On some inequalities for derivatives of algebraic polynomials in unbounded regions with angles

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**A B S T R A C T**

In this work we study Bernstein-Walsh-type estimations for the derivative of an arbitrary algebraic polynomial in regions with interior zero and exterior non zero angles.

**1. Introduction**

Let \( \mathbb{C} \) denote the complex plane and \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \); \( G \subset \mathbb{C} \) be a bounded Jordan region with boundary \( L = \partial G \) such that \( 0 \in G \);

Let \( \{ z_j \}_{j=1}^l \) be the fixed system of distinct points on the curve \( L \). We consider generalized Jacobi weight function \( h(z) \) which is defined as follows:

\[
h(z) = \prod_{j=1}^l |z - z_j|^{\gamma_j}, \quad z \in \mathbb{C},
\]

where \( \gamma_j > -2 \), for all \( j = 1, 2, \ldots, l \).

Let \( \mathcal{P}_n \) denotes the class of all algebraic polynomials \( P_n(z) \) of degree at most \( n \in \mathbb{N} \).

Let \( p > 0 \). For the Jordan region \( G \), we introduce:

\[
\|P_n\|_p := \|P_n\|_{A_p(h, G)} := \left( \int_G h(z)|P_n(z)|^p \, d\sigma_z \right)^{1/p}, 0 < p < \infty,
\]

\[
\|P_n\|_\infty := \|P_n\|_{A_\infty(1, G)} := \max_{z \in \overline{G}} |P_n(z)|, p = \infty,
\]

and \( A_p(1, G) \equiv A_p(G) \), where \( \sigma \) be the two-dimensional Lebesgue measure.

When \( L \) is rectifiable, for any \( p > 0 \), let

\[
\|P_n\|_{L_p(h, L)} := \left( \int_L h(z)|P_n(z)|^p \, |dz| \right)^{1/p}, 0 < p < \infty,
\]

\[
\|P_n\|_{L_\infty(1, L)} := \max_{z \in L} |P_n(z)|, p = \infty,
\]

and \( L_p(1, L) \equiv L_p(L) \).

Let us set \( \Omega := \overline{\mathbb{C}} \setminus \overline{G} = \text{ext} L; \Delta(w, R) := \{ w : |w| > R, R > 1 \}, \Delta := \Delta(0, 1) \) and let \( w = \Phi(z) \) be the univalent conformal
mapping of $\Omega$ onto $\Delta$ such that $\Phi(\infty) = \infty$ and $\lim_{z \to \infty} \frac{\Phi(z)}{z} > 0$; $\Psi = \Phi^{-1}$. For $R > 1$ we define $L_R := \{z : |\Phi(z)| = R\}$, $G_R := \text{int} L_R$, $\Omega_R := \text{ext} L_R$.

Well known Bernstein-Walsh Lemma [26] says that:

$$\|P_n\|_{C(\mathcal{L}_R)} \leq R^n \|P_n\|_{C(\mathcal{J})}. \quad (4)$$

Analogous estimation with respect to the quasinorm (4) for $p > 0$ was obtained in [19] for $h(z) \equiv 1$ (i.e., $\gamma_j = 0$ for all $j = 1, 2, \ldots, l$) and in [8, Lemma 2.4] for $h(z) \neq 1$, defined as in (1) as following:

$$\|P_n\|_{L_p(h, L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{L_p(h, L)} \quad \gamma^* = \max\{0; \gamma_j; 1 \leq j \leq l\}. \quad (5)$$

To give a similar estimation to (5) for the $A_p(h, G)$-norm, first of all we will give the following definition.

**Definition 1.** [20, p.97], [23] The Jordan arc (or curve) $L$ is called $K$-quasiconformal ($K \geq 1$), if there is a $K$-quasiconformal mapping $f$ of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms $f$ of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of $f$. Then $L$ is a quasiconformal curve, if $K_L < \infty$, and $L$ is a $K$-quasiconformal curve, if $K_L \leq K$.

A curve $L$ is called a quasiconformal, if it is a $K$-quasiconformal for some $K > 1$.

The Bernstein-Walsh type estimates for the norm (2), for the regions with quasiconformal boundary and weight function $h(z)$, defined in (1) with $\gamma_j > -2$, for all $p > 0$ as follows

$$\|P_n\|_{A_p(h, G_R)} \leq c_1 R^{\gamma_j + \frac{1}{n}} \|P_n\|_{A_p(h, G)}, \quad (6)$$

was found in [3] (see, also [2]), where $R^* := 1 + c_2(R - 1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from $n$ and $R$. It’s well known that quasiconformal curves can be non-rectifiable (see, for example, [16], [20, p.104]).

