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The Hosoya Polynomial of the Schreier Graphs of the Grigorchuk Group and the Basilica Group

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Abstract. The Grigorchuk group was first introduced by R. Grigorchuk in 1980. Also the Basilica group was introduced in 2002 by R. Grigorchuk and A. Zuk. In the following years, it was shown that these groups have deep connections with profinite group theory and complex dynamics. These groups have been proven to provide the self-similarity property, reflecting the fractalness of some limit objects associated with them. The Schreier graph codifies the intangible structure of a group. It establishes an equivalence relationship created by cosets. The Schreier graphs of the Grigorchuk group and the Basilica group are a combination of cycles arranged in a tree-like form due to the recursive expression of the generators of these groups. In this work, we study the Hosoya polynomial of these graphs and try to characterize them.

1. Introduction

The Hosoya polynomial of a graph was presented in 1988 by H. Hosoya [8]. The concept *distance* is one of the basic elements used in graph theory. This important concept has gained a wide place among the applications of graph theory in other disciplines. Hosoya polynomial is also defined with the help of this important concept. The main contribution of the Hosoya polynomial is that it provides important information for graph invariants defined with the help of the concept of distance. The value at point 1 of the first derivative of the Hosoya polynomial of a graph gives us the Wiener index, which is an important topological index [5]. The Hosoya polynomial has gained an important place in chemical graph theory studies [4].

In this work, we study the Hosoya polynomial of Schreier graphs associated with the motion of two automorphism groups of a binary rooted tree. These are the Grigorchuk group and the Basilica group. The Tutte polynomial of these graphs was calculated in 2010 [3]. The Grigorchuk group was first introduced by R. Grigorchuk in 1980. It gives a fairly simple solution to the Burnside problem and the first example of a finitely generated group of intermediate growth, see [7]. Also the Basilica group was introduced in 2002 by R. Grigorchuk and A. Zuk [6]. To the work of V. Nekrashevych, it was seen that this group can be defined as the iterated monodromy group of the complex polynomial $z^2 - 1$ [9]. Thus, a compact limit space that is homeomorphic to the Basilica fractal can be associated with it. It is also the first example of amenable group that does not belong to subexponentially amenable groups [2]. It is proved that these groups are very

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closely related to complex dynamics and profinite group theory [1]. These groups provide a self-similarity property that reflects the fractalness of some limit objects associated with them [9].

We are here doing some calculations over the Schreier graphs of the Grigorchuk group and the Basilica group. Moreover, we reckon the Wiener index of some these graphs. We carry out these calculations by deleting the loops in the graphs. The Schreier graph codifies the intangible structure of a group. It establishes an equivalence relationship created by cosets. The Schreier graphs of the Grigorchuk group and the Basilica group are a combination of cycles formed in a tree-like way. Because the recursive expressions of the generators of these groups cause these graphs to have a cactus structure.

2. Preliminaries

Definition 2.1. ([5]) *Let $G = (V, E)$ be connected and distance-based graph. The distance $d(u, v)$ between any two vertices u and v is the minimum of the lenghts of paths between u and v. The topological diameter $d(G)$ of a graph G (i.e. the longest topological distance in G) is defined as*

$$d(G) = \max_{u,v \in V(G)} \{d(u, v)\}.$$

Definition 2.2. ([10]) *Let $D_k = \{(u, v) |\ u, v \in V(G)$ and $d(u, v) = k\}$ be a set and we denote the number of elements of D_k by $|D_k|$ i.e. $d(G, k) = |D_k|$, $k \geq 0$.*

Let $d(G, k)$, $k \geq 0$, be the number of vertex pairs at distance k. The Hosoya polynomial of G is defined as follows:

$$H(G, y) = \sum_{k=0}^{d(G)} d(G, k) y^k$$

where $d(G, 0) = n$ such that n is the number of vertices in G.

The Grigorchuk group and the Basilica group are a self-similar group of automorphisms of the rooted binary tree generated by some elements which are the trivial and the non-trivial permutations in the symmetric group on 2 elements $Sym(2)$ [3]. The Schreier graphs of these groups are recursively constructed within the framework of certain rules, see [3] for more detailed information. The symbol Γ_n indicates the Schreier graphs of the Grigorchuk group, for $n = 1, 2, 3, \ldots$, as seen in Figure 1 [3]. The symbol B_n indicates the Schreier graphs of the Basilica group, for $n = 1, 2, 3, \ldots$, as seen in Figures 2 and 4 [3].

