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Efficient Numerical Technique for Heat Equation with Nonlocal Boundary Conditions

Zakia Hammouch[a], Anam Zahra[b], Aziz Rehman[c], Syed Ali Mardan[c]

[a]*Department of Mathematics Faculty of Sciences and Techniques, Moulay Ismail Meknes Morocco*

[b]*Department of Mathematics, Virtual University, 54 Lawrence Road Lahore, Pakistan.*

[c]*Department of Mathematics, University of the Management and Technology, C-II, Johar Town, Lahore-54590, Pakistan*

Abstract

A fourth order parallel splitting algorithm is proposed to solve one dimensional non-homogeneous heat equation with integral boundary conditions. we approximate the space derivative by fourth order finite difference approximation. This parallel splitting technique is combined with Simpson's 1/3 rule to tackle nonlocal part of this problem. The algorithm develop here is tested on two model problems. We conclude that our method provides better accuracy due to availability of real arithmetic.

Keywords: Finite difference scheme; Method of line; Pade's approximation; Simpson's 1/3; Partial differential equation; Boundary condition; Initial conditions.

1. Introduction

This section, contain some important definitions and basic material require to understand this paper. Partial differential equations (PDE) are the important class of differential equations. Parabolic PDE describes the diffusion of heat in a specific area over time and plays a vital role in different scientific research areas. Most general form of Parabolic PDE is diffusion equation, which is used in different scientific fields like chemical diffusion and other related processes. The diffusion equation is generated from Fouriers law and conservation of energy. Von Neumann developed this method in 1940. In this method, the starting line of the error is stated in terms of a Fourier series, as the important point of this series is it increase the power

Email addresses: hammouch.zakia@gmail.com (Zakia Hammouch), anam.zahra@vu.edu.pk, anamzahra381@gmail.com (Anam Zahra), aziz3037@yahoo.com (Aziz Rehman), syedalimardanazmi@yahoo.com (Syed Ali Mardan)

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of the function. This technique is only utilized if the linear difference scheme has constant coefficients and also the difference scheme with variable coefficient. This method can be applied at each grid point. The Neumann condition is compulsory only for three level difference scheme and this assumption is necessary for two level difference scheme.

PDE is an equation involving functions and their partial derivatives. It involves rates of change of dependent variable with respect to two or more independent variables. In PDE initial condition (IC) is known as seed value or cauchy condition. The IC is the value of unknown function u and its derivatives at the given point. The conditions which implies on the boundaries of the differential equation is known as boundary conditions (BCs). There are three types of BCs such as Dirichlet BCs, Neumann BCs and Robin BCs.

1. Dirichlet BCs are known as first kind of BCs. It is the certain values that a solution needs to take on along the boundary of the domain.

2. Neumann BCs are second type of BCs. It describes the values in which the rate of change of a solution is implemented within the boundary of the region.

3. Robin BCs are known as third kind of BCs and also known as mixed BCs. It is a requirement of a linear combination of the values of a function and the values of its derivatives on the boundary of the region.

The study of PDE with integral BCs is one of the most important issue of applied sciences. The development of numerical methods for the treatment of non-local boundary conditions (NLBC) is a greatly focoused research area as the appeared mathematical modeling of many real life problems such as chemical diffusion, thermoelasticity etc.

Finite difference (FD) scheme is an algebraic expression, which is used to approximate derivatives involve in PDE. A common usage of FD scheme is to solve differential equations numerically and approximating derivatives for root locating and numerical optimization. FD schemes are also used in many fields like engineering, business and science etc. While handling of differential equations, FD is the very efficient method due to ease of implementation on discretization based problem.

In 1963 Cannon [9] and Batten [8] presented the existence of NLBC independently. In 1964 Kamynin and Ionkin [13] studied homogeneous second order heat equation with Ic and NLBC. Taj and Twizzel [20, 22] introduced third order numerical techniques coupled with Pade approximation to find solution of one dimensional and higher dimensional heat equations. They [21] expand their work to fourth-order parallel splitting technique to get higher efficiency in previous results with much better error of approximation and also contain all the characteristic as in the third order. Dehghan [10, 11] start working on the idea of parallel splitting method and apply this technique for the results of parabolic and hyperbolic PDEs. Dehghan used method of lines (MOL) and semidiscretization approach to convert the PDE into a system of first-order ordinary differential equations [10, 11].

