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# The Narayana Polynomial and Narayana Hybrinomial Sequences

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**Abstract**

Hybrid numbers are generalization of complex, hyperbolic and dual numbers. In this paper we introduce the Narayana polynomial sequence(or polynomial sequence of Narayana's cows) and related Narayana hybrinomial sequence. We present Binet-like formula, generating function, exponential generating function of these sequences. In addition we give some identities such as Catalan-like identity, Cassini-like identity and Ocagne-like identity for these sequences.

***Keywords:*** *Hybrit numbers, Narayana sequence, Polynomials, Hybrinomials.*
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## 1. Introduction

Due to the important directions of well-known sequences such as the Fibonacci, Lucas, Pell, Pell Lucas, Padovan, Perrin, Narayana sequences in mathematics, statistics and other branches of sciences, so many researcher studied them. As we know complex numbers and hyperbolic numbers have fundamental rule in mathematics and engineering sciences like as computer sciences, electronic engineering, and mechanical engineering. For more information about Narayana sequence (or Narayana's cows sequence), Padovan and Perrin sequences and related number sequences we refer to [1, 2, 3, 4, 5, 6, 7] which have combined sequences with different topics. In [8], Özdemir introduced the hybrid numbers as a generalization of complex, hyperbolic and dual numbers.
The set $H$ of hybrid numbers $z$, is of the form

$$H = \{z = a+bi+c\varepsilon+dh;\ a,b,c,d \in \mathbb{R}\} \tag{1.1}$$

where $i,\varepsilon,h$ are operators such that

$$i^2=-1, \quad \varepsilon^2=0, \quad ih=-hi=\varepsilon+i.$$

For more information about these operators see [8]. The conjugate of hybrid number $z$ is defined by

$$\bar{z}=\overline{a+bi+c\varepsilon+dh}=a-bi-c\varepsilon-dh.$$

The real number $C(z)=z\bar{z}=\bar{z}z=a^2+b^2-2bc-d^2$ is called the character of the hybrid number $z$ and the real number $\sqrt{|C(z)|}$ will be called the norm of the hybrid number $z$ and will be denoted by $\|z\|$ (see [8]).
Jacobsthal and Jacobsthal-Lucas hybrid numbers are introduced by Liana and Wloch in [9], and also they investigated some of their properties. In [10, 11], the authors have mentioned the different properties of the Horadam hybrid numbers. In [12], modified $k-$Pell hybrid sequence is defined and some identities are obtained. In [13, 14], the authors examined the most common Narayana hybrid numbers and Van Der Laan hybrid numbers. Also Polatlı [15], defined hybrid numbers with Fibonacci and Lucas hybrid number coefficients. The first information about hybrinomial sequence is found in [16] and in [17], where the authors generalized the recurrence relations of the second type of hybrinomials. In [18], Wloch introduced Pell hybrinomial sequences and presented a lot of results about the sequence. Also, the same authors A. Szynal-Liana and I. Włoch considered generalized Fibonacci-Pell hybrinomials in [19].

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In this paper, firstly we consider the Narayana polynomial sequence (or polynomial sequence of Narayana's cows). We present Binet–like formula, generating function of this sequence. Then the special kind of hybrinomial sequence is introduced, called Narayana hybrinomial sequence. Binet's formula, generating function, character and norm of this sequence are investigated. In addition, we give some formulas for the Catalan-like identity, Cassini-like identity and d'Ocagne-like identity for these sequences.

## 2. Narayana Sequence (or Narayana's Cows Sequence)

The Narayana sequence (or Narayana's cows sequence) initially was introduced by the Indian mathematician Narayana in the 14th century, while studying the following problem of a herd of cows and calves: A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years? (see [20]). If $m$ is the year, then the Narayana problem can be modelled by the recurrence relation

$$N_{m+3} = N_{m+2} + N_m$$

for all $m \geq 0,$ with initial values $N_0 = 2, N_1 = 3, N_2 = 4$. This sequence is called Narayana sequence (or Narayana's cows sequence). The first values of $\{N_m\}$ are 2,3,4,6,9,13,19,28,41,60, 88,129,189,277. Also the Binet-like formula (see [13]), for the Narayana sequence is:

$$N_m = \frac{r_1^{m+1}}{(r_1 - r_2)(r_1 - r_3)} + \frac{r_2^{m+1}}{(r_2 - r_1)(r_2 - r_3)} + \frac{r_3^{m+1}}{(r_1 - r_3)(r_2 - r_3)}$$

where $r_1, r_2, r_3$ are the roots of the characteristic equation $x^3 - x^2 - 1 = 0$ of Narayana sequence $N_m$ ( one of these roots is real number and the others are complex and conjugate) and given by

$$\begin{aligned}
r_1 &= \frac{1}{3}\left(1 + \sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} + \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}\right), \\
r_2 &= \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}, \\
r_3 &= \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{29}{2} - \frac{3\sqrt{93}}{2}} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}.
\end{aligned}$$

Also in [21, 22], information about $k-$Narayana sequence is provided.
As a last piece of information, there is another sequence of Narayana to the same name has in the literature, the person who found these sequence is T. V. Narayana not an Indian mathematician and the terms of the sequence are determined by

$$N(n,k) = \frac{1}{n}\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} n \\ k-1 \end{pmatrix}$$

where $1 \leq k \leq n$.

## 3. Narayana Polynomial Sequence and Narayana Hybrinomial Sequence

In this section we define Narayana polynomial sequence (or polynomial sequence of Narayana's cows) and related Narayana hybrinomial sequence. We prove some theorems and give some identities for these sequences.

