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Another Version of Young Inequality Applying the Supplemental Young Inequality

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Abstract

In this note, we give the further reverses of the Young type inequalities for non-negative real scalars, using the supplemental Young inequality

$a^{\nu}b^{1-\nu} \geqslant \nu a+(1-\nu)b,$

where $a,b \geqslant 0$ and $\nu \notin [0,1]$. Making use of them, some matrix inequalities for Hilbert-Schmidt norm and trace norm are deduced.

Keywords: *Hilbert-Schmidt norm, Trace norm, positive semi-definite matrices, Reverse Young inequalities.*
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1. Introduction

The classical Young inequality for numbers, which is famous as the weighted AM-GM inequality, asserts that if $a,b \geqslant 0$ and $0 \leqslant \nu \leqslant 1$, then

$$a^{\nu}b^{1-\nu} \leqslant \nu a+(1-\nu)b, \tag{1.1}$$

with equality if and only if $a=b$. The supplemental Young inequality

$$a^{\nu}b^{1-\nu} \geqslant \nu a+(1-\nu)b, \tag{1.2}$$

is valid whenever $\nu \notin [0,1]$, for the proof see [2]. In recent years many authors have been studied (1.1) and have been obtained several refinements and reverses for it (for more information, we refer the interesting reader to [1], [4]-[8], [10]- [20] and references therein). Hu and Xue, in [9], gave reverses of the inequality (1.1) as follows:

$$\nu^2a^2+(1-\nu)^2b^2 \leqslant (1-\nu)^2(a-b)^2-r_0a(\sqrt{(1-\nu)b}-\sqrt{a})^2+[a^{\nu}((1-\nu)b)^{1-\nu}]^2, \quad 0\leqslant \nu \leqslant \frac{1}{2}, \tag{1.3}$$

where $r_0=\min\{2\nu,1-2\nu\}$.

$$\nu^2a^2+(1-\nu)^2b^2 \leqslant \nu^2(a-b)^2-r_0b(\sqrt{b}-\sqrt{\nu a})^2+[(\nu a)^{\nu}b^{1-\nu}]^2, \quad \frac{1}{2}\leqslant \nu \leqslant 1, \tag{1.4}$$

where $r_0=\min\{2\nu-1,2-2\nu\}$.
Let $M_n(\mathbb{C})$ be the algebra of all $n\times n$ matrices with entries in the complex field $\mathbb{C}$, and let $M_n^+(\mathbb{C})$ be the set of all positive semi-definite matrices in $M_n(\mathbb{C})$. For $A=[a_{ij}] \in M_n(\mathbb{C})$, the Hilbert-Schmidt norm (or Frobenious) and the trace norm of A are defined by

$$\|A\|_2=\sqrt{\sum_{j=1}^{n} s_j^2(A)}, \quad \|A\|_1=\sum_{j=1}^{n} s_j(A),$$

respectively, where $s_1(A)\geqslant s_2(A)\geqslant .... \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. The Hilbert-Schmidt norm is unitarily invariant, that is, $\|UAV\|_2=\|A\|_2$ for all $A\in M_n(\mathbb{C})$ and for all unitary matrices $U,V \in M_n(\mathbb{C})$.

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The matrix versions of the inequalities (1.3) and (1.4) for Hilbert-Schmidt norm and trace norm were proved by Hu and Xue in [9] are respectively in following form:

$$\|\nu AX+(1-\nu)XB\|_2^2+r_0\left[(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\|AX\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right]$$
$$\leqslant(1-\nu)^2\|AX-XB\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-\nu)^{2(1-\nu)}\|A^{\nu}XB^{1-\nu}\|_2^2, \tag{1.5}$$

for $0\leqslant\nu\leqslant\frac{1}{2}$ and $r_0=\min\{2\nu,1-2\nu\}$.