Rehman et al.[16] worked on the diffusion equation with initial and NLBC and they used fourth-order numerical method for non homogeneous diffusion equation with NLBC and develop L-acceptable. Fifth-order parallel splitting numerical technique was developed and applied on homogeneous and inhomogeneous diffusion equation with a NLBC. The results presented that the above described method contain L-acceptable as well as fifth order accuracy [23]. Soltanalizadeh [1, 19] introduced the new technique of matrix formulation for the system of equations and he focused on the wave equation with the NLBC [5]. Mardan et al. developed a hybrid numerical methods which is partially sixth-order accuracy in time and space, due to a combination of sixth-order finite approximations and fifth-order Pade approximation. They [24] approximate the second order spatial derivatives by sixth order FD approximation. The presence of NLBC makes such a problem applicable when modeling processes such as blood flow, underground water flow, population dynamics, and thermo-elasticity. We can also extend this work in fractional calculus for NLBC[14, 18, 2, 3, 4, 6, 7, 15].

2. Fourth order parallel splitting Method

In this section, we present the development of fourth order splitting method to get numerical solution of the non-homogeneous diffusion equation with NLBC. The spatial derivative is approximated by fourth-order FD approximations. Simpson's $\frac{1}{3}$ rule is used to tackle integral conditions and to remove additional variables to get a system of N equations with N variables.

Considered the diffusion equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v = p(x,t)\,, \quad 0 < x < X, \quad t > 0, \tag{1}$$

with the IC

$$v(x,0) = g(x) \quad x \in (0,1), \quad 0 < t < T, \tag{2}$$

and the NLBC

$$\int_0^1 v(x,t)dx = q_1(t), \quad 0 < t < T, \tag{3}$$

$$\int_0^1 b(x)v(x,t)dx = q_2(t). \quad 0 < t < T, \tag{4}$$

where $p(x,t), g(x), q_1(t), q_2(t), b(x)$ are known functions. Here we developed a method for the numerical solution of diffusion equation (1) under IC (2) and the NLBC (3) and (4) by using the FD. Simpson's 1/3 rule is use to over come the difficulty of the integral boundary conditions. The rectangular mesh of points is constructed with space variable $[0 < x < 1]$ and time variable $[t > 0]$

We divide the spatial interval into $N+1$ subintervals having length $h = 1/(N+1)$ and the open ended time interval is divided into length time step l results into rectangular mesh points with co-ordinates $(x_m, t_n) = (mh, nl)$ where $m = 0, 1, 2, ..., N, N+1$ and $n = 0, 1, 2, 3, ...$

To approximate second order spatial derivative the fourth order FD scheme is given by

$$\begin{aligned} \frac{\partial^2 v(x,t)}{\partial^2 x} &= \frac{1}{12}h^2\{-v_{i-2} + 16v_{i-1} - 30v_i + 16v_{i+1} - v_{i+2}\} \\ &+ \frac{h^4}{90}\frac{\partial^6 v(x,t)}{\partial x^6} + O(h^5), \end{aligned} \tag{5}$$

applying Eq. (5) with Eq. (1), we get system of ordinary differential equations as

$$\begin{aligned} \frac{dv_i}{dt} &= \frac{1}{12}h^2\{-v_{i-2} + 16v_{i-1} - (30+12h^2)v_i + 16v_{i+1} - v_{i+2}\} \\ &+ \frac{h^4}{90}\frac{\partial^6 v_i}{\partial x^6} + O(h^5), i = 2, ..., N-1 \end{aligned} \tag{6}$$

