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A NEW APPROACH TO THE BI-UNIVALENT ANALYTIC FUNCTIONS RELATED WITH q-ANALOGUE OF NOOR INTEGRAL OPERATOR

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Abstract. Recently, q-analogue of Noor integral operator and other special operators became importance in the field of Geometric Function Theory. In this study, by connecting this operators and the principle of subordination we introduced an interesting class of bi-univalent functions and obtained coefficient estimates for this new class.

1. Introduction

Let $\mathcal{A}$ indicates the family of analytic functions having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\mathfrak{D} = \{z : |z| < 1, z \in \mathbb{C}\}$ and let $S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathfrak{D}\}$.

According the Koebe one-quarter theorem [6], the image of $\mathfrak{D}$ under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. That is, every such univalent function has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \ \ (z \in \mathfrak{D})$$

and

$$f\left(f^{-1}(w)\right) = w \ \ \left(|w| < r_0(f) \ , \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

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If f and f^{-1} are univalent, then we say that f is bi-univalent function in $\mathfrak{D}$. The class of bi-univalent functions defined in $\mathfrak{D}$ is symbolized by Σ.

One can see important examples in the class in [20]. Although the functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}.\right)$ are in Σ, well known Koebe function is not in Σ. For example, $z-\frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are in $\mathcal{S}$ but not in Σ [20].

Given f , $g \in \mathcal{A}, f$ is said to be subordinate to g, symbolized

$$f(z) \prec g(z), \tag{3}$$

such that there is an analytic function w defined on $\mathfrak{D}$ with

$$w(0)=0 \quad \text{and} \quad |w(z)|<1$$

fulfilling the following condition:

$$f(z)=g\left(w(z)\right).$$

The aforecited subclasses of Σ were constructed and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in several recent studies (see [7], [8], [10], [20] , [21], [22]). In very nearly, they have been followed by many works (see also [5], [9], [12], [14], [19]).

Now, we give some basic definitions.

Definition 1. *[13] For $q \in (0,1)$, the q-derivative of function $f \in \mathcal{A}$ is defined by*

$$\partial_q f(z)=\frac{f(qz)-f(z)}{(q-1)z}, z \neq 0 \tag{4}$$

and

$$\partial_q f(0)=f'(0).$$

Thus we have

$$\partial_q f(z)=1+\sum_{k=2}^{\infty}[k,q]\,a_k z^{k-1} \tag{5}$$

where $[k,q]$ is given by

$$[k,q]=\frac{1-q^k}{1-q}, \qquad [0,q]=0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]!=\begin{cases}\prod\limits_{n=1}^{k}[n,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{7}$$

Also, the $q-$generalized Pochhammer symbol for $\mathfrak{p}\geq 0$ is given by

$$[\mathfrak{p},q]_k=\begin{cases}\prod\limits_{n=1}^{k}[\mathfrak{p}+n-1,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{8}$$

As $q \to 1$, then we get $[k,q] \to k$. Thus, by choosing the function $g(z) = z^k$, while $q \to 1$, then we obtain

$$\partial_q g(z) = \partial_q z^k = [k,q]\, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

Recently, function $F^{-1}_{q,\mu+1}(z)$ is defined by Arif et al. [4] by

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z\partial_q f(z), \quad (\mu > -1) \tag{9}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k-1,q]!} z^k, \quad z \in \mathcal{D}. \tag{10}$$

To the series defined in (10) is convergent absolutely in $\mathfrak{D}$, by using the definition of q-derivative through convolution, let us explain the integral operator $\zeta_q^\mu : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^\mu f(z) = F^{-1}_{q,\mu+1}(z) * f(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{D}) \tag{11}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{12}$$

From (11), one can readily have the identity

$$[\mu+1,q]\zeta_q^\mu f(z) = [\mu,q]\zeta_q^{\mu+1} f(z) + q^\mu z \partial_q(\zeta_q^{\mu+1} f(z)). \tag{13}$$

We can state that

$$\zeta_q^0 f(z) = z\partial_q f(z), \ \zeta_q' f(z) = f(z) \tag{14}$$

also

$$\lim_{q \to 1^-} \zeta_q^\mu f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{15}$$

This means that, by taking $q \to 1^-$, the operator defined in equation (11) reduces to the famous Noor integral operator given in ([15]). Moreover, for more detailed knowledge on the coefficient estimates of analytic bi-univalent functions given by q-analogue of differential and integral operators, see the work of [1], [2], [3], [4], [16], [17].

In this study, utilizing by the aforementioned works we introduce a general new subclass $\Im_\Sigma^{\mu,q}(\xi,\tau,\theta;k)$ of the function class Σ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in our new class $\Im_\Sigma^{\mu,q}(\xi,\tau,\theta;k)$. Also through this paper, f, g are given by (1) and (2) and ζ_q^μ is $q-$analogue of Noor integral operator.

