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A Note on Urysohn's Lemma under $mI\alpha g$-Normal Spaces and $mI\alpha g$-Regular Spaces in Ideal Minimal Spaces

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Abstract — This research article is concerned with the introduction of a new notion of normal spaces and regular spaces, namely $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces. We established their significant properties in ideal minimal spaces. Some equivalent conditions on $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces are proved. Urysohn's Lemma on $mI\alpha g$-normal spaces is also established.

Keywords — *$mI\alpha g$-closed sets, $mI\alpha g$-normal spaces, $mI\alpha g$-regular spaces, Urysohn's Lemma*

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1. Introduction

The perception of ideals was initiated by Kuratowski [1] and Vaidyanathaswamy [2]. A subset I of the universal set X is said to be an ideal, if there exist two subsets A and B of X satisfying i) $A \in I$ and $B \subset A$ then $B \in I$ ii)$A, B \in I$ implies $A \cup B \in I$. The notion of minimal structure and minimal spaces were established by Maki et al. [3]. They have explained the minimal spaces as the generalisation of classical topological spaces. M is referred as the minimal structure of the space X, if $\phi, X \in M$. The spaces (X, M_X) is called as the minimal structure space.The introduction of m-open sets in minimal structures was initialised by Maki et al. [3]. The members of minimal structure are called m-open sets. Generalised closed sets (briefly g-closed sets)were introduced by Levine [4]. The notion of αm-open sets was introduced by Min [5]. The idea of m-normal spaces and mg-normal spaces were established by Noiri et al. [6]. m-regular spaces was elaborately studied by Popa et al. [7]. m-continuous functions and its salient features in minimal structures were instigated by Popa et al. [8]. Singha et al. [9] proved Urysohn's lemma in minimal structures. An innovative approach on ideals in minimal spaces was given by Özbakır et al. [10]. They have defined a new type of local function A^*_m named as minimal local function in ideal minimal spaces. The conception of mIg-normal spaces and its characterisations were well established by Haining et al. [11]. Also they have proved Urysohn's lemma under mIg-normal spaces ideal minimal spaces. A new notion of generalised closed sets, called $mI\alpha g$-closed sets and its features in ideal minimal spaces were studied by Parimala et al. [12]. Local closedness under $mI\alpha g$-closed sets and few separation axioms under $mI\alpha g$-closed sets are intended by Parimala et al. in [13,14]. In this article, few properties of separating sets are studied in ideal minimal spaces. Two new separations namely $mI\alpha g$-normal spaces and $mI\alpha g$-regular spaces are initiated and their significant properties are established. As an application of $mI\alpha g$-normal spaces, we have proved Urysohn's lemma on $mI\alpha g$-normal spaces.

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In the present study, Section 2 provides preliminary definitions in minimal structure spaces and in ideal minimal spaces, Section 3 $mI\alpha g$-normal spaces and its characterisations in ideal minimal spaces, Section 4 Urysohn's lemma under $mI\alpha g$-normal spaces, Section 5 $mI\alpha g$-regular spaces in ideal minimal spaces, and Section 6 conclusion and future work.

2. Preliminary

In the following sequel, the following notations are used.

(i) MSS- minimal structure spaces

(ii) IMS - ideal minimal spaces

Definition 2.1. [8] The interior and closure in an MSS are defined as follows. Let (X, M) be a MSS and $A \subset X$, then

(a) $m\text{-}int(A) = \cup\{K : K \subseteq A, K \in M\}$

(b) $m\text{-}cl(A) = \cap\{N : A \subseteq N, X - N \in M\}$

Proposition 2.2. [8] Properties of *m-cl* and *m-int* are listed below.

(a) $m\text{-}int(X) = X$ and $m\text{-}cl(\phi) = \phi$

(b) $m\text{-}int(A) \subseteq A$ and $A \subseteq m\text{-}cl(A)$

(c) If $A \in M$, then $m\text{-}int(A) = A$ and if $X - F \in M$, then $m\text{-}cl(F) = F$.