In order to get same accuracy and dominant error term special formulas are needed for $i = 1$ and $i = N$ as

$$\begin{aligned} \frac{\partial^2 v(x,t)}{\partial^2 x} &= \frac{1}{12}h^2\{9v_{i-1} - 9v_i - 19v_{i+1} + 34v_{i+2} - 21v_{i+3} + 7v_{i+4} - v_{i+5}\} \\ &+ \frac{h^4}{90}\frac{\partial^6 v(x,t)}{\partial x^6} + O(h^5), \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial^2 v(x,t)}{\partial^2 x} &= \frac{1}{12}h^2\{-v_{i-5} + 7v_{i-4} - 21v_{i-3} + 34v_{i-2} - 19v_{i-1} - 9v_i \\ &+ 9v_{i+1}\} + \frac{h^4}{90}\frac{\partial^6 v(x,t)}{\partial x^6} + O(h^5), \end{aligned} \tag{8}$$

using Eq. (7) and Eq. (8) in Eq. (1) for $i = 1$ and $i = n$, we get

$$\begin{aligned}\frac{dv_1}{dt} &= \frac{1}{12}h^2\{9v_0 - (9+12h^2)v_1 - 19v_2 + 34v_2 - 21v_3 + 7v_4 - v_5\} \\ &+ \frac{h^4}{90}\frac{\partial^6 v(x,t)}{\partial x^6} + O(h^5),\end{aligned} \tag{9}$$

and

$$\begin{aligned}\frac{dv_N}{dt} &= \frac{1}{12}h^2\{-v_{N-5} + 7v_{N-4} - 21v_{N-3} + 34v_{N-2} - 19v_{N-1} \\ &- (9+12h^2)v_N + 9v_{N+1}\} + \frac{h^4}{90}\frac{\partial^6 v(x,t)}{\partial x^6} + O(h^5).\end{aligned} \tag{10}$$

The combination of Eq.(1) with Eqs.(6), (9) and (10) results in a mesh gride at time level $t = t_n$ construct a system of N linear equations in $N+2$ unknowns $v_0, v_1, v_2, ..., v_{N+1}$. Simpson's $1/3$ rule is used to approximate the integrals in (3) and (4) [17, 12]. Here Eq. (3) and Eq. (4)

$$q_1(t) = \frac{h}{3}\{v(0,t) + 4\sum_{n=1}^{\frac{N+1}{2}} v(2i-1)h, t + 2\sum_{n=1}^{\frac{N+1}{2}-1} v(2ih,t) + v(N+1)h, t\}, \tag{11}$$

and

$$\begin{aligned}q_2(t) &= \frac{h}{3}\{b(0,t)v(0,t) + 4\sum_{n=1}^{\frac{N+1}{2}} b((2i-1)h,t)v((2i-1)h,t) \\ &+2\sum_{n=1}^{\frac{N+1}{2}-1} b(2ih,t)v(2ih,t) + v(N+1)h, t\}.\end{aligned} \tag{12}$$

Solving Eq. (11) and Eq. (12) for V_0 and V_{N+1} and substituting their values in the above system we have a system of N linear ODE which can be written in vector matrix form as

$$\frac{dV(t)}{dt} = AV(t) + V(t), \qquad t > 0 \tag{13}$$

with initial distribution

$$V(0) = g \tag{14}$$

in which $V(t) = [v_1(t), v_2(t), ..., v_N(t)]^T$ and $g = [g(x_1), g(x_2), ..., g(x_N)]^T$, where transpose is denoted by T and order of matrix B is $N \times N$ which is given by

$$A = \frac{1}{12h^2}\begin{bmatrix} \eta_1^{'} & \eta_2^{'} & \eta_3^{'} & \eta_4^{'} & \eta_5^{'} & \cdots\cdots & \eta_N^{'} \\ \psi_1^{'} & \psi_2^{'} & \psi_3^{'} & \psi_4^{'} & \psi_5^{'} & \cdots\cdots & \psi_N^{'} \\ -1 & 16 & -30 & 16 & -1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & -1 & 16 & -30 & 16 & -1 & \\ \gamma_1^{'} & \gamma_2^{'} & \gamma_3^{'} & \gamma_4^{'} & \gamma_5^{'} & \cdots\cdots & \gamma_N^{'} \\ \delta_1^{'} & \delta_2^{'} & \delta_3^{'} & \delta_4^{'} & \delta_5^{'} & \cdots\cdots & \delta_N^{'} \end{bmatrix}_{N\times N} \tag{15}$$