**Definition 3.1.** *Narayana polynomial (or polynomial sequence of Narayana's cows) denoted by $\{N_m(x)\}$ is defined by*

$$N_m(x) = \begin{cases} 2 & m = 0 \\ 3 & m = 1 \\ 4 & m = 2 \\ xN_{m-1}(x) + N_{m-3}(x) & m \geq 3. \end{cases} \tag{3.1}$$

The first few terms of $\{N_m(x)\}$ are:

$2,\ 3,\ 4,\ 4x+2,\ 4x^2+2x+3,\ 4x^3+2x^2+3x+4,\ 4x^4+2x^3+3x^2+8x+2, \cdots.$

**Definition 3.2.** *Narayana hybrinomial sequence denoted by $\{N_m^{[H]}(x)\}$ is defined by*

$$N_m^{[H]}(x) = N_m(x) + N_{m+1}(x)i + N_{m+2}(x)\varepsilon + N_{m+3}(x)h, \tag{3.2}$$

*where $\{N_m(x)\}$ is the Narayana polynomial sequence (or polynomial sequence of Narayana's cows). Thus the initial values of Narayana hybrinomial sequence are:*

$$\begin{aligned}
N_0^{[H]}(x) &= 2 + 3i + 4\varepsilon + (4x+2)h \\
N_1^{[H]}(x) &= 3 + 4i + (4x+2)\varepsilon + (4x^2+2x+3)h \\
N_2^{[H]}(x) &= 4 + (4x+2)i + (4x^2+2x+3)\varepsilon + (4x^3+2x^2+3x+4)h \\
N_3^{[H]}(x) &= (4x+2) + (4x^2+2x+3)i + (4x^3+2x^2+3x+4)\varepsilon + (4x^4+2x^3+3x^2+8x+2)h.
\end{aligned}$$

According to the definition of Narayana hybrinomial sequence, for $x=1$ we see that

$$\begin{aligned} N_0^{[H]}(x) &= 2+3i+4\varepsilon+6h \\ N_1^{[H]}(x) &= 3+4i+6\varepsilon+9h \\ N_2^{[H]}(x) &= 4+6i+9\varepsilon+13h \\ N_3^{[H]}(x) &= 6+9i+13\varepsilon+19h. \end{aligned}$$

Therefore the Narayana hybrinomial sequence $\{N_m^{[H]}(x)\}$ is a generalization of the Narayana hybrid numbers $\{N_m^{[H]}\}$ which we introduced in [13].
Through the definition of the Narayana hybrinomial sequence and character of a hybrid numbers (see [8]), we get the following relation about the character of Narayana hybrinomial sequence:

$$C\left(N_n^{[H]}(x)\right) = N_n^2(x)+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x). \tag{3.3}$$

Now we have the following theorem about the norm of Narayana hybrinomial sequence.

**Theorem 3.3.** *Let $\{N_n^{[H]}(x)\}$ is the Narayana hybrinomial sequence. The norm of $\{N_n^{[H]}(x)\}$ is given by*

$$\left|\left|N_n^{[H]}(x)\right|\right|^2 = \left|N_n^2(x)+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x)\right|.$$

*Proof.* By (3.3) and (1.1), we find that

$$\left|\left|N_n^{[H]}(x)\right|\right| = \sqrt{C\left(N_n^{[H]}(x)\right)} = \sqrt{\left|N_n^2(x)+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x)\right|}.$$

Hence we get

$$\begin{aligned} \left|\left|N_n^{[H]}\right|\right|^2 &= \left|N_n^2(x)+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x)\right| \\ &= \left|(N_{n+3}(x)-xN_{n+2}(x))^2+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x)\right| \\ &= \left|N_{n+3}^2(x)+x^2N_{n+2}^2(x)-2xN_{n+3}(x)N_{n+2}(x)+N_{n+1}^2(x)-2N_{n+1}(x)N_{n+2}(x)-N_{n+3}^2(x)\right| \\ &= \left|N_{n+1}^2(x)+N_{n+2}(x)(x^2N_{n+2}(x)-2xN_{n+3}(x)-2N_{n+1}(x))\right| \\ &= \left|N_{n+1}^2(x)-N_{n+2}(x)(x^2N_{n+2}(x)+2N_{n+1}(x)+2xN_n(x))\right|. \end{aligned}$$

Thus the proof is completed. □

Özdemir in [8], defined the matrix representation of hybrid numbers by following relation:

$$M(a+bi+c\varepsilon+dh) = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix}.$$

Now we show the matrix representation of the Narayana hybrinomials.

**Theorem 3.4.** *Let $\{N_r^{[H]}(x)\}$ is the Narayana hybrinomial sequence. The matrix representation of $M_{N_r^{[H]}(x)}$ is defined as*

$$M(N_r^{[H]}(x)) = \begin{bmatrix} N_r(x)+N_{r+2}(x) & N_{r+1}(x)+(x-1)N_{r+2}(x)+N_r(x) \\ (x-1)N_{r+1}(x)+N_{r-1}(x)+N_{r+3}(x) & N_r(x)-N_{r+2}(x) \end{bmatrix}. \tag{3.4}$$