Analogous estimation was studied for $A_p(1, G)$-norm, $p > 0$, for arbitrary Jordan region in [4, Theorem 1.1] and for any $P_n \in \mathcal{P}_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary $R$, $R > R_1$, was obtain

$$\|P_n\|_{A_p(G_R)} \leq c R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_R)},$$

where $c = \left(\frac{2}{e^{p-1}}\right)^{\frac{1}{p}}\left[1 + O\left(\frac{1}{n}\right)\right]$, $n \to \infty$.

For a rectifiable quasiconformal curve $L$, N. Stylianopoulos [24] obtained the following estimate:

$$|P_n(z)| \leq c \sqrt{\frac{n}{d(z, L)}} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \quad (7)$$

where $d(z, L) := \inf\{|z - \xi| : \xi \in L\}$, a constant $c = c(L) > 0$ depending only on $L$.

Analogous results of (7)-type for $|P_n(z)|$ of different weight function $h$, unbounded region $\Omega$ were obtained in [17, p.418-428], [5], [6], [7], [8], [9], [10], [11], [15], [22] and others.

In this work, we study the pointwise estimates for the derivative $|P_n'(z)|$ in unbounded region $\Omega$ with zero angles as the following type

$$|P_n'(z)| \leq c_2 \eta_n(G, h, p, d(z, L), |\Phi(z)|) \|P_n\|_{p}, \quad z \in \Omega, \quad (8)$$

where $c_2 = c_2(G, p) > 0$ is a constant independent of $n, Z$ and $P_n$, and $\eta_n(G, h, p, d(z, L), |\Phi(z)|) \to \infty$, $n \to \infty$, depending on the properties of the $G$, $h$ and from the distance of point $z \in \Omega$ to the $\overline{G}$. 

2. Definitions and main results

Throughout this paper, \(c, c_0, c_1, c_2, \ldots \) are positive and \(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \) are sufficiently small positive constants (generally, different in different relations), which depends on \(G\) in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any \(k \geq 0\) and \(m > k\), notation \(i = \overline{k, m}\) means \(i = k, k+1, \ldots, m\). Let \(z = z(s)\), \(s \in [0, \text{mes} L]\) denote the natural representation of \(L\).

Definition 2. We say that \(L \in C_0\), if \(L\) has a continuous tangent \(\theta(z) := \theta(z(s))\) at every point \(z(s)\). Then we write \(G \in C_0 \Leftrightarrow \partial G \in C_0\).

According to the "three-point" criterion [13, p.100], every piecewise smooth curve (without any cusps) is quasiconformal. Moreover, according to [23], we have the following:

Corollary 3. If \(G \in C_0\), then \(\partial G\) is \((1 + \varepsilon)\)-quasiconformal for arbitrary small \(\varepsilon > 0\).

Now we give the definitions of regions with a piecewise smooth curve, which we present our main result and some notation that will be used later in the text.

Definition 4. [5] We say that a Jordan region \(G \in C_0(\lambda_1, \ldots, \lambda_l)\), \(0 < \lambda_j < 2\) is \(\gamma\)-quasiconformal (\(\gamma\)-q.c.) if \(\partial G\) consists of the union of finite smooth arcs \(\{l_j\}_{j=1}^l\), such that they have exterior (with respect to \(G\)) angles \(\lambda_j \pi\), \(0 < \lambda_j < 2\), at the corner points \(\{z_j\}_{j=1}^l \in L\), where two arcs meet.

Without loss of generality, we assume that these points on the curve \(L = \partial G\) are located in the positive direction such that, \(G\) has exterior \(\lambda_j \pi\), \(0 < \lambda_j < 2\), \(j = \overline{1, l}\), angle at the points \(\{z_j\}_{j=1}^l\), \(l \leq l\), and interior zero angle (i.e. \(\lambda_j = 2\) - interior cusps) at the points \(\{z_j\}_{j=1}^l\).

It is clear from Definition 4, the each region \(G \in C_0(\lambda_1, \ldots, \lambda_l)\), \(0 < \lambda_j < 2\), may have exterior nonzero \(\lambda_j \pi\), \(0 < \lambda_j < 2\), angles at the points \(\{z_j\}_{j=1}^l \in L\), and interior zero angles (\(\lambda_j = 2\)) at the the points \(\{z_j\}_{j=1}^l \in L\). If \(l = l = 0\), then the region \(G\) doesn’t have such angles, and in this case we will write: \(G \in C_0\); if \(l_1 = l \geq 1\), then \(G\) has only \(\lambda_j \pi\), \(0 < \lambda_j < 2\), \(i = \overline{1, l}\), exterior nonzero angles, and in this case we will write: \(G \in C_0(\lambda_i)\); if \(l_1 = 0\) and \(l \geq 1\), then \(G\) has only interior zero angles, and in this case we will write: \(G \in C_0(2)\).