2. The class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$

Definition 2. *Let $k:\mathfrak{D}\to\mathbb{C}$ be a convex univalent function such that*

$$k(0)=1, k(\overline{z})=\overline{k(z)},\quad (z\in\mathfrak{D};\mathcal{R}(k(z))>0). \tag{16}$$

For $f\in\Sigma$, the function f is said to be in the class of $\in\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$ if the following conditions are satisfied:

$$\begin{aligned}
&e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\prec k(z)\cos\theta+i\sin\theta,(z\in\mathfrak{D}),\\
&e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\prec k(w)\cos\theta+i\sin\theta,(w\in\mathfrak{D})
\end{aligned} \tag{17}$$

where $\xi\geq 1$, $\tau\neq 0$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$.

Remark 3. *Choosing*

$$k(z)=\frac{1+\mathcal{A}z}{1+\mathcal{B}z},(-1\leq\mathcal{B}<\mathcal{A}\leq 1) \tag{18}$$

in the class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$, we have $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;\mathcal{A},\mathcal{B})$ and defined as

$$\begin{aligned}
&e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\prec \frac{1+\mathcal{A}z}{1+\mathcal{B}z}\cos\theta+i\sin\theta,(z\in\mathfrak{D}),\\
&e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\prec \frac{1+\mathcal{A}w}{1+\mathcal{B}w}\cos\theta+i\sin\theta,(w\in\mathfrak{D})
\end{aligned} \tag{19}$$

where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\xi\geq 1$.

Remark 4. *Choosing*

$$k(z)=\frac{1+(1-2\gamma)z}{1-z},(0\leq\gamma<1) \tag{20}$$

in the class$\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$, we have $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta,\gamma)$ and defined as

$$\begin{aligned}
&R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}f(z)}{z}+\xi\partial_q(\zeta_q^{\mu}f(z))-1\right]\right)\right\}>\gamma\cos\theta,(z\in\mathfrak{D}),\\
&R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu}g(w)}{w}+\xi\partial_q(\zeta_q^{\mu}g(w))-1\right]\right)\right\}>\gamma\cos\theta,(w\in\mathfrak{D})
\end{aligned} \tag{21}$$

where $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\xi\geq 1$.

In the case of $k(z)=\frac{1+(1-2\gamma)z}{1-z}$, $(0\leq\gamma<1)$, by choosing different values instead of parameters, we obtain different subclasses:

1. Upon setting $q \to 1^-$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(\xi,\tau,\theta,\gamma) = \Im_{\Sigma}^{\mu}(\xi,\tau,\theta,\gamma)$$
if the following inequalities hold:
$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta^{\mu}f(z)}{z}+\xi(\zeta^{\mu}f(z))'-1\right]\right)\right\} > \gamma\cos\theta, (z \in \mathfrak{D}),$$
$$R\left\{e^{i\theta}\left(1+\frac{1}{\tau}\left[(1-\xi)\frac{\zeta^{\mu}g(w)}{w}+\xi(\zeta^{\mu}g(w))'-1\right]\right)\right\} > \gamma\cos\theta, (w \in \mathfrak{D}). \tag{22}$$
2. Upon setting $q \to 1^-$and for $\tau = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(\xi,1,\theta,\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}\left[(1-\xi)\frac{\zeta^{\mu}f(z)}{z}+\xi(\zeta^{\mu}f(z))'\right]\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}\left[(1-\xi)\frac{\zeta^{\mu}g(w)}{w}+\xi(\zeta^{\mu}g(w))'\right]\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{23}$$
3. Upon setting $q \to 1^-$, for $\tau = 1$ and $\xi = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{\mu,1}(1,1,\theta,\gamma) = \Im_{\Sigma}^{\mu}(\theta,\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}(\zeta^{\mu}f(z))'\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}(\zeta^{\mu}g(w))'\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{24}$$
4. Upon setting $q \to 1^-$, for $\tau = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{1,1}(\xi,1,\theta,\gamma) = \Im_{\Sigma}(\xi,\theta,\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}\left[(1-\xi)\frac{f(z)}{z}+\xi(f(z))'\right]\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}\left[(1-\xi)\frac{g(w)}{w}+\xi(g(w))'\right]\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{25}$$
5. Upon setting $q \to 1^-$,for $\tau = 1, \xi = 1$ and $\mu = 0$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{0}(1,1;\gamma) = \Im_{\Sigma}(\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}(z\partial f(z))'\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}(w\partial g(w))'\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{26}$$