$$\|\nu AX+(1-\nu)XB\|_2^2+r_0\left[\nu\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2+\|XB\|_2^2-2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]$$
$$\leqslant\nu^2\|AX-XB\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+\nu^{2\nu}\left\|A^{\nu}XB^{1-\nu}\right\|_2^2, \tag{1.6}$$

for $\frac{1}{2}\leqslant\nu\leqslant1$ and $r_0=\min\{2\nu-1,2-2\nu\}$.

$$\sqrt{\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2-(1-\nu)^2(\|A\|_2^2+\|B\|_2^2-2\|AB\|_1)+M_1}\leqslant(1-\nu)^{(1-\nu)}\|A^{\nu}\|_2\|B^{1-\nu}\|_2,\quad 0\leqslant\nu\leqslant\frac{1}{2}, \tag{1.7}$$

where $r_0=\min\{2\nu,1-2\nu\}$ and

$$M_1=r_0[(1-\nu)\|AB\|_1+\|A\|_2^2-2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1\|B^{\frac{1}{2}}\|_1],$$

$$\sqrt{\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2-\nu^2(\|A\|_2^2+\|B\|_2^2-2\|AB\|_1)+M_2}\leqslant\nu^{\nu}\|A^{\nu}\|_2\|B^{1-\nu}\|_2,\quad \frac{1}{2}\leqslant\nu\leqslant1, \tag{1.8}$$

where $r_0=\min\{2\nu-1,2-2\nu\}$ and

$$M_2=r_0[\nu\|AB\|_1+\|B\|_2^2-2\sqrt{\nu}\|A^{\frac{1}{2}}\|_1\|B^{\frac{1}{2}}\|_1].$$

It is a interesting question to ask whether there exist further refinements and improvements of the inequalities (1.3) and (1.4). In the present paper, motivated by the inequalities (1.3) and (1.4) and applying (1.2), we first obtain some new reverses of (1.1). Then, as an application of them, some norm inequalities such as Hilbert-Schmidt norm and trace norm give.

2. Main Results

2.1. Improved inequalities for scalars

Using the inequalities (1.3) and (1.4), we obtain the following result.

Corollary 2.1. *Suppose that $a,b\geqslant0$ and $0\leqslant\nu\leqslant1$.*

1. *If $0\leqslant\nu\leqslant\frac{1}{4}$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant(1-\nu)^2(a-b)^2-2\nu a(\sqrt{(1-\nu)b}-\sqrt{a})^2+[a^{\nu}((1-\nu)b)^{1-\nu}]^2. \tag{2.1}$$
2. *If $\frac{1}{4}\leqslant\nu\leqslant\frac{1}{2}$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant(1-\nu)^2(a-b)^2-(1-2\nu)a(\sqrt{(1-\nu)b}-\sqrt{a})^2[a^{\nu}((1-\nu)b)^{1-\nu}]^2. \tag{2.2}$$
3. *If $\frac{1}{2}\leqslant\nu\leqslant\frac{3}{4}$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant\nu^2(a-b)^2-(2\nu-1)b(\sqrt{b}-\sqrt{\nu a})^2+[b^{1-\nu}(\nu a)^{\nu}]^2. \tag{2.3}$$
4. *If $\frac{3}{4}\leqslant\nu\leqslant1$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant\nu^2(a-b)^2-(2-2\nu)b(\sqrt{b}-\sqrt{\nu a})^2+[(\nu a)^{\nu}b^{1-\nu}]^2. \tag{2.4}$$

Theorem 2.2. *Suppose that $a,b\geqslant0$ and $0\leqslant\nu\leqslant1$.*

1. *If $\nu\notin[\frac{1}{2},\frac{3}{4}]$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant(1-\nu)^2(a-b)^2-(1-2\nu)a(\sqrt{(1-\nu)b}-\sqrt{a})^2+[a^{\nu}((1-\nu)b)^{1-\nu}]^2. \tag{2.5}$$
2. *If $\nu\notin[\frac{1}{4},\frac{1}{2}]$, then*
$$\nu^2a^2+(1-\nu)^2b^2\leqslant\nu^2(a-b)^2-(2\nu-1)b(\sqrt{b}-\sqrt{\nu a})^2+[b^{1-\nu}(\nu a)^{\nu}]^2. \tag{2.6}$$