Throughout this work, we will assume that the points \(\{z_j\}_{j=1}^l \in L\) defined in (1) and Definition 4 are identical and \(w_j := \Phi(z_j)\).

For simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we will take \(l_1 = 1, l = 2\). Then, after this assumption, in the future we will have region \(G \in C_0(\lambda_1, 2)\), \(0 < \lambda_1 < 2\), such that at the point \(z_1 \in L\) region \(G\) have exterior nonzero \(\lambda_1 \pi\), \(0 < \lambda_1 < 2\), and at the point \(z_2 \in L\) - interior zero angle. Note that, the notation "\(G \in C_0(\lambda_1, \lambda_2)\), \(0 < \lambda_1, \lambda_2 < 2\)" means that the region \(G\) has two exterior nonzero \(\lambda_j \pi\), \(0 < \lambda_j < 2\), \(j = \overline{1, 2}\), angles at the point \(z_j \in L\).

For \(0 < \delta_j < \delta_0 := \frac{1}{4} \min\{\|z_1 - z_2\| : j = 1, 2\}\), \(\delta := \min_{1 \leq j \leq 2} \delta_j\), let

\[
\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}; \\
\Omega(\delta) := \bigcup_{j=1}^{2} \Omega(z_j, \delta), \Omega := \Omega \setminus \Omega(\delta).
\]

In this work, we study problem of (8) type in regions with piecewise smooth boundary without exterior cusps and generalized Jacobi weight function \(h(z)\), as defined in (1).

Now, we start to formulate the new results.

Theorem 5. Let \(p > 1; G \in C_0(\lambda_1, 2)\), for some \(0 < \lambda_1 < 2\); \(h(z)\) be defined as in (1). Then, for any \(P_n \in \varphi_n, n \in \mathbb{N}, \gamma_j > -2\) and arbitrary small \(\varepsilon > 0\)
\[ |P'_n(z)| \leq c_1 \left[ \frac{|\Phi(x)|^{n+1}}{a(x/\lambda_1)} G_{n,1}(z) + \frac{|\Phi(x)|^{2(n+1)}}{a(z/\lambda_1+1/n)} B_{n,1}(z) E_{n,1} \right] \|P_n\|_p \]

holds, where \(c_1 = c_1(G, \gamma, p, \epsilon) > 0;\)

\[ G_{n,1}(z) = \begin{cases} \frac{\gamma}{n^p}, & \text{if } \gamma \cdot \lambda > 1, \\ \frac{\gamma}{n^p}, & \text{if } \gamma < 1, \end{cases} \]

\[ E_{n,1} = \begin{cases} \frac{\gamma}{n^p}, & \text{if } \gamma \cdot \lambda > 1, \\ \frac{\gamma}{n^p}, & \text{if } \gamma < 1, \end{cases} \]

\[ \lambda^* = \{ \max\{1; \lambda\} + \epsilon, \text{ if } 0 < \lambda < 2, \} \]

\[ \gamma^* = \{ \gamma_1, \text{ if } \lambda = 2, \} \]

\[ \gamma_1 = \max\{0; \gamma_1\}, \quad i = 1,2; \quad \lambda_1 = \max\{1; \lambda_1\} + \epsilon. \]

**Theorem 6.** Let \( p > 1; G \in C_0(\lambda_1, \lambda_2), \text{ for some } 0 < \lambda_1 < 2, j = 1,2; h(z) \text{ be defined as in (1).} \) Then, for any \( P_n \in \mathcal{P}_n, \quad n \in \mathbb{N}, \quad \gamma_j > -2 \text{ and arbitrary small } \epsilon > 0 \)

\[ |P'_n(z)| \leq c_2 \|P_n\|_p \left[ \frac{|\Phi(x)|^{n+1}}{a(x/\lambda_1)} G_{n,1}(z) + \frac{|\Phi(x)|^{2(n+1)}}{a(z/\lambda_1+1/n)} B_{n,2}(z) E_{n,2} \right] \]

holds, where \(c_2 = c_2(G, \gamma, p, \epsilon) > 0;\)

\[ G_{n,2}(z) = \begin{cases} \frac{\gamma}{n^p}, & \text{if } \gamma \cdot \lambda > 1, \\ \frac{\gamma}{n^p}, & \text{if } \gamma < 1, \end{cases} \]

\[ E_{n,2} = \begin{cases} \frac{\gamma}{n^p}, & \text{if } \gamma \cdot \lambda > 1, \\ \frac{\gamma}{n^p}, & \text{if } \gamma < 1, \end{cases} \]

\[ \lambda^* = \{ \max\{1; \lambda\} + \epsilon, \text{ if } 0 < \lambda < 2, \} \]

\[ \gamma^* = \{ \gamma_1, \text{ if } \lambda = 2, \} \]

\[ \gamma_1 = \max\{0; \gamma_1\}, \quad i = 1,2; \quad \lambda_1 = \max\{1; \lambda_1\} + \epsilon. \]