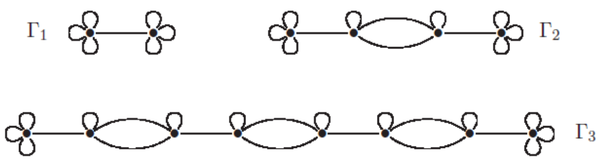

Figure 1: Some the Schreier graphs of the Grigorchuk group

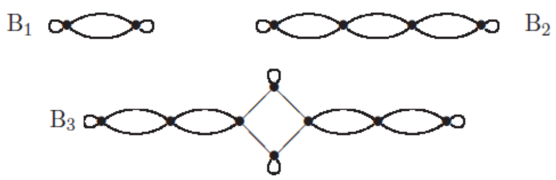

Figure 2: Some the Schreier graphs of the Basilica group

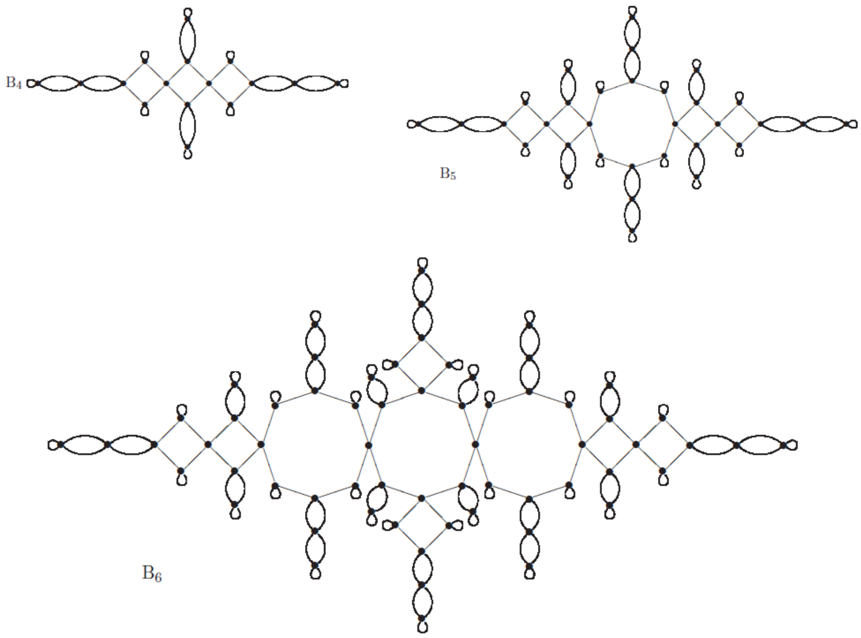

Figure 3: Some the Schreier graphs of the Basilica group

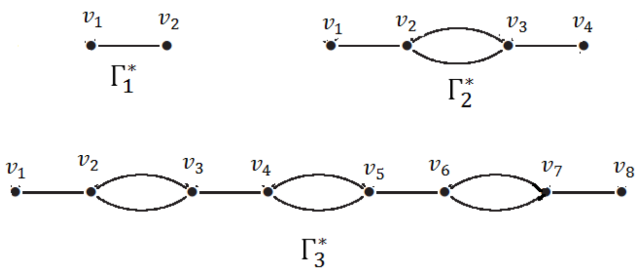

Figure 4: Some the Schreier graphs without loops of the Grigorchuk group

Since many calculations are inconclusive for graphs containing loops, we will consider the graphs obtained by deleting loops from these graphs, as seen in Figures 4 and 5.

The graphs Γ_n^* and B_n^* contain the values specified in the table below.

$d(\Gamma_1^*) = 1$	$d(\Gamma_2^*) = 3$	$d(\Gamma_3^*) = 7$	$d(\Gamma_4^*) = 15$	$d(\Gamma_5^*) = 31$
$d(B_1^*) = 1$	$d(B_2^*) = 3$	$d(B_3^*) = 6$	$d(B_4^*) = 10$	$d(B_5^*) = 16$