Proof. It is clear that $(3-4\nu)\notin[0,1]$ for $\nu\notin[\frac{1}{2},\frac{3}{4}]$. Then, utilizing (1.2), we have

$$\begin{aligned}\nu^2a^2+(1-\nu)^2b^2-(1-\nu)^2(a-b)^2+(1-2\nu)a(\sqrt{(1-\nu)b}-\sqrt{a})^2 &= (3-4\nu)((1-\nu)ab)+(4\nu-2)a\sqrt{(1-\nu)ab}\\ &\leqslant [a^\nu(b(1-\nu))^{1-\nu}]^2.\end{aligned}$$

Similarly, $(4\nu-1)\notin[0,1]$ for $\nu\notin[\frac{1}{4},\frac{1}{2}]$. Again, applying (1.2), we get

$$\begin{aligned}\nu^2a^2+(1-\nu)^2b^2-\nu^2(a-b)^2+(2\nu-1)b(\sqrt{b}-\sqrt{\nu a})^2 &= (2-4\nu)b\sqrt{\nu ab}+(4\nu-1)(\nu ab)\\ &\leqslant (\nu ab)^{4\nu-1}(b\sqrt{\nu ab})^{2-4\nu}\\ &= [b^{1-\nu}(\nu a)^\nu]^2.\end{aligned}$$

□

In the following, we compare Theorem 2.2 with Corollary 2.1 to show advantage of Theorem 2.2.

Remark 2.3. *1. Let $\nu\notin[\frac{1}{2},\frac{3}{4}]$, then $\nu\in[0,\frac{1}{4}]\bigcup[\frac{1}{4},\frac{1}{2}]\bigcup[\frac{3}{4},1]$. Notice that, in the special case $\nu\in[\frac{1}{4},\frac{1}{2}]$, the inequality (2.5) becomes the inequality (2.2). For $\nu\in[0,\frac{1}{4}]$, one can easily show that the right hand side of the inequality (2.5) is less than or equal to the right hand side of the inequality (2.1). Moreover, an simple computation show that the right hand side of the inequality (2.5) is less than or equal to the right hand side of the inequalitiy (2.4). Therefore, the range of (2.5) is wider than of (2.2).*

2. Let $\nu\notin[\frac{1}{4},\frac{1}{2}]$, then $\nu\in[0,\frac{1}{4}]\bigcup[\frac{1}{2},\frac{3}{4}]\bigcup[\frac{3}{4},1]$. For $\nu\in[\frac{1}{2},\frac{3}{4}]$, the inequality (2.6) corresponds the inequality (2.3). For the cases of $\nu\in[0,\frac{1}{4}]$ and $\nu\in[\frac{3}{4},1]$, the right hand side of the inequality (2.6) is less than or equal to the right hand side of the inequalities (2.1) and (2.4), respectively. Thus, the right hand side of the inequality (2.6) is the better bound respect to the right hand side of the inequalities (2.1) and (2.4), respectively. This show the range of (2.6) is wider than of (2.3). Thus for all cases, the upper bounds in the inequalities (2.5) and (2.6) are better than those in Corollary 2.1.