*Proof.* By definition of the Narayana hybrinomial sequence (3.2) and the matrix representation of hybrid numbers introduced by Özdemir [8], we have

$$\begin{aligned} M(N_r^{[H]}(x)) &= N_r(x)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + N_{r+1}(x)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + N_{r+2}(x)\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + N_{r+3}(x)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} N_r(x)+N_{r+2}(x) & N_{r+1}(x)-N_{r+2}(x)+N_{r+3}(x) \\ -N_{r+1}(x)+N_{r+2}(x)+N_{r+3}(x) & N_r(x)-N_{r+2}(x) \end{bmatrix} \\ &= \begin{bmatrix} N_r(x)+N_{r+2}(x) & N_{r+1}(x)-N_{r+2}(x)+xN_{r+2}(x)+N_r(x) \\ -N_{r+1}(x)+xN_{r+1}(x)+N_{r-1}(x)+N_{r+3}(x) & N_r(x)-N_{r+2}(x) \end{bmatrix} \\ &= \begin{bmatrix} N_r(x)+N_{r+2}(x) & N_{r+1}(x)+(x-1)N_{r+2}(x)+N_r(x) \\ (x-1)N_{r+1}(x)+N_{r-1}(x)+N_{r+3}(x) & N_r(x)-N_{r+2}(x) \end{bmatrix}. \end{aligned}$$

□

**Corollary 3.5.** *Let $\{N_r^{[H]}(x)\}$ is the Narayana hybrinomial sequence. We know that $M_{N_r^{[H]}(x)}$ is the matrix representation of $\{N_r^{[H]}(x)\}$, then*
$\det\left(M(N_r^{[H]}(x))\right) = C\left(N_n^{[H]}(x)\right) = N_r^2(x) + N_{r+1}^2(x) - 2N_{r+1}(x)N_{r+2}(x) - N_{r+3}^2(x).$

*Proof.* We know from Theorem 3.2 that, (3.4) is the matrix representation of Narayana hybrinomial sequence. Then we have

$$\begin{aligned}\det\left(M(N_r^{[H]}(x))\right) &= \det\left(\begin{bmatrix} N_r(x)+N_{r+2}(x) & N_{r+1}(x)+(x-1)N_{r+2}(x)+N_r(x) \\ (x-1)N_{r+1}(x)+N_{r-1}(x)+N_{r+3}(x) & N_r(x)-N_{r+2}(x)\end{bmatrix}\right) \\ &= \left|(N_r(x)+N_{r+2}(x))(N_r(x)-N_{r+2}(x)) - (N_{r+1}(x)+(x-1)N_{r+2}(x)+N_r(x))((x-1)N_{r+1}(x)+N_{r-1}(x)+N_{r+3}(x))\right|.\end{aligned}$$

After some computations we have

$$\det\left(M(N_r^{[H]}(x))\right) = N_r^2(x) + N_{r+1}^2(x) - 2N_{r+1}(x)N_{r+2}(x) - N_{r+3}^2(x) = C\left(N_n^{[H]}(x)\right).$$

□

**Lemma 3.6.** *Let $\{N_n^{[H]}(x)\}$ is the Narayana hybrinomial sequence and $n \geq 0$ be an integer. Then the recurrence relation of $\{N_n^{[H]}(x)\}$ is*
$N_n^{[H]}(x) = xN_{n-1}^{[H]}(x) + N_{n-3}^{[H]}(x).$

*Proof.* By (3.2), we get

$$\begin{aligned} N_n^{[H]}(x) - xN_{n-1}^{[H]}(x) - N_{n-3}^{[H]}(x) &= (N_n(x) - xN_{n-1}(x) - N_{n-3}(x)) + (N_{n+1}(x) - xN_n(x) - N_{n-2}(x))i \\ &+ (N_{n+2}(x) - xN_{n+1}(x) - N_{n-1}(x))\varepsilon + (N_{n+3}(x) - xN_{n+2}(x) - N_n(x))h,\end{aligned}$$

Since $\{N_m(x)\}$ is a Narayana polynomial sequence, consequently the right side of the above equality is equal to zero. Therefore we conclude that

$$N_n^{[H]}(x) - xN_{n-1}^{[H]}(x) - N_{n-3}^{[H]}(x) = 0$$

Thus the proof is completed. □

**Theorem 3.7.** *Let $\{N_n^{[H]}(x)\}$ is the Narayana hybrinomial sequence. The generating function for $\{N_n^{[H]}(x)\}$ is*

$$\sum_{n=0}^{\infty} N_n^{[H]}(x)t^n = \frac{N_0^{[H]}(x) + \left(N_1^{[H]}(x) - xN_0^{[H]}(x)\right)t + \left(N_2^{[H]}(x) - xN_1^{[H]}(x)\right)t^2}{1 - tx - t^3}.$$

*Proof.* Suppose that the generating function for the Narayana hybrinomials, $\{N_n^{[H]}(x)\}$ has the following formal power series

$$g(t) = \sum_{n=0}^{\infty} N_n^{[H]}(x)t^n = N_0^{[H]}(x) + N_1^{[H]}(x)t + N_2^{[H]}(x)t^2 + N_3^{[H]}(x)t^3 + \cdots.$$

Then we have

$$\begin{aligned} xtg(t) &= xN_0^{[H]}(x)t + xN_1^{[H]}(x)t^2 + xN_2^{[H]}(x)t^3 + xN_3^{[H]}(x)t^4 + \cdots \\ t^3g(t) &= N_0^{[H]}(x)t^3 + N_1^{[H]}(x)t^4 + N_2^{[H]}(x)t^5 + N_3^{[H]}(x)t^6 + \cdots\end{aligned}$$