(d) If $A \subseteq B$, then $m\text{-}int(A) \subseteq m\text{-}int(B)$ and $m\text{-}cl(A) \subseteq m\text{-}cl(B)$.

Definition 2.3. [10] Let (X, M, I) be an IMS with an ideal I. The power set of X is denoted by $P(X)$. A mapping $(.)^*_m$ is defined from $P(X)$ to itself. For a subset $A \subset X$, the minimum local function is $A^*_m(I, M) = \{x \in X : U_m \cap A \notin I;$ for every $U_m \in U_{m(x)}\}$. The minimal $*$-closure operator $m\text{-}cl^*$ is defined as $m\text{-}cl^*(A) = A \cup A^*_m$. A minimal structure via $m\text{-}cl^*$ is termed as $M^*(I, M)$ (briefly M^*) and is described as $M^* = \{U \subset X : m\text{-}cl(X - U) = X - U\}$. The members of $M^*(I, M)$ are termed as m^*-open sets. The interior of m^*-open sets is denoted by $m\text{-}int^*$.

Theorem 2.4. [10] In an MSS (X, M), let I, J be two ideals on X. $P, Q \subset X$. Then,

(a) $P \subset Q \Rightarrow P^*_m \subset Q^*_m$

(b) $P^*_m \cup Q^*_m \subset (P \cup Q)^*_m$

(c) $(P^*_m)^*_m \subset P^*_m$

(d) $P^*_m = m\text{-}cl(P^*_m) \subset m\text{-}cl(P)$

(e) $I \subset J \Rightarrow P^*_m(J) \subset P^*_m(I)$

Remark 2.5. [10] The MSS (X, M) is said to exhibit the property $[U]$ if the union of any number of m-open sets is an m-open set and the property $[I]$ if the intersection of finite number of m-open sets is an m-open set.

Remark 2.6. [10] If (X, M) has the property $[U]$, then (b) of Theorem 2.4. can be stated as $P^*_m \cup Q^*_m = (P \cup Q)^*_m$.

Proposition 2.7. [10] Significant features of $m\text{-}cl^*$ are as follows. Let $P_1, P_2 \subseteq X$. Then,

(a) $m\text{-}cl^*(P_1) \cup m\text{-}cl^*(P_2) \subseteq m\text{-}cl^*(P_1 \cup P_2)$

(b) If $P_1 \subseteq P_2$, then $m\text{-}cl^*(P_1) \subseteq m\text{-}cl^*(P_2)$.

(c) When $A \subseteq X$, then $A \subseteq m\text{-}cl^*(A)$.

(d) $m\text{-}cl^*(\phi) = \phi$ and $m\text{-}cl^*(X) = X$

Definition 2.8. In an MSS (X, M, I), let A be a non empty subset of X. A is defined to be an

(a) m^*-closed set [10] if A^*_m is a subset of A. $(A^*_m \subset A)$.

(b) minimal generalised closed set (mg-closed) [3] if $m\text{-}cl(A) \subseteq U$, $A \subseteq U$ and U is an m-open set.

(c) minimal α-open set (αm-open set) [5] if $A \subseteq m\text{-}int(m\text{-}cl(m\text{-}int(A)))$. The complement of αm-open set is called an αm-closed set.

(d) minimal ideal α generalised closed set ($mI\alpha g$-closed set) [12] if $A^*_m \subseteq U$ whenever $A \subseteq U$ and U is an αm-open set.

Proposition 2.9. [5] In an MSS (X, M_X) if a subset A is an m-open set, then it is an αm-open set.