Remark 2.4. *The obtained inequalities in Theorem 2.2 are equivalent to the following inequalities:*

1. *If $\nu\notin[\frac{1}{2},\frac{3}{4}]$, then*

$$\begin{aligned}(\nu a+(1-\nu)b)^2 &\leqslant (1-\nu)^2(a-b)^2+2\nu(1-\nu)ab\\ &\quad -(1-2\nu)a(\sqrt{(1-\nu)b}-\sqrt{a})^2+[a^\nu((1-\nu)b)^{1-\nu}]^2.\end{aligned}\tag{2.7}$$

2. *If $\nu\notin[\frac{1}{4},\frac{1}{2}]$, then*

$$\begin{aligned}(\nu a+(1-\nu)b)^2 &\leqslant \nu^2(a-b)^2+2\nu(1-\nu)ab\\ &\quad -(2\nu-1)b(\sqrt{b}-\sqrt{\nu a})^2+[b^{1-\nu}(\nu a)^\nu]^2.\end{aligned}\tag{2.8}$$

Remark 2.5. *The equality in the inequality(2.5) and the equality in the inequality (2.6) hold if and only if $a=(1-\nu)b$ and $\nu a=b$, respectively.*

3. Matrix inequalities

Applying Corollary 2.1 and the unitarily invariant property of $\|.\|_2$, the following result is trivial.

Corollary 3.1. *Let $A,B\in M_n^+(\mathbb{C})$, $X\in M_n(\mathbb{C})$ and $0\leqslant\nu\leqslant 1$.*

1. *If $0\leqslant\nu\leqslant\frac{1}{4}$, then*

$$\begin{aligned}&\|\nu AX+(1-\nu)XB\|_2^2\\ &\leqslant (1-\nu)^2\|AX-XB\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-\nu)^{2(1-\nu)}\|A^\nu XB^{1-\nu}\|_2^2\\ &-2\nu\left[\|AX\|_2^2+(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right].\end{aligned}\tag{3.1}$$

2. *If $\frac{1}{4}\leqslant\nu\leqslant\frac{1}{2}$, then*

$$\begin{aligned}&\|\nu AX+(1-\nu)XB\|_2^2\\ &\leqslant (1-\nu)^2\|AX-XB\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-\nu)^{2(1-\nu)}\|A^\nu XB^{1-\nu}\|_2^2\\ &-(1-2\nu)\left[\|AX\|_2^2+(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right].\end{aligned}\tag{3.2}$$

3. *If $\frac{1}{2} \leqslant \nu \leqslant \frac{3}{4}$, then*

$$\begin{aligned} &\|\nu AX + (1-\nu)XB\|_2^2 \\ &\leqslant \nu^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \nu^{2\nu}\|A^{\nu}XB^{1-\nu}\|_2^2 \\ &-(2\nu-1)\left[\|XB\|_2^2 + \nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]. \end{aligned} \tag{3.3}$$

4. *If $\frac{3}{4} \leqslant \nu \leqslant 1$, then*

$$\begin{aligned} &\|\nu AX + (1-\nu)XB\|_2^2 \\ &\leqslant \nu^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \nu^{2\nu}\|A^{\nu}XB^{1-\nu}\|_2^2 \\ &-(2-2\nu)\left[\|XB\|_2^2 + \nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]. \end{aligned} \tag{3.4}$$

Now, by the unitarily invariant property of $\|.\|_2$, we obtain the matrix version of the inequalities (2.7)-(2.8) as follows.

Theorem 3.2. *Let $A,B \in M_n^+(\mathbb{C})$, $X \in M_n(\mathbb{C})$ and $0 \leqslant \nu \leqslant 1$.*

1. *If $\nu \notin [\frac{1}{2}, \frac{3}{4}]$, then*

$$\begin{aligned} &\|\nu AX + (1-\nu)XB\|_2^2 \\ &\leqslant (1-\nu)^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + (1-\nu)^{2(1-\nu)}\|A^{\nu}XB^{1-\nu}\|_2^2 \\ &-(1-2\nu)\left[\|AX\|_2^2 + (1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right]. \end{aligned} \tag{3.5}$$