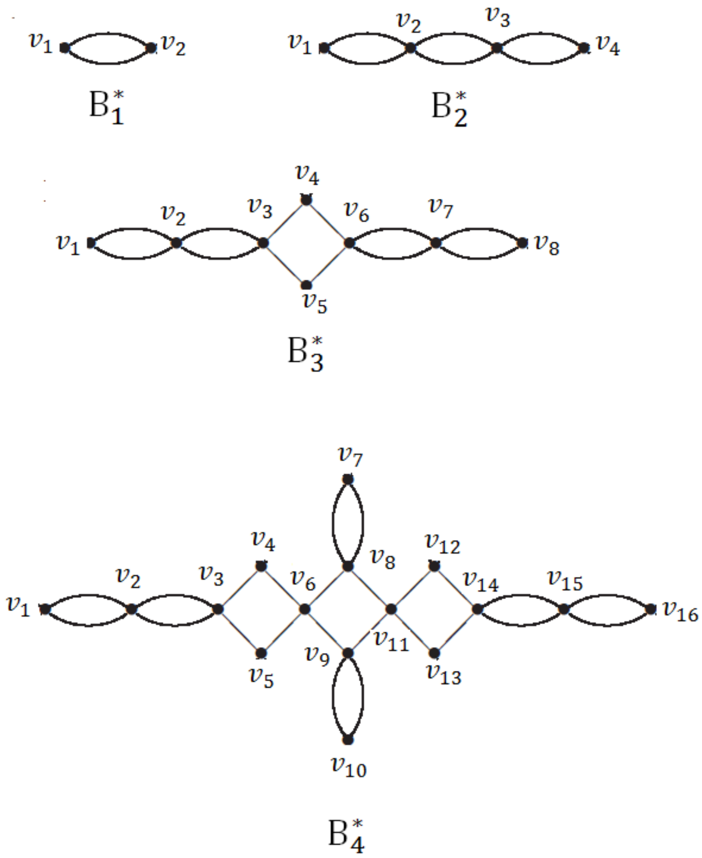

Figure 5: Some the Schreier graphs without loops of the Basilica group

3. Main Results

In this section, we will compute the Hosoya polynomials of the graphs Γ_n^* and B_n^*, $(n = 1, 2, 3, \dots)$.

For Γ_1^*:

$$D_0 = \{v_1, v_2\} \Rightarrow |D_0| = d(\Gamma_1^*, 0) = 2,$$
$$D_1 = \{(v_1, v_2)\} \Rightarrow |D_1| = d(\Gamma_1^*, 1) = 1,$$
$$\Rightarrow H(\Gamma_1^*, y) = 2y^0 + 1y^1$$
$$H(\Gamma_1^*, y) = 2 + y \tag{1}$$

For Γ_2^*:

$$D_0 = \{v_1, v_2, v_3, v_4\} \Rightarrow |D_0| = d(\Gamma_2^*, 0) = 4,$$
$$D_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\} \Rightarrow |D_1| = d(\Gamma_2^*, 1) = 3,$$
$$D_2 = \{(v_1, v_3), (v_2, v_4)\} \Rightarrow |D_2| = d(\Gamma_2^*, 2) = 2,$$
$$D_3 = \{(v_1, v_4)\} \Rightarrow |D_3| = d(\Gamma_2^*, 3) = 1,$$

$$\Rightarrow H(\Gamma_2^*, y) = 4y^0 + 3y^1 + 2y^2 + 1y^3$$
$$H(\Gamma_2^*, y) = 4 + 3y + 2y^2 + y^3$$

For Γ_3^*:

$$D_0 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \Rightarrow |D_0| = d(\Gamma_3^*, 0) = 8,$$
$$D_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_7, v_8)\} \Rightarrow |D_1| = d(\Gamma_3^*, 1) = 7,$$
$$D_2 = \{(v_1, v_3), (v_2, v_4), (v_3, v_5), (v_4, v_6), (v_5, v_7), (v_6, v_8)\} \Rightarrow |D_2| = d(\Gamma_3^*, 2) = 6,$$
$$D_3 = \{(v_1, v_4), (v_2, v_5), (v_3, v_6), (v_4, v_7), (v_5, v_8)\} \Rightarrow |D_3| = d(\Gamma_3^*, 3) = 5,$$
$$D_4 = \{(v_1, v_5), (v_2, v_6), (v_3, v_7), (v_4, v_8)\} \Rightarrow |D_4| = d(\Gamma_3^*, 4) = 4,$$
$$D_5 = \{(v_1, v_6), (v_2, v_7), (v_3, v_8)\} \Rightarrow |D_5| = d(\Gamma_3^*, 5) = 3,$$
$$D_6 = \{(v_1, v_7), (v_2, v_8)\} \Rightarrow |D_6| = d(\Gamma_3^*, 6) = 2,$$
$$D_7 = \{(v_1, v_8)\} \Rightarrow |D_7| = d(\Gamma_3^*, 7) = 1,$$
$$\Rightarrow H(\Gamma_3^*, y) = 8y^0 + 7y^1 + 6y^2 + 5y^3 + 4y^4 + 3y^5 + 2y^6 + 1y^7$$
$$H(\Gamma_3^*, y) = 8 + 7y + 6y^2 + 5y^3 + 4y^4 + 3y^5 + 2y^6 + y^7$$

Theorem 3.1. *The Hosoya polynomial of the Schreier graphs of the Grigorchuk group is defined as*

$$H(\Gamma_n^*, y) = \sum_{i=1}^{2^n} i y^{2^n - i}, \tag{2}$$

where $n = 1, 2, 3, \ldots$.