Definition 2.10. [5] α-closure and α-interior of a set A are defined as follows:

(a) $\alpha m\text{-}cl(A) = \cap\{F : A \subseteq F, \text{F is } \alpha m\text{-closed in } X\}$

(b) $\alpha m\text{-}int(A) = \cup\{U : U \subseteq A, U \text{ is } \alpha m\text{-open in } X\}$

Let (X, M, I) be an IMS, then we have the following theorems.

Proposition 2.11. [12] When $I = \{\phi\}$, then an $mI\alpha g$-closed set ($mI\alpha g$-open set) is an mg-closed set (mg-open set).

Proposition 2.12. [12] An m^*-closed set in an IMS is an $mI\alpha g$-closed set.

Theorem 2.13. [12] The necessary and sufficient condition for a subset to be an $mI\alpha g$-closed set in (X, M, I) is that every αm open set in X is an m^*-closed set.

Theorem 2.14. [12] Theorem 3.4(d), in an IMS X a subset A is an $mI\alpha g$-closed set if and only if every m open set is an m^*-closed set.

Proof. Obvious, since every m open set is an αm open set. □

Theorem 2.15. [12] Consider $A \subseteq X$, then A is $mI\alpha g$-open if $S \subseteq m\text{-}int^*(A)$, S is αm-closed and $S \subseteq A$. Sufficiency is also true.

Definition 2.16. [6] An MSS (X, M) is called m-normal (resp.mg-normal) if for every pair of m-closed subsets (resp. mg-closed subsets) A and B such that $A \cap B = \phi$, there exists m-open sets U and V such that $U \cap V = \phi$ and $A \subset U$, $B \subset V$.

Definition 2.17. [7] An MSS (X, M) is termed to be a m-regular space if for every m-closed set F and an element $x \notin F$, there are m-open sets U and V such that $x \in U$, $F \subseteq V$ and also $U \cap V = \phi$.

Definition 2.18. [15] A m-T_1 space we mean, for all distinct points $x_1, x_2 \in X$ there exists an m-open set X such that $x_1 \in X$, but $x_2 \notin X$ and an m-open set Y such that $x_1 \notin Y$, $x_2 \in Y$.

Theorem 2.19. [7] Consider an m-T_1 space (X, M, I) with $I = \{\phi\}$ then the following statements below given are equivalent.

(a) (X, M, I) is an m-regular space.

(b) For every m-open set V such that $x \in X$, there exists an m-open set U of X satisfying $x \in U \subseteq m\text{-}cl(U) \subseteq V$.

Proposition 2.20. [16] Every m-closed set is an $mI\alpha g$-closed set. (Every m-open set is an $mI\alpha g$-open set.)

Proposition 2.21. [16] Every mg-closed set is an $mI\alpha g$-closed set.

Definition 2.22. [8] Let (X, M_X) and (Y, M_Y) be two MSS. The function $f : (X, M_X) \to (Y, M_Y)$ is defined to be an m-continuous function, if for $x \in X$ and $V \in M(f(x))$, there exist $U \in M(x)$ satisfying $f(U) \subseteq V$.

3. $mI\alpha g$-Normal Spaces

Definition 3.1. $mI\alpha g$-normal space we mean, if for every pair of $mI\alpha g$ closed sets K_1, K_2 such that $K_1 \cap K_2 = \phi$, there exists at least a pair of m-open sets U and V of X such that $U \cap V = \phi$ satisfying $A \subseteq U$ and $B \subseteq V$.

Theorem 3.2. An $mI\alpha g$-normal space is an m-normal space (mg-normal space).

Proof. Obvious, since every m-closed set(mg-closed set) is an $mI\alpha g$ set with references to Proposition 2.20., and Proposition 2.21. The example given below shows that the converse of the above theorem is not true. □

Example 3.3. (X, M, I) be an IMS with $X = \{a, b, c, d\}$, $M = \{\phi, X, \{b\}, \{a, b\}, \{a, c, d\}\}$ and $I = \{\phi, \{a\}, \{c\}, \{a, c\}\}$ and $M^c = \{X, \phi, \{a, c, d\}, \{c, d\}, \{b\}\}$. Here X is an m-normal space. Since for the disjoint $mI\alpha g$-closed sets $\{a\}$ and $\{c\}$ there does not exists disjoint m open sets containing them, X is not an $mI\alpha g$-normal space.