2. *If $\nu \notin [\frac{1}{4}, \frac{1}{2}]$, then*

$$\begin{aligned} &\|\nu AX + (1-\nu)XB\|_2^2 \\ &\leqslant \nu^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \nu^{2\nu}\|A^{\nu}XB^{1-\nu}\|_2^2 \\ &-(2\nu-1)\left[\|XB\|_2^2 + \nu\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 - 2\sqrt{\nu}\left\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\right\|_2^2\right]. \end{aligned} \tag{3.6}$$

Proof. It is well known, that every positive definite matrix is unitarily diagonalizable. Since $A,B \in M_n^+(\mathbb{C})$, then by the spectral decomposition, there are unitary matrices $U,V \in M_n(\mathbb{C})$ so that $A = UDU^*$ and $B = VEV^*$, where $D = diag(\lambda_1,\ldots,\lambda_n)$ and $E = diag(\mu_1,\ldots,\mu_n)$ with the eigenvalues $\lambda_i, \mu_i \geqslant 0$ for $1 \leqslant i \leqslant n$.
Let $Y = U^*XV = [y_{ij}]$, then we have

$$\begin{aligned} A^{\nu}XB^{1-\nu} &= (UDU^*)^{\nu}X(VEV^*)^{1-\nu} \\ &= UD^{\nu}(U^*XV)E^{1-\nu}V^* \\ &= U(D^{\nu}YE^{1-\nu})V^* \\ &= U(\lambda_i^{\nu}\mu_j^{1-\nu}y_{ij})V^*. \end{aligned}$$

Using unitarily invariant property of $\|.\|_2$, it follows that

$$\begin{aligned} \|A^{\nu}XB^{1-\nu}\|_2^2 &= \|U(D^{\nu}YE^{1-\nu})V^*\|_2^2 \\ &= \|D^{\nu}YE^{1-\nu}\|_2^2 \\ &= \sum_{i,j=1}^{n}(\lambda_i^{\nu}\mu_j^{1-\nu})^2|y_{ij}|^2. \end{aligned}$$

Analogously, one can prove that

$$\|\nu AX + (1-\nu)XB\|_2^2 = \sum_{i,j=1}^{n}(\nu\lambda_i + (1-\nu)\mu_j)^2|y_{ij}|^2, \quad \|AX - XB\|_2^2 = \sum_{i,j=1}^{n}(\lambda_i - \mu_j)^2|y_{ij}|^2$$

and

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|_2^2 = \sum_{i,j=1}^{n}\left(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\right)^2|y_{ij}|^2.$$

If $\nu \notin [\frac{1}{2}, \frac{3}{4}]$, then in view of the unitary invariance of the Hilbert-Schmidt norm and the inequality (2.7), we have

$$\|\nu AX+(1-\nu)XB\|_2^2 = \sum_{i,j=1}^{n} (\nu\lambda_i+(1-\nu)\mu_j)^2|y_{ij}|^2$$
$$\leqslant (1-\nu)^2\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|y_{ij}|^2+2\nu(1-\nu)\sum_{i,j=1}^{n}(\lambda_i\mu_j)^{\frac{1}{2}}|y_{ij}|^2$$
$$-(1-2\nu)\sum_{i,j=1}^{n}\lambda_i(\sqrt{\lambda_i}-\sqrt{(1-\nu)\mu_j})^2|y_{ij}|^2+\sum_{i,j=1}^{n}[\lambda_i^{\nu}((1-\nu)\mu_j)^{1-\nu}]^2|y_{ij}|^2$$
$$=(1-\nu)^2\|AX-XB\|_2^2+2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2+(1-\nu)^{2(1-\nu)}\|A^{\nu}XB^{1-\nu}\|_2^2$$
$$-(1-2\nu)\left[\|AX\|_2^2+(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2-2\sqrt{1-\nu}\left\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\right\|_2^2\right].$$

Similar to the inequality (3.5), we can prove the inequality (3.6). □

Remark 3.3. *By a similar method as in Remark 2.3, it is easy to verify that the range of the inequalities (3.5) and (3.6) are the wider than of the range of the inequalities (3.1)-(3.4), respectively.*

Now, we are going to present the matrix versions of the inequalities (2.5) and (2.6) for the trace norm. To do this work, we need to state the following Lemmas.