Therefore we get

$$\begin{aligned} g(t) - xtg(t) - t^3g(t) &= \left(N_0^{[H]}(x) + N_1^{[H]}(x)t + N_2^{[H]}(x)t^2 + N_3^{[H]}(x)t^3 + \cdots\right) - \left(xN_0^{[H]}(x)t + xN_1^{[H]}(x)t^2 + xN_2^{[H]}(x)t^3 + xN_3^{[H]}(x)t^4 + \cdots\right) \\ &- \left(N_0^{[H]}(x)t^3 + N_1^{[H]}(x)t^4 + N_2^{[H]}(x)t^5 + N_3^{[H]}(x)t^6 + \cdots\right) \\ &= N_0^{[H]}(x) + (N_1^{[H]}(x) - xN_0^{[H]}(x))t + (N_2^{[H]}(x) - xN_1^{[H]}(x))t^2 + (N_3^{[H]}(x) - xN_2^{[H]}(x) - N_0^{[H]}(x))t^3 + \cdots \\ &+ (N_m^{[H]}(x) - xN_{m-1}^{[H]}(x) - N_{m-3}^{[H]}(x))t^n + \cdots.\end{aligned}$$

By Lemma 3.6 we find that

$$g(t)(1 - tx - t^3) = N_0^{[H]}(x) + (N_1^{[H]}(x) - xN_0^{[H]}(x))t + (N_2^{[H]}(x) - xN_1^{[H]}(x))t^2.$$

Therefore we get

$$\sum_{n=0}^{\infty} N_n^{[H]}(x)t^n = \frac{N_0^{[H]}(x) + \left(N_1^{[H]}(x) - xN_0^{[H]}(x)\right)t + \left(N_2^{[H]}(x) - xN_1^{[H]}(x)\right)t^2}{1 - tx - t^3}.$$

□

**Lemma 3.8.** *Let $\{N_r(x)\}$ is the Narayana polynomial sequence and $r \geq 0$ be an integer. The Binet-like formula for $\{N_r(x)\}$ is*

$$N_r(x) = \frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r + \frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^r + \frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r,$$

*where*

$$\begin{aligned} k_1 &= 2\alpha^2 + (3-2x)\alpha + 4 - 3x, \\ k_2 &= 2\beta^2 + (3-2x)\beta + 4 - 3x, \\ k_3 &= 2\gamma^2 + (3-2x)\gamma + 4 - 3x, \end{aligned}$$

*and $\alpha, \beta, \gamma$ are the roots of the characteristic equation $1 - tx - t^3 = 0$.*

*Proof.* We know that the recurrence relation $N_m(x) = xN_{m-1}(x) + N_{m-3}(x)$ has the characteristic equation $f(t) = t^3 - xt^2 - 1 = 0$. For an arbitrary value of $x$ we know that this equation has three distinct roots $\alpha, \beta, \gamma$. Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of $h(t) = f(\frac{1}{t}) = 1 - xt - t^3$. In exact we have

$$h(x) = 1 - xt - t^3 = (1-\alpha t)(1-\beta t)(1-\gamma t).$$

According to the generating function of Narayana polynomial sequence we have

$$\begin{aligned} G(t) &= \frac{2 + (3-2x)t + (4-3x)t^3}{1 - xt - t^3} \\ &= \frac{A}{1-\alpha t} + \frac{B}{1-\beta t} + \frac{C}{1-\gamma t} \\ &= A\sum_{r=0}^{\infty}(\alpha t)^r + B\sum_{r=0}^{\infty}(\beta t)^r + C\sum_{r=0}^{\infty}(\gamma t)^r. \end{aligned} \tag{3.5}$$

Thus we have

$$\begin{aligned} G(t) &= \frac{2 + (3-2x)t + (4-3x)t^2}{1 - xt - t^3} \\ &= \frac{A(1-\beta t)(1-\gamma t) + B(1-\alpha t)(1-\gamma t) + C(1-\alpha t)(1-\beta t)}{(1-\alpha t)(1-\beta t)(1-\gamma t)}. \end{aligned}$$

Therefore by comparison of the left and right sides of this equality we get that

$$2 + (3-2x)t + (4-3x)t^2 = A(1-\beta t)(1-\gamma t) + B(1-\alpha t)(1-\gamma t) + C(1-\alpha t)(1-\beta t).$$

If we substitute $t$ by $1/\alpha$ we find that

$$2 + (3-2x)\frac{1}{\alpha} + (4-3x)(\frac{1}{\alpha})^2 = A(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha}).$$

Hence we get

$$2\alpha^2 + (3-2x)\alpha + 4 - 3x = A(\alpha-\beta)(\alpha-\gamma).$$

Consequently we obtain

$$A = \frac{2\alpha^2 + (3-2x)\alpha + 4 - 3x}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly we have

$$B = \frac{2\beta^2 + (3-2x)\beta + 4 - 3x}{(\beta-\alpha)(\beta-\gamma)},\ C = \frac{2\gamma^2 + (3-2x)\gamma + 4 - 3x}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus by (3.5), we get

$$\begin{aligned} G(t) &= \sum_{r=0}^{\infty}\frac{(2\alpha^2 + (3-2x)\alpha + 4 - 3x)\alpha^r}{(\alpha-\beta)(\alpha-\gamma)}t^r + \sum_{r=0}^{\infty}\frac{(2\beta^2 + (3-2x)\beta + 4 - 3x)\beta^r}{(\beta-\alpha)(\beta-\gamma)}t^r + \sum_{r=0}^{\infty}\frac{(2\gamma^2 + (3-2x)\gamma + 4 - 3x)\gamma^r}{(\gamma-\alpha)(\gamma-\beta)}t^r \\ &= \sum_{r=0}^{\infty}\left\{ \begin{array}{c} \frac{(2\alpha^2 + (3-2x)\alpha + 4 - 3x)\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2 + (3-2x)\beta + 4 - 3x)\beta^r}{(\beta-\alpha)(\beta-\gamma)} \\ + \frac{(2\gamma^2 + (3-2x)\gamma + 4 - 3x)\gamma^r}{(\gamma-\alpha)(\gamma-\beta)} \end{array} \right\} t^r. \end{aligned}$$