Theorem 3.4. In an IMS (X, M, I) the equivalent statements on $mI\alpha g$-normal-spaces are given below.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For each $mI\alpha g$-closed set K and an $mI\alpha g$-open set F such that $K \subseteq F$, there exists an m-open set $U \subseteq X$ such that $K \subseteq U \subseteq m\text{-}cl(U) \subseteq F$.

Proof.
(a) $\Rightarrow$ (b): Assume K be an $mI\alpha g$-closed set and F be an $mI\alpha g$-open set such that $K \subset F$. Then $X - F$ is an $mI\alpha g$-closed set. Therefore $K \cap (X - F) = \phi$. Referring the hypothesis (a), for a pair of disjoint m-open sets U and V such that $K \subseteq U$ and $X - F \subseteq V$ and $U \cap V = \phi$. But $U \subseteq (X - V)$ implies $m\text{-}cl(U) \subseteq (X - V)$. Hence $K \subseteq U \subseteq m\text{-}cl(U) \subseteq (X - V) \subseteq F$ which proves (b).

(b) $\Rightarrow$ (a): Let K and F be two disjoint $mI\alpha g$-closed sets such that $K \subseteq (X - F)$. Hypothesis (b) of this theorem infers the existence of the m-open subset U of X such that $K \subseteq U \subseteq m\text{-}cl(U) \subseteq (X - F)$. Let $V = X - m\text{-}cl(U)$, since $m\text{-}cl(U)$ is a m-closed set V is m-open. These U and V are the m-open sets which contains K and F which proves (a). □

Corollary 3.5. In an IMS (X, M, I) the statements below given are equivalent.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For every $mI\alpha g$-closed set A and $mI\alpha g$-open set B such that $A \subseteq B$, there exists an αm-open set $U \subseteq X$ satisfies $A \subseteq U \subseteq \alpha m\text{-}cl(U) \subseteq B$).

Proof. By referring Proposition 2.9., every m-open set is an αm-open set. Apply this result in Theorem 3.4., the proof follows. □

Corollary 3.6. In an IMS (X, M, I) the following statements are equivalent on mg-normal spaces when $I = \{\phi\}$.

(a) Consider (X, M, I) be an mg-normal space

(b) For a pair of mg-closed set A and an mg-open set B such that $A \subseteq B$, there exists an m-open set $U \subseteq X$ satisfies $A \subseteq U \subseteq m\text{-}cl(U) \subseteq B$.

Proof. When $I = \{\phi\}$, Proposition 2.11., infers that every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.4., the proof follows. □

Theorem 3.7. In an IMS (X, M, I) the following statements are equivalent.

(a) (X, M, I) is an $mI\alpha g$-normal space.

(b) For every pair of $mI\alpha g$ closed subsets A and B of X, there corresponds an m-open set U of X satisfies $A \subseteq U$, then $m\text{-}cl(U) \cap B = \phi$.

(c) For every pair of $mI\alpha g$-closed subsets A and B such that $A \cap B = \phi$, there corresponds an m-open set U satisfying $A \subseteq U$ and an m-open set V satisfying $B \subseteq V$ then $m\text{-}cl(U) \cap m\text{-}cl(V)$ is an empty set.

Proof.
(a) $\Rightarrow$ (b): Consider a pair of $mI\alpha g$-closed subsets A, B such that $A \cap B = \phi$ then $A \subseteq (X - B)$ where $X - B$ is an $mI\alpha g$-open set. Referring Theorem 3.4., there corresponds an m-open set U such that $A \subseteq U \subseteq m\text{-}cl(U) \subseteq X - B$. Therefore, $m\text{-}cl(U)$ and B are disjoint sets. Hence, U is the required m-open set that satisfies (b).