Lemma 3.4. *[3] (Cauchy-Schwarz inequality). Suppose $a_i, b_i \geqslant 0, (1\leqslant i\leqslant n)$.*

$$\sum_{i=1}^{n} a_ib_i \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}. \tag{3.7}$$

Lemma 3.5. *[3] Let $A,B\in M_n(\mathbb{C})$, then*

$$\sum_{j=1}^{n} s_j(AB) \leqslant \sum_{j=1}^{n} s_j(A)s_j(B). \tag{3.8}$$

Using Corollary 2.1 and the trace norm, we have the following result:

Corollary 3.6. *Let $A,B\in M_n^+(\mathbb{C})$, $X\in M_n(\mathbb{C})$ and $0\leqslant\nu\leqslant 1$.*

1. *If $0\leqslant\nu\leqslant\frac{1}{4}$, then*

$$\begin{aligned}&\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2\\&\leqslant(1-\nu)^2[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1]\\&-2\nu[\|A\|_2^2+(1-\nu)\|AB\|_1-2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1\|B^{\frac{1}{2}}\|_1]\\&+(1-\nu)^{2(1-\nu)}\|A^{\nu}\|_2^2\|B^{1-\nu}\|_2^2,\end{aligned} \tag{3.9}$$

2. *If $\frac{1}{4}\leqslant\nu\leqslant\frac{1}{2}$, then*

$$\begin{aligned}&\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2\\&\leqslant(1-\nu)^2[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1]\\&-(1-2\nu)[\|A\|_2^2+(1-\nu)\|AB\|_1-2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1\|B^{\frac{1}{2}}\|_1]\\&+(1-\nu)^{2(1-\nu)}\|A^{\nu}\|_2^2\|B^{1-\nu}\|_2^2,\end{aligned} \tag{3.10}$$

3. *If $\frac{1}{2}\leqslant\nu\leqslant\frac{3}{4}$, then*

$$\begin{aligned}\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2\leqslant{}&\nu^2[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1]\\&-(2\nu-1)[\|B\|_2^2+\nu\|AB\|_1-2\sqrt{\nu}\|A^{\frac{1}{2}}\|_1\|B^{\frac{3}{2}}\|_1]\\&+\nu^{2\nu}\|A^{\nu}\|_2^2\|B^{1-\nu}\|_2^2,\end{aligned} \tag{3.11}$$

4. *If $\frac{3}{4}\leqslant\nu\leqslant 1$, then*

$$\begin{aligned}\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2\leqslant{}&\nu^2[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1]\\&-(2-2\nu)[\|B\|_2^2+\nu\|AB\|_1-2\sqrt{\nu}\|A^{\frac{1}{2}}\|_1\|B^{\frac{3}{2}}\|_1]\\&+\nu^{2\nu}\|A^{\nu}\|_2^2\|B^{1-\nu}\|_2^2,\end{aligned} \tag{3.12}$$

Theorem 3.7. *Let $A,B \in M_n^+(\mathbb{C})$, $X \in M_n(\mathbb{C})$ and $0 \leqslant \nu \leqslant 1$.*

1. *If $\nu \notin [\frac{1}{2}, \frac{3}{4}]$, then*

$$\begin{aligned} &\nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2 \\ &\leqslant (1-\nu)^2[\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1] \\ &+ (1-\nu)^{2(1-\nu)}\|A^\nu\|_2^2\|B^{1-\nu}\|_2^2 \\ &- (1-2\nu)[\|A\|_2^2 + (1-\nu)\|AB\|_1 - 2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1\|B^{\frac{1}{2}}\|_1], \end{aligned} \tag{3.13}$$