6. Upon setting $q \to 1^-$, for $\tau = 1, \xi = 1$ and $\mu = 1$, it is simply to see that $f \in \Sigma$ is in
$$\Im_{\Sigma}^{1,1}(1,1,\theta,\gamma) = \Im_{\Sigma}(\theta,\gamma)$$
if the following inequalities hold:
$$\begin{aligned} R\left\{e^{i\theta}(f(z))'\right\} &> \gamma\cos\theta, (z \in \mathfrak{D}), \\ R\left\{e^{i\theta}(g(w))'\right\} &> \gamma\cos\theta, (w \in \mathfrak{D}). \end{aligned} \tag{27}$$
We state that
$$\begin{aligned} \Im_{\Sigma}(\xi 0,\gamma) &= \mathcal{B}_{\Sigma}(\alpha,\lambda) \qquad \text{(see [10])} \\ \Im_{\Sigma}(0,\gamma) &= \mathcal{H}_{\Sigma}(\alpha,\lambda) \qquad \text{(see [20])}. \end{aligned}$$

We need the following lemma to derive our main result.

Lemma 5. *[18] Let the function $k(z)$ defined with*
$$k(z) = \sum_{n=1}^{\infty} \mathcal{B}_n z^n$$
be convex in $\mathfrak{D}$. Assume also that the function $\Psi(z)$ given by
$$\Psi(z) = \sum_{n=1}^{\infty} c_n z^n$$
is holomorphic in $\mathfrak{D}$. If $\Psi(z) \prec k(z), (z \in \mathfrak{D})$, then
$$|c_n| \leq |\mathcal{B}_1|, (n \in \mathbb{N}). \tag{28}$$
Now, we give our general results involving the new class $\Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k)$.

Theorem 6. *Let $f \in \Im_{\Sigma}^{\mu,q}(\xi,\tau,\theta;k), (\xi \geq 1, \theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and $\tau \neq 0$, with*
$$k(z) = 1 + B_1 z + B_2 z^2 + \cdots. \tag{29}$$
Then
$$|a_2| \leq \min\left\{\frac{|\tau B_1|\cos\theta}{(1+\xi q)\,\phi_1}, \sqrt[2]{\frac{|\tau B_1|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}}\right\} \tag{30}$$
and
$$|a_3| \leq \frac{|\tau B_1|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}. \tag{31}$$

Proof. According the inequality (17), we can write that
$$e^{i\theta}\left(1 + \frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu} f(z)}{z} + \xi\partial_q(\zeta_q^{\mu} f(z)) - 1\right]\right) = r(z)\cos\theta + i\sin\theta, (z \in \mathfrak{D})$$
$$e^{i\theta}\left(1 + \frac{1}{\tau}\left[(1-\xi)\frac{\zeta_q^{\mu} g(w)}{w} + \xi\partial_q(\zeta_q^{\mu} g(w)) - 1\right]\right) = s(w)\cos\theta + i\sin\theta\ , (w \in \mathfrak{D}) \tag{32}$$

where $r(z) \prec k(z)$ and $s(w) \prec k(w)$ have the following series expansions

$$r(z) = 1 + r_1 z + r_2 z^2 + ... \tag{33}$$

$$s(w) = 1 + s_1 w + s_2 w^2 + ... \tag{34}$$

respectively: By equating the coefficients of the two equations in (32), we have

$$e^{i\theta}\frac{1}{\tau}\left(1+\xi q\right)\phi_1 a_2 = r_1 \cos\theta \quad ... \tag{35}$$

$$e^{i\theta}\frac{1}{\tau}(1+\xi q+\xi q^2)\phi_2 a_3 = r_2 \cos\theta \ ... \tag{36}$$

and

$$-e^{i\theta}\frac{1}{\tau}\left(1+\xi q\right)\phi_1 a_2 = s_1 \cos\theta \ ... \tag{37}$$

$$e^{i\theta}\frac{1}{\tau}\left[(1+\xi q+\xi q^2)\phi_2\left(2a_2^2 - a_3\right)\right] = s_2 \cos\theta \ .... \tag{38}$$

From (35) and (37), we have

$$r_1 = -s_1 \quad ... \tag{39}$$

and

$$a_2^2 = \frac{(r_1^2 + s_1^2)e^{-2i\theta}\cos^2\theta}{2\left[\frac{1}{\tau}\left(1+\xi q\right)\phi_1\right]^2} \quad ... \tag{40}$$

Also from (36) and (38), we get

$$a_2^2 = \frac{(r_2 + s_2)e^{-i\theta}\cos\theta}{2(1+\xi q+\xi q^2)\phi_2}\ \tau \ .... \tag{41}$$