Consequently we obtain

$$N_r(x) = \left\{ \begin{array}{c} \frac{(2\alpha^2 + (3-2x)\alpha + 4 - 3x)\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(2\beta^2 + (3-2x)\beta + 4 - 3x)\beta^r}{(\beta-\alpha)(\beta-\gamma)} \\ + \frac{(2\gamma^2 + (3-2x)\gamma + 4 - 3x)\gamma^r}{(\gamma-\alpha)(\gamma-\beta)} \end{array} \right\}.$$

By taking

$$\begin{aligned} k_1 &= 2\alpha^2+(3-2x)\alpha+4-3x, \\ k_2 &= 2\beta^2+(3-2x)\beta+4-3x, \\ k_3 &= 2\gamma^2+(3-2x)\gamma+4-3x, \end{aligned}$$

we get

$$N_r(x)=\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^r+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r.$$

Thus the proof is completed. □

**Theorem 3.9.** *Let $\{N_r^{[H]}(x)\}$ is the Narayana hybrinomial sequence and $r\geq 0$ be an integer. The Binet-like formula for $\{N_r^{[H]}(x)\}$ is*

$$N_r^{[H]}(x) = \left(\frac{k_1(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^r+\left(\frac{k_2(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^r+\left(\frac{k_3(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^r, \tag{3.6}$$

*where*

$$\begin{aligned} k_1 &= 2\alpha^2+(3-2x)\alpha+4-3x, \\ k_2 &= 2\beta^2+(3-2x)\beta+4-3x, \\ k_3 &= 2\gamma^2+(3-2x)\gamma+4-3x. \end{aligned}$$

*Proof.* By definition of Narayana hybrinomial sequence (3.2), we have

$$N_r^{[H]}(x)=N_r(x)+N_{r+1}(x)i+N_{r+2}(x)\varepsilon+N_{r+3}(x)h$$

By Lemma 3.2 we have

$$N_r(x)=\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^r+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r,$$

where

$$\begin{aligned} k_1 &= 2\alpha^2+(3-2x)\alpha+4-3x, \\ k_2 &= 2\beta^2+(3-2x)\beta+4-3x, \\ k_3 &= 2\gamma^2+(3-2x)\gamma+4-3x. \end{aligned}$$

Thus we get

$$\begin{aligned} N_r^{[H]}(x) &= \left(\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^r+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r\right) \\ &\quad+\left(\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+1}+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+1}+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+1}\right)i \\ &\quad+\left(\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+2}+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+2}+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+2}\right)\varepsilon \\ &\quad+\left(\frac{k_1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r+3}+\frac{k_2}{(\beta-\alpha)(\beta-\gamma)}\beta^{r+3}+\frac{k_3}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r+3}\right)h. \end{aligned}$$

Consequently by some computations we have

$$N_r^{[H]}(x) = \left(\frac{k_1(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^r+\left(\frac{k_2(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^r+\left(\frac{k_3(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^r.$$

□

**Corollary 3.10.** *Let $\{N_n^{[H]}(x)\}$ is the Narayana hybrinomial sequence. The exponential generating function for $\{N_n^{[H]}(x)\}$ is*

$$\sum_{n=0}^{\infty}N_n^{[H]}(x)\frac{t^n}{n!} = \left(\frac{k_1(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)e^{\alpha t}+\left(\frac{k_2(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)e^{\beta t}+\left(\frac{k_3(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)e^{\gamma t},$$

*where*

$$\begin{aligned} k_1 &= 2\alpha^2+(3-2x)\alpha+4-3x, \\ k_2 &= 2\beta^2+(3-2x)\beta+4-3x, \\ k_3 &= 2\gamma^2+(3-2x)\gamma+4-3x. \end{aligned}$$

*Proof.* By using the Binet-like formula for the Narayana hybrinomial sequence, we have

$$\begin{aligned} \sum_{n=0}^{\infty}N_n^{[H]}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left[\left(\frac{k_1(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^n+\left(\frac{k_2(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^n+\left(\frac{k_3(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^n\right]\frac{t^n}{n!} \\ &= \left(\frac{k_1(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h)}{(\alpha-\beta)(\alpha-\gamma)}\right)\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}+\left(\frac{k_2(1+\beta i+\beta^2\varepsilon+\beta^3 h)}{(\beta-\alpha)(\beta-\gamma)}\right)\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!}+\left(\frac{k_3(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h)}{(\gamma-\alpha)(\gamma-\beta)}\right)\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!}. \end{aligned}$$

Thus we get the result. □

## 4. Some Identities for Polynomial Sequences

**Lemma 4.1.** *Let $\{N_k(x)\}$ is the Narayana polynomial sequence and $\{N_k^{[H]}(x)\}$ is the Narayana hybrinomial sequence respectively. Then*

1. *The sum of $\{N_k(x)\}$ is*
$$\sum_{k=0}^{n} N_k(x) = \frac{1}{x}\left(N_{n+1}(x)+N_n(x)+N_{n-1}(x)-9\right)+5$$
*for $n \geq 0$ be an integer.*
2. *The sum of $\{N_k^{[H]}(x)\}$ is*
$$\sum_{k=0}^{n} N_k^{[H]}(x) = \frac{1}{x}\left(N_{n+1}^{[H]}(x)+N_n^{[H]}(x)+N_{n-1}^{[H]}(x)\right)+5+3i-4h$$
*for $n \geq 0$ be an integer.*