(b) $\Rightarrow$ (c): Hypothesis (b) of this theorem implies that $m\text{-}cl(U)$ and B are disjoint $mI\alpha g$-closed subsets of X. Therefore, there exists an m-open set V containing B such that $m\text{-}cl(U) \cap m\text{-}cl(V) = \phi$ which proves (c).

(c) $\Rightarrow$ (a): Hypothesis (c) proves (a). □

Corollary 3.8. In an IMS (X, M, I) the statements given below are equivalent when $I = \{\phi\}$.

(a) The IMS X is an mg-normal space.

(b) All pairs of subsets of X consisting mg closed sets A, B there corresponds an m-open set U of X such that $A \subseteq U$, then $m\text{-}cl(U)$ and B are disjoint sets.

(c) Every pair of mg-closed sets A, B of X such that $A \cap B = \phi$ there corresponds an m-open set U such that $A \subseteq U$ and an m-open set V such that $B \subseteq V$ then $m\text{-}cl(U)$ and $m\text{-}cl(V)$ are disjoint sets.

Proof. When $I = \{\phi\}$, every $mI\alpha g$-open set is an mg-open set by Proposition 2.11. Apply this result in Theorem 3.7., we get the proof. □

Theorem 3.9. Let (X, M, I) be an $mI\alpha g$-normal space. If A and B are $mI\alpha g$-closed sets that containing no common elements, then there exists a pair of m-open sets U and V such that $U \cap V = \phi$ and satisfies $m\text{-}cl^*(A) \subseteq U$ and $m\text{-}cl^*(B) \subseteq V$.

Proof. Consider a pair of $mI\alpha g$-closed sets A and B. Referring Theorem 3.7 (3)., there exist m-open sets U and V such that $A \subset U$ and $B \subset V$ satisfying $m\text{-}cl(U) \cap m\text{-}cl(V) = \phi$. As A is an $mI\alpha g$-closed set, we have $A^*_m \subseteq U$ and also $A \subseteq U$. Therefore, $A \cup A^*_m = m\text{-}cl^*(A) \subseteq U$. Similarly, $m\text{-}cl^*(B) \subseteq V$. □

Corollary 3.10. Let (X, M, I) be an $mI\alpha g$-normal space with $I = \{\phi\}$ and A and B are mg-closed sets of X and $A \cap B = \phi$, then there are disjoint m-open sets U and V such that $m\text{-}cl^*(A)$ is contained in U and $m\text{-}cl^*(B)$is contained in V.

Proof. When $I = \{\phi\}$, referring Proposition 2.11., we know that every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.9., we get the proof. □

Theorem 3.11. Let (X, M, I) be an $mI\alpha g$-normal space. If A and B are $mI\alpha g$-closed and $mI\alpha g$-open sets respectively and also $A \subset B$, then there corresponds an m-open set U such that $A \subseteq m\text{-}cl^*(A) \subseteq U \subseteq m\text{-}int^*(B) \subseteq B$.

Proof. Suppose that A is an $mI\alpha g$-closed set and B is an $mI\alpha g$-open set such that $A \subseteq B$. Then, $A \cap (X - B) = \phi$. That is, A and $X - B$ are disjoint $mI\alpha g$-closed sets. Referring Theorem 3.9., there exist disjoint m-open sets K_1 and K_2 such that $m\text{-}cl^*(A) \subseteq K_1$ and $m\text{-}cl^*(X - B) \subseteq K_2$. Also, $X - (m\text{-}int^*(B)) = m\text{-}cl^*(X - B) \subseteq K_2$. So $m\text{-}cl^*(X - B) \subseteq K_2$ implies that $X - K_2 \subseteq m\text{-}int^*(B)$. Also, since K_1 and K_2 are disjoint m-open sets, we get $A \subseteq m\text{-}cl^*(A) \subseteq K_1 \subseteq (X - K_2) \subseteq m\text{-}int^*(B) \subseteq B$. □

Corollary 3.12. Let (X, M, I) be an mg-normal space and $I = \{\phi\}$. For each mg-closed subset A and an mg-open subset U containing A there exists an m-open subset V such that $A \subseteq m\text{-}cl^*(A) \subseteq V \subseteq m\text{-}int^*(U) \subseteq U$.

