1)$ if and only if*

$$\lim_r \left(\sup_k \sum_{n=r}^{\infty} |\tilde{a}_{nk}| \right) = 0.$$

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