where
$\eta_1^{'} = 9B_1 - 9$, $\eta_2^{'} = 9B_2 - 19$, $\eta_3^{'} = 9B_3 + 34$, $\eta_4^{'} = 9B_4 - 21$,
$\eta_5^{'} = 9B_5 + 7$, $\eta_6^{'} = 9B_6 - 1$, $\eta_j^{'} = 9B_j$ for $j \geq 7$,
$\psi_1^{'} = -B_1 + 16$, $\psi_2^{'} = -B_2 - 30$, $\psi_3^{'} = -B_3 + 16$, $\psi_4^{'} = -B_4 - 1$, $\psi_j^{'} = -B_j$ for $j \geq 5$,
$\gamma_j^{'} = -n_j \;\; for 1 \leq j \leq 5$, $\gamma_6^{'} = -n_6 - 1$, $\gamma_7^{'} = -n_7 + 16$, $\gamma_8^{'} = -n_8 - 30$, $\gamma_9^{'} = -n_9 + 16$,

$\delta_j^{'} = 9n_j \ \ for 1 \leq j \leq 3$, $\delta_4^{'} = 9n_4 - 1$, $\delta_5^{'} = 9n_5 + 7$, $\delta_6^{'} = 9n_6 - 21$, $\delta_7^{'} = 9n_7 + 34$, $\delta_8^{'} = 9n_8 - 19$, $\delta_9^{'} = 9n_9 - 9$, in which

$$m_j = \begin{cases} \frac{\frac{4h}{3}(C_4 - C_2 b_j)}{C_1 C_4 - C_2 C_3}, & for \quad j = 1, 3, 5, ..., N \\ \frac{\frac{2h}{3}(C_4 - C_2 b_j)}{C_1 C_4 - C_2 c_3}, & for \quad j = 2, 4, 6, ..., N - 1 \end{cases}$$

and

$$n_j = \begin{cases} \frac{\frac{4h}{3}(C_3 - C_1 b_j)}{C_1 C_4 - C_2 C_3}, & for \quad j = 1, 3, 5, ..., N \\ \frac{\frac{2h}{3}(C_3 - C_1 b_j)}{C_1 C_4 - C_2 c_3}, & for \quad j = 2, 4, 6, ..., N - 1 \end{cases}$$

Here $c_1 = \frac{-h}{3}, c_2 = \frac{-h}{3}, c_3 = \frac{-h}{3} b_0$ and $c_4 = \frac{-h}{3} b_{10}$, also $\Upsilon_i = \Upsilon(ih, t)$ and $\Delta_i = \Delta(ih, t)$. The column matrix v(t) contains the factors from the functions $p(x, t), q_1(t)$ and $q_2(t)$ and is given as

$$v(t) = [\frac{9l_1}{12h^2} + p_1, \frac{-l1}{12h^2} + p_2, p_3, ..., p_{N-2}, \frac{-l_2}{12h^2} + p_{N-1}, + \frac{9l_2}{12h^2} + p_N]^T \tag{16}$$

where

$$l_1 = \frac{c_2 q_2(t) - c_4 q_1(t)}{c_1 c_4 - c_2 c_3},$$

and

$$l_2 = \frac{c_3 q_1(t) - c_1 q_2(t)}{c_1 c_4 - c_2 c_3}.$$

The output of the system (13) with (14) is given by

$$V(t) = exp(tA)f + \int_0^t exp[(t - r)A]v(r)dr \tag{17}$$

satisfies the recurrence relation

$$V(t + l) = exp(lA)V(t) + \int_t^{t+l} exp[(t + l - r)A]v(r)dr, t = 0, l, 2l, ... \tag{18}$$