Analogously, we also can give a theorem for the regions such as \( G \in C_0(2,2). \)
3. Some auxiliary results

**Lemma 1.** [1] Let $L$ be a $K$–quasiconformal curve, $z_1 \in L, z_2, z_3 \in \Omega \cap \{z: |z - z_1| < d(z_1, L_{R_0})\}; w_j = \Phi(z_j), (z_2, z_3 \in G \cap \{z: |z - z_1| < d(z_1, L_{R_0})\}; w_j = \varphi(z_j), j = 1, 2, 3.$ Then

a) The statements $|z_1 - z_2| < |z_1 - z_3|$ and $|w_1 - w_2| < |w_1 - w_3|$ are equivalent.

So are $|z_1 - z_2| \leq |z_1 - z_3|$ and $|w_1 - w_2| \leq |w_1 - w_3|$.

b) If $|z_1 - z_2| < |z_1 - z_3|$, then

$$\frac{|w_1 - w_3|^2}{|w_1 - w_2|^2} < \frac{|z_1 - z_3|^2}{|z_1 - z_2|^2} < \frac{|w_1 - w_3|^2}{|w_1 - w_2|^2},$$

where $\varepsilon < 1, c > 1, R_0 > 1$ are constants, depending on $G$.

**Corollary 7.** Under the assumptions of Lemma 1, if $z_3 \in L_{R_0}$, then

$$|w_1 - w_2|^2 < |z_1 - z_2| < |w_1 - w_2|^{-2}.$$

**Corollary 8.** If $L \in C_0$, then

$$|w_1 - w_2|^{1+\varepsilon} < |z_1 - z_2| < |w_1 - w_2|^{1-\varepsilon},$$

for all $\varepsilon > 0$.

The following lemma is a consequence of the results given in [18], [21], [27] and of estimate for the $|\Psi'|$ (see, for example, [14, Th.2.8]):

$$|\Psi'(r)| \approx \frac{d(\psi(r), L)}{|r|^{-1}} \quad (10)$$

Let $w_j = \Phi(z_j), \varphi_j = \arg w_j$. Without loss of generality, we will assume that $\varphi_1 < 2\pi$. Additionally to the notations (9), for $\eta_j = \min_{t \in \Phi(\ell(z_j, \delta_j))}\{t - w_j\} > 0$ and $\eta = \min\{\eta_j, j = 1, l\}$ let us set: $\Delta(\eta_j) = \{t: |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j, \delta_j)), \Delta(\eta) = \bigcup_{j=1}^l \Delta_j(\eta), \tilde{\Delta}(\eta_j) = \Delta(\eta_j); \tilde{\Delta}(\eta) = \Delta(\eta), \Delta_1 = \Delta_1(1), \Delta_j = \Delta_j(\rho) = \{t = R \cdot e^{i\theta}; R \geq \rho > 1, \frac{q_0 + q_1}{2} \leq \theta < \frac{q_1 + q_2}{2}\}, j = 2, 3, \ldots, l$, where $\varphi_0 = 2\pi - \varphi_l; \Omega_j = \Psi(\Delta_j), L_{\rho_j} = \Omega_j \cap \Delta_j.$ Clearly, $\Omega = \bigcup_{j=1}^l \Omega_j$.

The following lemma is a consequence of the results given in [27] and [18].

**Lemma 2.** Let $G \in C_0(\lambda, \ldots, \lambda_l), 0 < \lambda_j < 2, j = 1, 2, \ldots, l$. Then

i) for any $w \in \Delta_j, |w - w_j|^{1+\varepsilon} < |\Psi(w) - \Psi(w_j)| < |w - w_j|^4 - \varepsilon, |w - w_j|^4 - 1 + \varepsilon < |\Psi'(w)| < |w - w_j|^{4-1-\varepsilon},$

ii) for any $w \in \tilde{\Delta}_j, (|w| - 1)^{1+\varepsilon} < d(\Psi(w), L) < (|w| - 1)^1 - \varepsilon, (|w| - 1)1 - \varepsilon < |\Psi'(w)| < (|w| - 1)^{-\varepsilon}.$

Let $\{z_j\}_{j=1}^l$ be a fixed system of distinct points on curve $L$ ordered in the positive direction and the weight function $h(z)$ be defined as in (1).