*Proof.* **(1)**From the definition of Narayana polynomial sequence we now that

$$N_n(x) = \frac{1}{x}\left(N_{n+1}(x)-N_{n-2}(x)\right).$$

Thus we have

$$\begin{aligned}
N_2(x) &= \frac{1}{x}\left(N_3(x)-N_0(x)\right), \\
N_3(x) &= \frac{1}{x}\left(N_4(x)-N_1(x)\right), \\
&\vdots \\
N_{n-3}(x) &= \frac{1}{x}\left(N_{n-2}(x)-N_{n-5}(x)\right), \\
N_{n-2}(x) &= \frac{1}{x}\left(N_{n-1}(x)-N_{n-4}(x)\right), \\
N_{n-1}(x) &= \frac{1}{x}\left(N_n(x)-N_{n-3}(x)\right), \\
N_n(x) &= \frac{1}{x}\left(N_{n+1}(x)-N_{n-2}(x)\right).
\end{aligned}$$

Therefore we get

$$\sum_{k=0}^{n} N_k(x) - N_0(x) - N_1(x) = \frac{1}{x}\left(N_{n+1}(x)+N_n(x)+N_{n-1}(x)-9\right).$$

Consequently we obtain the result.

**(2)**As we know $\sum_{k=0}^{n} N_k^{[H]}(x) = N_0^{[H]}(x)+N_1^{[H]}(x)+N_2^{[H]}(x)+\cdots+N_n^{[H]}(x)$. Thus we obtain

$$\begin{aligned}
\sum_{k=0}^{n} N_k^{[H]}(x) &= (N_0(x)+N_1(x)i+N_2(x)\varepsilon+N_3(x)h)+(N_1(x)+N_2(x)i+N_3(x)\varepsilon+N_4(x)h)+\cdots+(N_n(x)+N_{n+1}(x)i+N_{n+2}(x)\varepsilon+N_{n+3}(x)h) \\
&= (N_0(x)+N_1(x)+N_2(x)+\cdots+N_n(x))+(N_1(x)+N_2(x)+\cdots+N_{n+1}(x)+N_0(x)-N_0(x))\,i \\
&\quad +(N_2(x)+N_3(x)+\cdots+N_{n+2}(x)+N_0(x)+N_1(x)-N_0(x)-N_1(x))\,\varepsilon \\
&\quad + \left\{ \begin{array}{c} N_3(x)+N_4(x)+\cdots+N_{n+3}(x)+N_0(x)+N_1(x)+N_2(x) \\ -N_0(x)-N_1(x)-N_2(x) \end{array} \right\} h \\
&= \sum_{k=0}^{n} N_k(x) + \left(\sum_{k=0}^{n+1} N_k(x)-2\right) i + \left(\sum_{k=0}^{n+2} N_k(x)-2-3\right)\varepsilon + \left(\sum_{k=0}^{n+3} N_k(x)-2-3-4\right) h
\end{aligned}$$

Consequently using Lemma 4.1 we get,

$$\begin{aligned}
\sum_{k=0}^{n} N_k^{[H]}(x) &= \left(\frac{1}{x}(N_{n+1}(x)+N_n(x)+N_{n-1}(x)-9)+5\right)+\left(\left(\frac{1}{x}(N_{n+2}(x)+N_{n+1}(x)+N_n(x)-9)+5\right)-2\right) i \\
&\quad +\left(\left(\frac{1}{x}(N_{n+3}(x)+N_{n+2}(x)+N_{n+1}(x)-9)+5\right)-5\right)\varepsilon+\left(\left(\frac{1}{x}(N_{n+4}(x)+N_{n+3}(x)+N_{n+2}(x)-9)+5\right)-9\right) h \\
&= \frac{1}{x}\left(N_{n+1}^{[H]}(x)+N_n^{[H]}(x)+N_{n-1}^{[H]}(x)\right)+5+3i-4h
\end{aligned}$$

Thus the proof is completed. □

The following Theorem will be proved with the help of Lemma 4.1.

**Theorem 4.2.** *Let $\{N_m^{[H]}(x)\}$ is the Narayana hybrinomial sequence and $\{N_m(x)\}$ is the Narayana polynomial sequence respectively . Let m and r be arbitrary positive integers such that $m \geq r$ . Then,*

1. *The Catalan-like identity for $\{N_m^{[H]}(x)\}$ is*

$$\begin{aligned} N_{m+r}^{[H]}(x)N_{m-r}^{[H]}(x) - \left(N_m^{[H]}\right)^2(x) &= (l_1 l_2 \alpha^m \beta^m (\alpha^r \beta^{-r} - 2)\alpha^{m+1}\beta^{m-1} + l_2 l_1 \beta^{m+1}\alpha^{m-1}) \\ &\quad + (l_1 l_3 \alpha^m \gamma^m (\alpha^r \gamma^{-r} - 2) + l_3 l_1 \gamma^{m+1}\alpha^{m-1}) \\ &\quad + (l_2 l_3 \beta^m \gamma^m (\beta^r \gamma^{-r} - 2) + l_3 l_2 \gamma^{m+1}\beta^{m-1}, \end{aligned}$$

*where*

$$\begin{aligned} l_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}(1+\alpha i + \alpha^2\varepsilon + \alpha^3 h), \\ l_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}(1+\beta i + \beta^2\varepsilon + \beta^3 h), \\ l_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}(1+\gamma i + \gamma^2\varepsilon + \gamma^3 h). \end{aligned}$$