By using Pade's approximation to approximate the matrix exponential function in (18) where a_1, a_2, a_3 are three parameters and a real scalar θ given by

$$E_4\theta = \frac{1 + (1 - a_1)\theta + (1/2 - a_1 + a_2)\theta^2 + (\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3)\theta^3}{(1 - a_1\theta + a_2\theta^2 - a_3\theta^3) + (\frac{-1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3)} = \frac{p(\theta)}{q(\theta)}, \tag{19}$$

with error constant $C = \frac{1}{30} - \frac{1}{8}a_1 + \frac{1}{3}a_2 - \frac{1}{2}a_3$. Stability of this technique is guaranteed by [11].
The quadrature term in (18) is approximated by

$$\int_t^{t+l} exp[(t + l - r)A]v(r)dr = W_1 v(r_1) + W_2 v(r_2) + W_3 v(r_3) + W_4 v(r_4), \tag{20}$$

where $r_1 \neq r_2 \neq r_3 \neq r_4$ and W_1, W_2, W_3 and W_4 are matrices. Taking $r_1 = t$, $r_2 = t + \frac{l}{3}$, $r_3 = t + \frac{2l}{3}$,

$r_4 = t + l.$

$$
\begin{aligned}
W_1 &= \frac{l}{24}\{3I - (19 - 78a_1 + 216a_2 - 324a_3)lA \\
&+ (3 - 8a_1 + 12a_2)l^2A^2\}P, \\
W_2 &= \frac{3l}{16}\{2I + (16 - 56a_1 + 144a_2 - 216a_3)lA \\
&+ (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}P, \\
W_3 &= \frac{3l}{8}\{I - (7 - 26a_1 + 72a_2 - 108a_3)lA \\
&- (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}P, \\
W_4 &= \frac{l}{48}\{6I + (44 - 168a_1 + 432a_2 - 648a_3)lA \\
&+ (11 - 44a_1 + 132a_2 - 216a_3)l^2A^2 \\
&+ (2 - 8a_1 + 24a_2 - 48a_3)l^3A^3\}P.
\end{aligned}
\tag{21}
$$

2.1. Calculation of the above Method For N=11

By using above system of equations construct a matrix.

$$R_1 = \begin{bmatrix} -a & -19 & 34 & -21 & 7 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 16 & -d & 16 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} -1 & -16 & -d & 16 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 0 & -1 & 16 & -d & 16 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_5 = \begin{bmatrix} 0 & 0 & -1 & 16 & -d & 16 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_6 = \begin{bmatrix} 0 & 0 & 0 & -1 & 16 & -d & 16 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$R_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 16 & -d & 16 & -1 & 0 & 0 \end{bmatrix},$$

$$R_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 16 & -d & 16 & -1 & 0 \end{bmatrix},$$

$$R_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 16 & -d & 16 & -1 \end{bmatrix},$$

$$R_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 16 & -d & 16 \end{bmatrix},$$

$$R_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 7 & -21 & 34 & -19 & -a \end{bmatrix},$$

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ R_{10} \\ R_{11} \end{bmatrix},$$

$$A_1 = \frac{1}{12h^2}A.$$

where $a = 9 + 12h^2$ and $d = 30 + 12h^2$.

By using MATLAB calculate the values of g_1, g_2, l_1 and l_2.