**Lemma 3.** [6] Let $L$ is a $K$–quasiconformal curve; $R = 1 + \frac{\varepsilon}{n}$. Then, for any fixed $\varepsilon \in (0, 1)$ there exist a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in p_n, n \in \mathbb{N}$:

$$\|P_n\|_{L_p(\frac{h}{|\Psi'|L_{1+\varepsilon(R-1)}})} < \frac{1}{n\pi^2}\|P_n\|_{p}, \quad p > 0. \quad (11)$$

**Lemma 4.** [6] Let $L$ be a $K$–quasiconformal curve; $h(z)$ be defined as in (1). Then, for arbitrary $P_n(z) \in p_n, n \in \mathbb{N}$ and $n = 1, 2, \ldots$, we have
\[
\|P_n\|_{A_p(b,G)} < R^{n+1} \|P_n\|_{A_p(b,G)}, \quad p > 0,
\]

where \( R = 1 + c(R - 1) \) and \( c \) is independent from \( n \) and \( R \).

**Lemma 5.** Let \( G \in C_0(\lambda_1, \ldots, \lambda_j), \quad 0 < \lambda_j \leq 2, \ j = 1, \ldots, l. \) Then, for arbitrary \( P_n(z) \in \mathcal{D}_n \) and any \( p > 0 \), we have:

\[
\|P_n\|_{A_p(b,G_{1+c/n})} < \|P_n\|_{A_p(b,G)}.
\]

4. **Proof of Theorems**

4.1. **Proof of Theorems 5 and 6.**

**Proof.** We will prove both theorems simultaneously. Suppose that \( G \in C_0(\lambda; 2) \) \((C_0(\lambda_1; \lambda_2))\) for some \( 0 < \lambda < 2; \ h(z) \) be defined as in (1). For \( z \in \Omega \), we define:

\[
T_n(z) := \frac{p_n(z)}{\Phi^{n+1}(z)},
\]

Then

\[
T_n'(z) = \frac{p_n'(z)}{\Phi^{n+1}(z)} + P_n(z) \left( \frac{1}{\Phi^{n+1}(z)} \right)', \quad z \in \Omega.
\]

For any \( R > 1 \) and \( R_1 := 1 + \frac{R-1}{2} \), Cauchy integral representation for the region \( \Omega_{R_1} \) gives

\[
T_n'(z) = -\frac{1}{2\pi i} \int_{L_{R_1}} \frac{\Phi^{n+1}(\zeta) d\zeta}{(\zeta - z)^2}, \quad z \in \Omega_{R_1},
\]

and

\[
\left( \frac{1}{\Phi^{n+1}(z)} \right)' = -\frac{1}{2\pi i} \int_{L_{R_1}} \frac{\Phi^{n+1}(\zeta) d\zeta}{(\zeta - z)^2}, \quad z \in \Omega_{R_1}.
\]

Then, from (15), we get

\[
P_n'(z) = \Phi^{n+1}(z) \left[ T_n'(z) - P_n(z) \left( \frac{1}{\Phi^{n+1}(z)} \right)' \right]
\]

\[
= \Phi^{n+1}(z) \left[ -\frac{1}{2\pi i} \int_{L_{R_1}} \frac{p_n(\zeta)}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^2} + \frac{p_n(z)}{2\pi i} \int_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^2} \right], \quad z \in \Omega_{R_1}.
\]

Therefore,

\[
|P_n'(z)| \leq \frac{|\Phi^{n+1}(z)|}{2\pi} \left| \int_{L_{R_1}} \left| \frac{p_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta-z|^2} \right| + |P_n(z)| \left| \int_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)} \frac{|d\zeta|}{|\zeta-z|^2} \right|.
\]

Since \( |\Phi(\zeta)| > 1, \) for \( \zeta \in L_{R_1}, \) then, we have:

\[
|P_n'(z)| \leq \frac{|\Phi(\zeta)|^{n+1}}{2\pi} \left| \int_{L_{R_1}} \left| P_n(\zeta) \right| \frac{|d\zeta|}{|\zeta-z|^2} \right| + |P_n(z)| \left| \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta-z|^2} \right|.
\]

(16)

Denote by

\[
A_n(z) := \int_{L_{R_1}} \left| P_n(\zeta) \right| \frac{|d\zeta|}{|\zeta-z|^2}; \quad B_n(z) := \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta-z|^2}.
\]