Due to the fact $r, s \in h(\mathfrak{D})$, by virtue of Lemma 5, we obtain

$$\begin{aligned} |r_k| &= \left|\frac{r^{(k)}(0)}{k!}\right| \le |B_1|, \\ |s_k| &= \left|\frac{s^{(k)}(0)}{k!}\right| \le |B_1|, (k \in \mathbb{N}). \end{aligned} \tag{42}$$

If we apply (42) and Lemma 5 for coefficients r_1, r_2, s_1 and s_2, from (40) and (41), we have

$$|a_2|^2 \le \frac{|\tau B_1|^2 \cos^2\theta}{|(1+\xi q)\,\phi_1|^2} \tag{43}$$

and

$$|a_2|^2 \le \frac{|\tau B_1| \cos\theta}{|(1+\xi q+\xi q^2)\phi_2|}. \tag{44}$$

Thus, we obtain desired result for $|a_2|$.

Next, in order to find the bound on the coefficient $|a_3|$, if we subtract (38), from (36), then we get

$$a_3 - a_2^2 = \frac{e^{-i\theta}(r_2 - s_2)\tau\cos\theta}{2(1+\xi q+\xi q^2)\phi_2}. \tag{45}$$

By substituting $\mathcal{A}_2^2$ from (41) into (45), it is obtained that

$$a_3 = \frac{e^{-2i\theta}(r_1^2 + s_1^2)\tau^2\cos^2\theta}{2(1+\xi q)^2\phi_1^2} + \frac{e^{-i\theta}(r_2 - s_2)\tau\cos\theta}{2(1+\xi q+\xi q^2)\phi_2}. \tag{46}$$

Taking absolute value of the equation (46), we get

$$|a_3| \le \frac{\cos^2\theta \ \left|\tau B_1\right|^2}{(1+\xi q)^2\phi_1^2} + \frac{\cos\theta\,|\tau B_1|}{(1+\xi q+\xi q^2)\phi_2} \tag{47}$$

Thus,

$$|a_3| \le \frac{\cos\theta\,|\tau B_1|}{(1+\xi q+\xi q^2)\phi_2}.$$

□

3. Corollaries and Consequences

According the Remark 1 and Remark 2, choosing

$$k(z) = \frac{1+\mathcal{A}z}{1+\mathcal{B}z}, (-1 \le \mathcal{B} < \mathcal{A} \le 1)$$

and

$$k(z) = \frac{1+(1-2\gamma)z}{1-z} \ (0 \le \gamma < 1)$$

in Theorem 6, Corollaries 7, 8, and 9 can be readily deduced, respectively.

Corollary 7. *If* $f \in \Im_\Sigma^{\mu,q}(\xi,\theta,\tau;\mathcal{A},\mathcal{B})$, $(\theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \xi \ge 1, \tau \ne 0, -1 \le \mathcal{B} < \mathcal{A} \le 1)$, *then we have*

$$|a_2| \le \min\left\{\frac{|\tau|\,(\mathcal{A}-\mathcal{B})\cos\theta}{(1+\xi q)\,\phi_1}, \sqrt{\frac{|\tau\mathcal{B}|\,(\mathcal{A}-\mathcal{B})\cos\theta}{(1+\xi q+\xi q^2)\phi_2}}\right\}$$

and

$$|a_3| \le \frac{(\mathcal{A}-\mathcal{B})\,|\tau|\cos\theta}{(1+\xi q+\xi q^2)\phi_2}.$$

Corollary 8. *If* $f \in \Im_\Sigma^{\mu}(\xi,\theta,\gamma)$, $(\theta \in (-\frac{\pi}{2},\frac{\pi}{2}), \xi \ge 1, 0 \le \gamma < 1)$, *then we have*

$$|a_2| \le \min\left\{\frac{2\tau(1-\gamma)\cos\theta}{(1+\xi)\,\phi_1}, \sqrt{\frac{2\,|\tau|\,(1-\gamma)\cos\theta}{(1+2\xi)\phi_2}}\right\}$$

and

$$|a_3| \le \frac{2\tau(1-\gamma)\cos\theta}{((1+2\xi)\phi_2}.$$

When $\theta = 0$ in Corollary 8, we obtain a new result:

Corollary 9. *If* $f \in \Im_{\Sigma}^{\mu}(\xi, \gamma), \quad (\xi \geq 1, 0 \leq \gamma < 1)$*, then we have*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{(1+2\xi)\phi_2}}$$

and

$$|a_3| \leq \frac{2\tau(1-\gamma)}{((1+2\xi)\phi_2}.$$

Corollary 10. *If* $f \in \Im_{\Sigma}(\gamma)$*, then we have*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3\phi_2}}$$

and

$$|a_3| \leq \frac{2(1-\gamma)}{3\phi_2}.$$

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