$$g_1 = 1.499600040000000e + 000,\ g_2 = 8.331333533333334e - 001$$
$$l_1 = 2.399280072000000e + 001,\ l_2 = 2.999280072000000e + 001$$

values of m_j, for $j = 1, 2, 3, ..., 11$

$$m_j = \begin{bmatrix} -3.666666666666667e+000 \\ -1.666666666666666e+000 \\ -3.000000000000000e+000 \\ -1.333333333333333e+000 \\ -2.333333333333334e+000 \\ -1.000000000000000e+000 \\ -1.666666666666667e+000 \\ -6.666666666666666e-001 \\ -1.000000000000000e+000 \\ -3.333333333333336e-001 \\ -3.333333333333333e-001 \end{bmatrix},$$

values of n_j, for $j = 1, 2, 3, ..., 11$

$$n_j = \begin{bmatrix} -3.333333333333333e-001 \\ -3.333333333333333e-001 \\ -1.000000000000000e+000 \\ -6.666666666666666e-001 \\ -1.666666666666666e+000 \\ -1.000000000000000e+000 \\ -2.333333333333333e+000 \\ -1.333333333333333e+000 \\ -3.000000000000000e+000 \\ -1.666666666666666e+000 \\ -3.666666666666666e+000 \end{bmatrix},$$

and calculated values of p_j for $(j = 1, 2, 3, ..., 20)$

$$p_1 = -5.386114882221006e + 001,\ p_2 = 1.079999999999073e + 002,$$
$$p_3 = -5.335030674440576e + 001,\ p_4 = 2.114555667085107e - 001,$$
$$p_5 = 1.417949572500437e + 002,\ p_6 = -8.099999999984026e + 002,$$
$$p_7 = 7.640302208857925e + 002,\ p_8 = -9.282517813743382e + 001,$$
$$p_9 = -2.779587258928380e + 002,\ p_{10} = 9.719999999984564e + 002,$$
$$p_{11} = -7.602263303810352e + 002,\ p_{12} = 6.818505627541670e + 001,$$
$$p_{13} = 1.244022140366359e + 002,\ p_{14} = -4.859999999991785e + 002,$$
$$p_{15} = 3.999612344557147e + 002,\ p_{16} = -3.736344849317209e + 001,$$
$$p_{17} = -1.164891815551947e + 003,\ p_{18} = 4.211999999993268e + 003,$$
$$p_{19} = -3.296460697860340e + 003,\ p_{20} = 2.553525134190170e + 002.$$

Table 1: Relative error for example 1

$l = 0.00001$	Exact Solution	Approximate Solution	Relative Error
$N = 7$	1.12498	1.1249	8.3942×10^{-6}
$N = 9$	1.09998	1.0999	8.4048×10^{-6}
$N = 11$	1.08331	1.0833	8.3984×10^{-6}
$N = 13$	1.14283	1.1428	8.3876×10^{-6}
$N = 15$	1.12498	1.1249	8.4103×10^{-6}

3. Experiments

Now we apply the develop method on some examples from literature.This section contain some experiments taken from literature.

Experiment 1: Consider the diffusion equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v = p(x,t), \ \ 0 < x < X, \quad t > 0,$$

with the IC

$$v(x,0) = x, \ \ x\epsilon(0,1), \quad 0 < t < T,$$

and the NLBC

$$\int_0^1 v(x,t)dx = \frac{1}{2} + t^2, \ \ 0 < t < T,$$
$$\int_0^1 xv(x,t)dx = \frac{1}{3} + \frac{1}{2}t^2, \ \ 0 < t < T.$$

and exact solution to this problem is $v(x,t) = x + t^2$.

By using algorithm which developed above, this problem is solved for $l = 0.00001$ and $n = 7, 9, 11, 13, 15$. In Table (3.1), calculate the exact solution, approximate solution and relative error. These results show that the method behave smoothly over the interval $0 \leq x \leq 1$. There is no oscillation are observed. It is worth noting that the results obtained from this method are more accurate and precise than the methodology in literature. Moreover, this method is forth-order precise except for few estimation of l and h when truncation error is high.