(17)
and will be estimate these integrals separately. To estimate $A_n(z)$, first of all replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrant by $\prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)|^{\frac{\gamma_j}{p}}|\Psi'(\tau)|^\frac{2}{p}$ and applying the Hölder inequality, we obtain:

$$A_n(z) = \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|}$$

$$= \sum_{i=1}^3 \int_{L_{R_1}} \frac{\prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))\Psi'(\tau)|^{\frac{2}{p}} |\Psi'(\tau)|^{\frac{2}{p}}}{\prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)| |\Psi'(\tau)|^{\frac{2}{p}}} \, d\tau$$

$$\leq \sum_{i=1}^3 \left( \int_{L_{R_1}} \prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |P_n(\Psi(\tau))|^{\frac{2}{p}} |\Psi'(\tau)|^{\frac{2}{p}} \, d\tau \right)^{\frac{1}{p}}$$

$$\times \left( \int_{L_{R_1}} \left( \frac{|\Psi'(\tau)|^{\frac{2}{p}}}{\prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)| |\Psi'(\tau)|^{\frac{2}{p}}} \right)^{\frac{1}{q}} \, d\tau \right)^{\frac{1}{q}}$$

$$=: \sum_{i=1}^3 A_n^i(z),$$

where $F_n^j = \Phi(L_{R_1}) = \Delta_j \cap \{ \tau: |\tau| = R_1 \}, j = 1, 2, 3$ and $F_n^3 = \Phi(L_{R_1}) \setminus (F_n^1 \cup F_n^2)$, and

$$A_n^i(z) = \left( \int_{F_n^i} |f_n(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \left( \int_{F_n^i} \prod_{j=1}^3 |\Psi(\tau) - \Psi(w_j)|^{\gamma_j} |\Psi'(\tau)|^{\frac{2}{p}} \, d\tau \right)^{\frac{1}{q}}$$

$$=: f_{n,1} \cdot f_{n,2}(z),$$

$$f_{n,1}(\tau) = h^\frac{1}{p}(\Psi(\tau))P_n(\Psi(\tau))(\Psi'(\tau))^\frac{2}{p}, \ |\tau| = R_1.$$

Applying to Lemmas 3, 4 and 5, we get:

$$J_{n,1}^i < n^\frac{1}{p} R_{n,1}, \ i = 1, 2, 3. \quad (19)$$

For the estimation of the integral $J_{n,2}(z)$, for $i = 1, 2, 3$, and $j = 1, 2$, we set:

$$E_{R_1}^{ij}(w_j) = \{ \tau: \tau \in F_n^j, \ |\tau - w_j| < c_j(R_1 - 1) \},$$

$$E_{R_1}^{ij}(w_j) = \{ \tau: \tau \in F_n^j, \ c_j(R_1 - 1) \leq |\tau - w_j| < \eta \},$$

$$E_{R_1}^{ij}(w_j) = \{ \tau: \tau \in \Phi(L_{R_1}), \ |\tau - w_j| \geq \eta \},$$

where $0 < c_j < \eta$ is chosen so that $\{ \tau: |\tau - w_j| < c_j(R_1 - 1) \} \cap \Delta \neq \emptyset$ and $\Phi(L_{R_1}) = \cup_{j=1}^3 E_{R_1}^{ij}(w_j)$. Taking into consideration these notations, (19) can be written as:

$$\sum_{i=1}^3 J_{n,2}^i(z) = J_2(z) = \sum_{i=1}^3 \sum_{j=1}^2 J_2(E_{R_1}^{ij}(w_j), z)$$

$$=: \sum_{i=1}^3 \sum_{j=1}^2 J_{2,1}^i(z)$$

$$A_n(z) < n^\frac{1}{p} R_n \cdot \sum_{i=1}^3 \sum_{j=1}^2 J_{2,1}^i(z) =: \sum_{i=1}^3 \sum_{j=1}^2 A_{n,1}^i(z), \quad (20)$$

and, consequently,

$$A_n(z) < n^\frac{1}{p} R_n \cdot \sum_{i=1}^3 J_{2,1}^i(z) =: \sum_{i=1}^3 A_{n,1}^i(z),$$

where

$$A_{n,1}^i(z) = n^\frac{1}{p} R_n \cdot J_{2,1}^i(z), \ i = 1, 2, 3; j = 1, 2. \quad (22)$$
(21)\[\int_{z_{i}}^{4} = \int_{z_{i}}^{4} |\psi'(-\gamma)|^{2} [dt_{i}]^{2} |\psi(\gamma)|^{q}[|\psi(\gamma)|^{q} - |\psi(\gamma)|^{q}]
\]

\[= \sum_{i=1}^{\gamma} \int_{z_{i}}^{4} |\psi'(-\gamma)|^{2} [dt_{i}]^{2} |\psi(\gamma)|^{q}[|\psi(\gamma)|^{q} - |\psi(\gamma)|^{q}], \quad i = 1,2,3,
\]
since the points \(w_{1}\) and \(w_{2}\) are isolated.