Proof. We will make the proof of the theorem by the induction method on n. Firstly, it is clear that the expression is $H(\Gamma_1^*, y) = y + 2$ for $n = 1$ and it is obvious. It follows from the equation (1). Then, for $n = k$, let us assume that the expression, i.e. the equation

$$H(\Gamma_k^*, y) = y^{2^k - 1} + 2y^{2^k - 2} + 3y^{2^k - 3} + \cdots + 2^{k-1} y^{2^k - 2^{k-1}} + 2^k y^{2^k - 2^k}$$
$$H(\Gamma_k^*, y) = y^{2^k - 1} + 2y^{2^k - 2} + 3y^{2^k - 3} + \cdots + 2^{k-1} y^{2^k - 2^{k-1}} + 2^k,$$

is true. The correctness of the expression will now be shown for $n = k + 1$. For $n = 1, 2, 3, \ldots$ Γ_n^* has a linear shape formed by alternating bridges and 2-cycles. Moreover, for $n = 1, 2, 3, \ldots$ there are 2^n vertices and $3.2^{n-1} - 2$ edges in Γ_n^* and the diameter of Γ_n^* is equal to $2^n - 1$. It means that there are $\frac{2^{k+1}}{2}$ bridges among the edges in Γ_{k+1}^* and the remaining $(3.2^{n-1} - 2 - \frac{2^{k+1}}{2})$ edges in Γ_{k+1}^* are two by two parallel. For $n = k + 1$, there must be 2^{k+1} terms in the expansion of the expression. Therefore by the concept of distance in graphs and the definition of the Hosoya polynomial, for $n = k + 1$ it is obtained that

$$H(\Gamma_{k+1}^*, y) = y^{2^{k+1} - 1} + 2y^{2^{k+1} - 2} + 3y^{2^{k+1} - 3} + \cdots + 2^k y^{2^{k+1} - 2^k} + 2^{k+1}.$$

Thus the proof is completed. □

Proposition 3.2. *The Wiener index of the Schreier graphs of the Grigorchuk group is defined as*

$$W(\Gamma_n^*) = \sum_{i=1}^{2^n} i(2^n - i),$$

where $n = 1, 2, 3, \ldots$.

Proof. By applying the equation (2), The Wiener index of the Schreier graphs of the Grigorchuk group is obtained. It is reckoned as the first derivative of the polynomial of $H(\Gamma_n^*, y)$ at $y = 1$, i.e.,

$$\begin{aligned} \Big(H(\Gamma_n^*, y)\Big)' &= \Big(\sum_{i=1}^{2^n} i y^{2^n - i}\Big)' \\ &= \sum_{i=1}^{2^n} i(2^n - i) y^{2^n - i - 1}, \\ \Big(H(\Gamma_n^*, 1)\Big)' &= \sum_{i=1}^{2^n} i(2^n - i) = W(\Gamma_n^*). \end{aligned}$$

So the proof is completed. □

Now let us give a few examples of calculating the Hosoya polynomials of B_n^*. According to the definition of the Hosoya polynomial, the following results are obtained by applying the method applied in the above calculations.

For $n = 1$,

$$H(B_1^*, y) = 2 + y.$$

For $n = 2$,

$$H(B_2^*, y) = 4 + 3y + 2y^2 + y^3.$$

For $n = 3$,

$$H(B_3^*, y) = 8 + 8y + 8y^2 + 6y^3 + 3y^4 + 2y^5 + y^6$$

For $n = 4$,

$$H(B_4^*, y) = 16 + 18y + 24y^2 + 24y^3 + 17y^4 + 14y^5 + 11y^6 + 6y^7 + 3y^8 + 2y^9 + y^{10}.$$

For $n = 5$,

$$H(B_5^*, y) = 32+36y+49y^2+62y^3+62y^4+64y^5+55y^6+42y^7+36y^8+30y^9+18y^{10}+14y^{11}+11y^{12}+6y^{13}+3y^{14}+2y^{15}+y^{16}.$$

Conclusion 3.3. *In the calculations for the Hosoya polynomial of the Schreier graphs of the Basilica group, as can be seen in the examples given above, the following can be stated: in the expansion of polynomials to be obtained for each n, although some values such as the number of terms, the degree of the terms, some of its beginning and last terms are known a general characterization of these polynomials is not possible in this way. Because there is no clarity for the coefficients of the polynomials. However, it is predicted that this problem can be solved by conducting a study on the array of the shape of the graph obtained for each n as different from the method followed here. This prediction stands as an open problem.*

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