2. *If $\nu \notin [\frac{1}{4}, \frac{1}{2}]$, then*

$$\begin{aligned} &\nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2 \\ &\leqslant \nu^2[\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1] \\ &+ \nu^{2\nu}\|A^\nu\|_2^2\|B^{1-\nu}\|_2^2 \\ &- (2\nu-1)[\|B\|_2^2 + \nu\|AB\|_1 - 2\sqrt{\nu}\|A^{\frac{1}{2}}\|_1\|B^{\frac{3}{2}}\|_1]. \end{aligned} \tag{3.14}$$

Proof. For $\nu \notin [\frac{1}{2}, \frac{3}{4}]$, by the inequality (2.5), it follows that

$$\begin{aligned} &\nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2 \\ &= tr(\nu^2A^2 + (1-\nu)^2B^2) \end{aligned}$$

$$= \nu^2 trA^2 + (1-\nu)^2 trB^2$$

$$\begin{aligned} &= \sum_{j=1}^n (\nu^2 s_j^2(A) + (1-\nu)^2 s_j^2(B)) \\ &\leqslant (1-\nu)^2\left[\sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2\sum_{j=1}^n s_j(A)s_j(B)\right] \\ &+ (1-\nu)^{2(1-\nu)}\sum_{j=1}^n [s_j(A^\nu)s_j(B^{1-\nu})]^2 \\ &- (1-2\nu)\left[\sum_{j=1}^n s_j^2(A) + (1-\nu)\sum_{j=1}^n s_j(A)s_j(B) - 2\sqrt{1-\nu}\left(\sum_{j=1}^n s_j^{\frac{3}{2}}(A)s_j^{\frac{1}{2}}(B)\right)\right] \\ &\leqslant (1-\nu)^2\left[\sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2\sum_{j=1}^n s_j(AB)\right] (\text{by } (3.8)) \\ &+ (1-\nu)^{2(1-\nu)}[\sum_{j=1}^n s_j^2(A^\nu)\sum_{j=1}^n s_j^2(B^{1-\nu})] \\ &- (1-2\nu)\left[\sum_{j=1}^n s_j^2(A) + (1-\nu)\sum_{j=1}^n s_j(A)s_j(B) - 2\sqrt{1-\nu}\left(\sum_{j=1}^n s_j^{\frac{3}{4}}(A)s_j^{\frac{1}{4}}(B)\right)^2\right] \\ &\leqslant (1-\nu)^2\left[\sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2\sum_{j=1}^n s_j(AB)\right] (\text{by } (3.8)) \\ &+ (1-\nu)^{2(1-\nu)}[\sum_{j=1}^n s_j^2(A^\nu)\sum_{j=1}^n s_j^2(B^{1-\nu})] \\ &- (1-2\nu)\left[\sum_{j=1}^n s_j^2(A) + (1-\nu)\sum_{j=1}^n s_j(A)s_j(B)\right. \\ &\left. - 2\sqrt{1-\nu}\left(\sum_{j=1}^n s_j^{\frac{1}{2}}(A)\sum_{j=1}^n s_j^{\frac{1}{2}}(B)\right)\right] (\text{by } (3.7)) \\ &\leqslant (1-\nu)^{2(1-\nu)}[\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1] \\ &+ (1-\nu)^{2(1-\nu)}\|A^\nu\|_2^2\|B^{1-\nu}\|_2^2 \\ &- (1-2\nu)\left[\|A\|_2^2 + (1-\nu)\|AB\|_1 - 2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1\|B^{\frac{1}{2}}\|_1\right]. \end{aligned}$$

This estimate completes the proof of (3.13). The proof (3.14) is similar. So we omit its details. This completes the proof.

□

Remark 3.8. *Obviously, the range of the inequalities (3.13) and (3.14) are the wider than the range of the inequalities (3.9)-(3.12), respectively. Therefore, the right hand side of the inequalities (3.13) and (3.14) are the better bounds respect to the right hand side of the inequalities (3.9)-(3.12)*

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