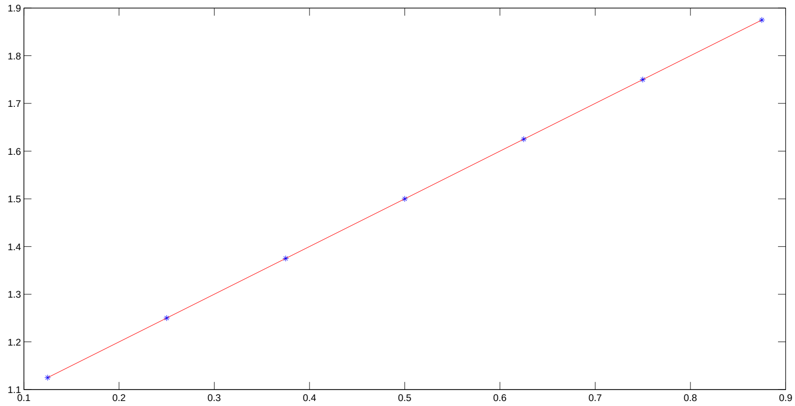

Figure 1: Numerical solution of problem 1, for h=0.1, l=0.00001, n=7

Experiment 2:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + v = (10-2x)e^t, \quad x\epsilon(0,1), \ \ 0<t<T,$$

with the IC

$$v(x,0) = 5-x, \ \ x\epsilon(0,1), \quad 0<t<T,$$

and the NLBC

$$\int_0^1 v(x,t)dx = \frac{9}{2}e^t, \quad 0<t<T,$$

$$\int_0^1 xv(x,t)dx = \frac{13}{6}e^t, \quad 0<t<T.$$

The exact solution to this problem is $v(x,t) = (5-x)e^t$.

Table 2: Relative error for example 2

$l=0.00001$	Exact solution	Approximate Solution	Relative Error
$N=7$	1.2232×10^1	1.2232×10^1	5.3935×10^{-6}
$N=9$	1.2232×10^1	1.2232×10^1	5.3934×10^{-6}
$N=11$	1.2232×10^1	1.2232×10^1	5.3935×10^{-6}
$N=13$	1.2232×10^1	1.2232×10^1	5.3934×10^{-6}
$N=15$	1.2232×10^1	1.2232×10^1	5.3935×10^{-6}

By using algorithm which described above, this problem is solved for $l=0.00001$ and $n=7,9,11,13,15$. In Table (3.2) calculate the exact solution, approximate solution and relative error. These results show that the method behave smoothly over the interval $0\le x\le 1$. There is no oscillation are observed. It is worth noting that the results obtained from this method are more accurate and precise than the methodology in literature. Moreover, this method is forth-order precise except for few estimation of l and h when truncation error is high.

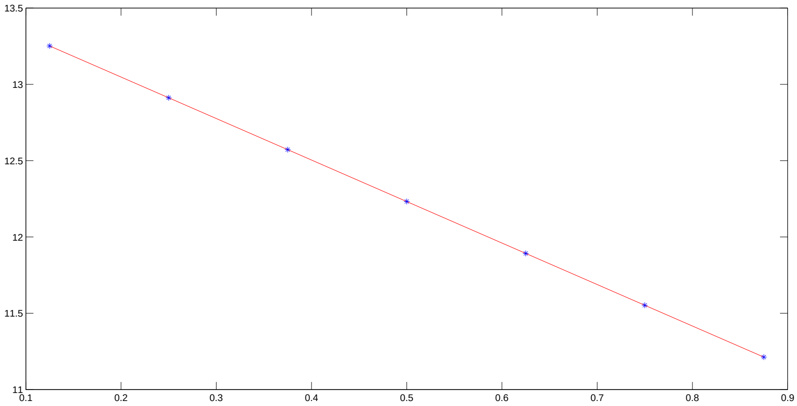

Figure 2: Numerical solution of problem 2, for h=0.1, l=0.00001, n=7

4. Summary and Conclusion

In this paper, we developed the forth order parallel splitting technique for getting more accurate results of two dimensional parabolic PDEs. This paper start from the general introduction of PDEs, some explanation of FD schemes, heat equation and their applications are discussed. Finally, a new method has been developed for numerical solution of the second order diffusion equation with nonlocal integral conditions. The results obtained by using this new technique are highly accurate, when compared to the exact solution. This method is time efficient due to availability of real arithmetic only. This method can be extended in to the higher space dimensions.

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