2. *The Catalan-like identity for $\{N_m(x)\}$ is*

$$\begin{aligned} N_{m+r}(x)N_{m-r}(x) - N_m^2(x) &= w_1 w_2 (\alpha^r - \beta^r)^2 (\alpha\beta)^{m-r} + w_1 w_3 (\alpha^r - \gamma^r)^2 (\alpha\gamma)^{m-r} \\ &\quad + w_2 w_3 (\beta^r - \gamma^r)^2 (\beta\gamma)^{m-r}, \end{aligned}$$

*where*

$$\begin{aligned} w_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}, \\ w_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}, \\ w_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

*Proof.* **(1)**According to the Binet-like formula of Narayana hybrinomial sequence (3.6), we can write

$$N_{m+r}^{[H]}(x)N_{m-r}^{[H]}(x) - \left(N_m^{[H]}\right)^2(x) = (l_1\alpha^{m+r} + l_2\beta^{m+r} + l_3\gamma^{m+r}) \times (l_1\alpha^{m-r} + l_2\beta^{m-r} + l_3\gamma^{m-r}) - (l_1\alpha^m + l_2\beta^m + l_3\gamma^m)^2,$$

where

$$\begin{aligned} l_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}(1+\alpha i + \alpha^2\varepsilon + \alpha^3 h), \\ l_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}(1+\beta i + \beta^2\varepsilon + \beta^3 h), \\ l_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}(1+\gamma i + \gamma^2\varepsilon + \gamma^3 h)). \end{aligned}$$

Now we get

$$\begin{aligned} & N_{m+r}^{[H]}(x)N_{m-r}^{[H]}(x) - \left(N_m^{[H]}\right)^2(x) \\ = \; & (l_1^2\alpha^{2m} + l_1 l_2 \alpha^{m+r}\beta^{m-r} + l_1 l_3 \alpha^{m+r}\gamma^{m-r} \\ & + l_2 l_1 \beta^{m+r}\alpha^{m-r} + l_2^2\beta^{2m} + l_2 l_3 \beta^{m+r}\gamma^{m-r} \\ & + l_3 l_1 \gamma^{m+r}\alpha^{m-r} + l_3 l_2 \gamma^{m+r}\beta^{m-r} + l_3^2\gamma^{2m}) \\ & - (l_1^2\alpha^{2m} + l_2^2\beta^{2m} + l_3^2\gamma^{2m} + 2l_1 l_2 \alpha^m\beta^m \\ & + 2l_1 l_3 \alpha^m \gamma^m + 2l_2 l_3 \beta^m\gamma^m) \\ = \; & (l_1 l_2 \alpha^{m+r}\beta^{m-r} + l_2 l_1 \beta^{m+r}\alpha^{m-r} - 2l_1 l_2 \alpha^m \beta^m) \\ & + (l_1 l_3 \alpha^{m+r}\gamma^{m-r} + l_3 l_1 \gamma^{m+r}\alpha^{m-r} - 2l_1 l_3 \alpha^m\gamma^m) \\ & + (l_2 l_3 \beta^{m+r}\gamma^{m-r} + l_3 l_2 \gamma^{m+r}\beta^{m-r} - 2l_2 l_3 \beta^m \gamma^m). \end{aligned}$$

Thus the proof is completed.
**(2)**It can be proved similar to (1). □

**Corollary 4.3.** *Let $\{N_m^{[H]}(x)\}$ is the Narayana hybrinomial sequence and $\{N_m(x)\}$ is the Narayana polynomial sequence respectively and $m \geq 1$ . Then,*

1. *The Cassini-like identity for Narayana hybrinomial sequence is*

$$\begin{aligned} N_{m+1}^{[H]}(x)N_{m-1}^{[H]}(x) - \left(N_m^{[H]}\right)^2(x) &= (l_1 l_2 \alpha^m \beta^m (\alpha\beta - 2)\alpha^{m+1}\beta^{m-1} + l_2 l_1 \beta^{m+1}\alpha^{m-1}) \\ &+ (l_1 l_3 \alpha^m \gamma^m (\alpha\gamma - 2) + l_3 l_1 \gamma^{n+1}\alpha^{m-1}) \\ &+ (l_2 l_3 \beta^m \gamma^m (\beta\gamma - 2) + l_3 l_2 \gamma^{n+1}\beta^{m-1} - 2 l_2 l_3 \beta^m \gamma^n), \end{aligned}$$

*where*

$$\begin{aligned} l_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}(1 + \alpha i + \alpha^2 \varepsilon + \alpha^3 h), \\ l_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}(1 + \beta i + \beta^2 \varepsilon + \beta^3 h), \\ l_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}(1 + \gamma i + \gamma^2 \varepsilon + \gamma^3 h). \end{aligned}$$

2. *The Cassini-like identity for Narayana polynomial sequence is*

$$\begin{aligned} N_{m+1}(x)N_{m-1}(x) - N_m^2(x) &= (w_1 w_2 (\alpha-\beta)^2)(\alpha\beta)^{m-1} + (w_1 w_3 (\alpha-\gamma)^2)(\alpha\gamma)^{m-1} \\ &+ (w_2 w_3 (\beta-\gamma)^2)(\beta\gamma)^{m-1}, \end{aligned}$$

*where*

$$\begin{aligned} w_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}, \\ w_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}, \\ w_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

*Proof.* (1)It can be proved by replacing $r = 1$ in Catalan-like identity for Narayana hybrinomial sequence. (2)It can be proved by replacing $r = 1$ in Catalan-like identity for Narayana polynomial sequence. □

**Theorem 4.4.** *Let $\{N_n^{[H]}(x)\}$ is the Narayana hybrinomial sequence and $\{N_n(x)\}$ is the Narayana polynomial sequence respectively . Let n be a nonnegative integer and m a natural number, for $m \geq n+1$ . Then,*