Therefore, we need to estimate the quantity (21). In case of \(j = 1\), for any \(p > 1, 0 < \lambda_{1} < 2, \gamma > -2\), and for all sufficiently small \(\varepsilon > 0\), in [12] is proved following estimate:

\[\Sigma_{i=1}^{\gamma} A_{n,i}^{1}(z) < ||P_{n}||_{p} \cdot \begin{cases} \frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & 0 < \lambda_{1} < 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} = \frac{1}{\lambda_{1}}, \lambda_{1} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
1, & \text{otherwise}
\end{cases},
\]

where \(\lambda_{1} = \max\{1; \lambda_{1}\} + \varepsilon\).

Similarly to the case \(j = 1\), for the case \(j = 2\), we obtain:

\[\Sigma_{i=1}^{\gamma} A_{n,i}^{2}(z) < ||P_{n}||_{p} \cdot \begin{cases} \frac{\gamma_{2}^{2} + \varepsilon}{n^{p}}, & \gamma_{2} \geq -\frac{3}{2}, \\
\frac{\gamma_{2}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} > -2, \\
1, & \text{otherwise}
\end{cases},
\]

Combining (23) and (24), for the region \(G \subseteq C_{\Omega}(\lambda_{1}, 2)\), any \(p > 1, \gamma_{1} > -2, 0 < \lambda_{2} < 2\), and for all sufficiently small \(\varepsilon > 0\), we obtain:

\[A_{n}(z) = \Sigma_{k=1}^{\gamma} A_{n,k}^{1} < ||P_{n}||_{p} \times \begin{cases} \frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{2}{\lambda_{1}}, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2,
\end{cases},
\]

If the angle at point \(z_{2}\) is equals \(\lambda_{2}\pi\) such that \(0 < \lambda_{1}, \lambda_{2} < 2\), then, analogously to (25), for the region \(G \subseteq C_{\Omega}(\lambda_{1}, \lambda_{2})\) all \(0 < \lambda_{1}, \lambda_{2} < 2\), we have:

\[A_{n}(z) = \Sigma_{k=1}^{\gamma} A_{n,k}^{1} < ||P_{n}||_{p} \times \begin{cases} \frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{2}{\lambda_{1}}(\gamma_{2} + 2), \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2, \\
\frac{\gamma_{1}^{2} + \varepsilon}{n^{p}}, & \gamma_{1} \geq \frac{1}{\lambda_{1}} - 2.
\end{cases}
\]
Now, let us estimate $B_n(z)$. Let $G \in C_0(\Lambda_1, 2)$. By replacing the variable $r = \Phi(\xi)$ and according to (10) and Lemma 2, we obtain:

$$B_n(z) = \int_{|r| = \tau} \left| \frac{d\xi}{r^2} \right| = \int_{|r| = \tau} \left| \frac{h''(\xi)}{\psi'(\xi) - \psi'(w)} \right| |\Psi'(r)| = \frac{d(\Psi(\xi), L)}{|r| - 1}.$$  \hfill (27)

Let us set:

$$F_1 := \{ |r| = \tau_1 \cap \Delta_1 : |r - w_1| \geq |r - w| \}, \quad F_2 := \{ |r| = \tau_1 \cap \Delta_2 : |r - w_2| \geq |r - w| \}.$$

Under this notations we have:

$$B_n^1(z) = \int_{F_1} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr + \int_{F_2} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr = \left\{ \begin{array}{ll} \frac{A_{n-1}^{1-\epsilon}}{n}, & \text{if } \lambda_1 \geq 1, \\ \int_{F_1} \left| \frac{dr}{r^2(\Lambda_1 + \epsilon)} \right| + \int_{F_2} \left| \frac{dr}{r^2(\Lambda_1 + \epsilon)} \right|, & \text{if } \lambda_1 < 1, \end{array} \right.$$

$$B_n^2(z) = \int_{F_1} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr + \int_{F_2} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr = \left\{ \begin{array}{ll} \frac{A_{n-1}^{1-\epsilon}}{n}, & \text{if } \lambda_1 \geq 1, \\ \int_{F_1} \left| \frac{dr}{r^2(\Lambda_1 + \epsilon)} \right| + \int_{F_2} \left| \frac{dr}{r^2(\Lambda_1 + \epsilon)} \right|, & \text{if } \lambda_1 < 1, \end{array} \right.$$

$$B_n^3(z) = \int_{F_1} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr + \int_{F_2} \left| \frac{A_{n-1}^{1-\epsilon}}{r^2(\Lambda_1 + \epsilon)} \right| dr < n^{1+\epsilon}, \forall \epsilon > 0.$$

So, from (27), we have:

$$B_n^3(z) < B_n^2(z) = \left\{ \begin{array}{ll} \frac{n^2}{n^{1+\epsilon}}, & \text{if } \lambda_1 \geq 1, \\ \frac{n^{1+\epsilon}}{n^{1+\epsilon}}, & \text{if } \lambda_1 < 1, \end{array} \right.$$
\[ B_n(z) < B_{n,2}(z) = \begin{cases} n^{\gamma}, & z \in \Omega(\delta), \forall \varepsilon > 0. \\ n^{1+\varepsilon}, & z \in \hat{\Omega}(\delta), \end{cases} \tag{29} \]

Now, combining (16), (17), (25), (28) and (29) for the region, any \( p > 1, \gamma_1 > -2, 0 < \lambda_1 < 2, \) and for all sufficiently small \( \varepsilon > 0, \) we obtain:

\[ |P_n'(z)| < |\Phi(z)|^{n+1} \left[ \frac{1}{d(z, L_{R_1})} \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} + \frac{|\Phi(z)||d\zeta|}{|\zeta - z|^2} \right] \]

\[ < |\Phi(z)|^{n+1} \|P_n\|_p \left[ \frac{1}{d(z, L_{R_1})} G_{n,1}(z) + |P_n(z)||B_{n,1}(z)| \right], \text{if } G \in C_\Theta(\lambda_1, 2), \]

\[ |P_n(z)| < |\Phi(z)|^{n+1} \|P_n\|_p \left[ \frac{1}{d(z, L_{R_1})} G_{n,2}(z) + |P_n(z)||B_{n,2}(z)| \right], \text{if } G \in C_\Theta(\lambda_1, \lambda_2), \]

Now, using estimates for \( |P_n(z)| \) ((25, Theorem 1 and Corollary 1)) for the cases \( G \in C_\Theta(\lambda_1, 2) \) and \( G \in C_\Theta(\lambda_1, \lambda_2), \) we get:

\[ |P_n'(z)| < \|P_n\|_p \left[ \frac{|\Phi(z)|^{n+1}}{d(z, L_{R_1})} G_{n,1}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d(z, L_{R_1})^{n+1}} E_{n,1} B_{n,1}(z) \right], \text{if } G \in C_\Theta(\lambda_1, 2), \]

and

\[ |P_n(z)| < \|P_n\|_p \left[ \frac{|\Phi(z)|^{n+1}}{d(z, L_{R_1})} G_{n,2}(z) + \frac{|\Phi(z)|^{2(n+1)}}{d(z, L_{R_1})^{n+1}} E_{n,2} B_{n,2}(z) \right], \text{if } G \in C_\Theta(\lambda_1, \lambda_2), \]

where

\[ E_{n,1} = \begin{cases} \frac{\gamma \lambda}{n^p}, & \text{if } \bar{\gamma} \cdot \hat{\lambda} \geq 1, \\ \frac{1}{n^p}, & \text{if } \bar{\gamma} \cdot \hat{\lambda} < 1, \end{cases} \]

\[ E_{n,2} = \begin{cases} \frac{\gamma \lambda}{n^p}, & \text{if } \bar{\gamma} \cdot \hat{\lambda} \geq 1, \\ \frac{1}{n^p}, & \text{if } \bar{\gamma} \cdot \hat{\lambda} < 1, \end{cases} \]

\[ \bar{\lambda}: = \begin{cases} \max \{1; \lambda\} + \varepsilon, & \text{if } 0 < \lambda < 2, \\ 2, & \text{if } \lambda = 2, \end{cases} \]

\[ \bar{\gamma}: = \begin{cases} \gamma_1, & \text{if } 0 < \lambda < 2, \\ \gamma_2, & \text{if } \lambda = 2, \end{cases} \]

\[ \bar{\gamma}_i: = \max \{0; \gamma_i\}, \quad i = 2, 3; \quad \bar{\lambda}_i: = \max \{1; \lambda_1, \lambda_2\} + \varepsilon. \]

Therefore, we complete the proof of Theorems 5 and 6.

References