1. *The Ocagne-like identity for Narayana hybrinomial sequence is*

$$\begin{aligned} N_m^{[H]}(x)N_{n+1}^{[H]}(x) - N_{m+1}^{[H]}(x)N_n^{[H]}(x) &= l_1 l_2 \alpha^m \beta^n (\beta-\alpha) + l_2 l_1 \beta^m \alpha^n (\alpha-\beta) + l_2 l_3 \beta^m \gamma^n (\gamma-\beta) + l_3 l_2 \gamma^m \beta^n (\beta-\gamma) \\ &+ l_1 l_3 \alpha^m \gamma^n (\gamma-\alpha) + l_3 l_1 \gamma^m \alpha^n (\alpha-\beta), \end{aligned}$$

*where*

$$\begin{aligned} l_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}(1 + \alpha i + \alpha^2 \varepsilon + \alpha^3 h), \\ l_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}(1 + \beta i + \beta^2 \varepsilon + \beta^3 h), \\ l_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}(1 + \gamma i + \gamma^2 \varepsilon + \gamma^3 h). \end{aligned}$$

2. *The Ocagne-like identity for Narayana polynomial sequence is*

$$\begin{aligned} N_m(x)N_{n+1}(x) - N_{m+1}(x)N_n(x) &= w_1 w_2 (\alpha-\beta)(\alpha^n \beta^m - \alpha^m \beta^n) + w_1 w_3 (\alpha-\gamma)(\alpha^n \gamma^m - \alpha^m \gamma^n) \\ &+ w_2 w_3^2 (\beta-\gamma)(\beta^n \gamma^m - \beta^m \gamma^n) \end{aligned}$$

*where*

$$\begin{aligned} w_1 &= \frac{2\alpha^2 + (3-2x)\alpha + (4-3x)}{(\alpha-\beta)(\alpha-\gamma)}, \\ w_2 &= \frac{2\beta^2 + (3-2x)\beta + (4-3x)}{(\beta-\alpha)(\beta-\gamma)}, \\ w_3 &= \frac{2\gamma^2 + (3-2x)\gamma + (4-3x)}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

*Proof.* **(1)**According to the Binet-like formula of Narayana hybrinomial sequence (3.6), we have

$$N_m^{[H]}(x)N_{n+1}^{[H]}(x)-N_{m+1}^{[H]}(x)N_n^{[H]}(x)=(l_1\alpha^m+l_2\beta^m+l_3\gamma^m)(l_1\alpha^{n+1}+l_2\beta^{n+1}+l_3\gamma^{n+1})-(l_1\alpha^{m+1}+l_2\beta^{m+1}+l_3\gamma^{m+1})(l_1\alpha^n+l_2\beta^n+l_3\gamma^n)$$

where

$$\begin{aligned} l_1 &= \frac{2\alpha^2+(3-2x)\alpha+(4-3x)}{(\alpha-\beta)(\alpha-\gamma)}(1+\alpha i+\alpha^2\varepsilon+\alpha^3 h), \\ l_2 &= \frac{2\beta^2+(3-2x)\beta+(4-3x)}{(\beta-\alpha)(\beta-\gamma)}(1+\beta i+\beta^2\varepsilon+\beta^3 h), \\ l_3 &= \frac{2\gamma^2+(3-2x)\gamma+(4-3x)}{(\gamma-\alpha)(\gamma-\beta)}(1+\gamma i+\gamma^2\varepsilon+\gamma^3 h). \end{aligned}$$

Hence we get

$$\begin{aligned} N_m^{[H]}(x)N_{n+1}^{[H]}(x)-N_{m+1}^{[H]}(x)N_n^{[H]}(x) &= (l_1^2\alpha^{m+n+1}+l_1l_2\alpha^m\beta^{n+1}+l_1l_3\alpha^m\gamma^{n+1} \\ &\quad +l_2l_1\beta^m\alpha^{n+1}+l_2^2\beta^{m+n+1}+l_2l_3\beta^m\gamma^{n+1} \\ &\quad +l_3l_1\gamma^m\alpha^{n+1}+l_3l_2\gamma^m\beta^{n+1}+l_3^2\gamma^{m+n+1}) \\ &\quad -(l_1^2\alpha^{m+n+1}+l_1l_2\alpha^{m+1}\beta^n+l_1l_3\alpha^{m+1}\gamma^n+l_2l_1\beta^{m+1}\alpha^n+l_2^2\beta^{m+n+1} \\ &\quad +l_2l_3\beta^{m+1}\gamma^n+l_3l_1\gamma^{m+1}\alpha^n+l_3l_2\gamma^{m+1}\beta^n+l_3^2\gamma^{m+n+1}) \\ &= l_1l_2\alpha^m\beta^n(\beta-\alpha)+l_1l_3\alpha^m\gamma^n(\gamma-\alpha)+l_2l_1\beta^m\alpha^n(\alpha-\beta) \\ &\quad +l_2l_3\beta^m\gamma^n(\gamma-\beta)+l_3l_1\gamma^m\alpha^n(\alpha-\beta)+l_3l_2\gamma^m\beta^n(\beta-\gamma). \end{aligned}$$

**(2)** It can be proved similar to (1) □

## 5. Conclusion

In this paper we introduced the Narayana polynomial sequence and Narayana hybrinomial sequence. We obtained Binet-like formula, generating function, exponential generating function for these sequences. Finally in last section we investigated some interesting identities and summation formulas about Narayana polynomial and Narayana hybrinomial sequence. The subject of this paper has the potential to motivate future researches to work about the applications of this sequence of polynomials in matrix theory, combinatorial number theory and other areas in matrix algebras.

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