Proof. Let $I = \{\phi\}$. With reference to Proposition 2.11., every $mI\alpha g$-open set is an mg-open set. Apply this result in Theorem 3.11., we get the proof. □

4. Urysohn's Lemma on $mI\alpha g$-Normal Spaces

Theorem 4.1. The necessary and sufficient condition for an IMS (X, M, I) to be an $mI\alpha g$-normal space is that, for every pair of $mI\alpha g$-closed sets A and B and $A \cap B = \phi$, it is possible to define a m-continuous mapping $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof.

Necessary Part: Consider a $mI\alpha g$-normal space (X, M, I) and let $A, B \subset X$ be a pair of $mI\alpha g$-closed sets such that $A \cap B = \phi$. As B is an $mI\alpha g$-closed set, $X - B$ is $mI\alpha g$-closed and also $A \subset (X - B)$. Here, A is an $mI\alpha g$-closed set and $X - B$ is an $mI\alpha g$-open set in X. Referring Theorem 3.4., there exists an m-open set namely $U_{1/2}$ satisfies $A \subseteq U_{1/2} \subseteq m\text{-}cl(U_{1/2}) \subseteq (X - B)$. With reference to Theorem 2.19., $U_{1/2}$ is a $mI\alpha g$-open set. That is, $U_{1/2}$ and $X - B$ are the $mI\alpha g$-open sets such that $A \subseteq U_{1/2}$ and $m\text{-}cl(U_{1/2}) \subseteq (X - B)$, where A and $m\text{-}cl(U_{1/2})$ are $mI\alpha g$-closed sets. Therefore, with reference to Theorem 3.4., there exist m-open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subseteq U_{1/4} \subseteq m\text{-}cl(U_{1/4}) \subseteq U_{1/2} \quad \text{and} \quad m\text{-}cl(U_{1/2}) \subseteq U_{3/4} \subseteq m\text{-}cl(U_{3/4}) \subseteq (X - B)$$

Continuing in this way, for every rational number in the open interval $(0, 1)$ of the form $t = \frac{m}{2^n}, n = 1, 2, 3, ...$ and $m = 1, 2, 3, ...2^{n-1}$, we obtain m-open sets of the form U_t such that for each $s < t$,

$$A \subseteq U_s \subseteq m\text{-}cl(U_s) \subseteq U_t \subseteq m\text{-}cl(U_t) \subseteq (X - B)$$

Let us denote the set of all rational number by $\mathcal{Q}$. Also, $Q(x) = \{t : t \in \mathcal{Q}$ and $x \in U_t\}$, this set contains no number less than 0, since no x is in U_t for $t < 0$ and it contains every number greater than 1. Let us define $f : X \to [0, 1]$ as $f(x) = 1$, if $x \notin U_t$ and $f(x) = inf\{t : t \in \mathcal{Q}$ and $x \in U_t\}$. For each $x \in B$, $x \notin X - B$ implies $x \notin U_t$. Therefore, $f(B) = \{1\}$. For each $x \in A$, $x \in U_t$ and $t \in \mathcal{Q}$. By definition $f(x) = inf\{t : t \in \mathcal{Q}$ and $x \in U_t\} = inf\mathcal{Q} = 0$. Hence, $f(A) = 0$. To prove f is an m-continuous mapping, let the intervals of the form $[0, a)$ and $(b, 1]$ where $a, b \in (0, 1)$ forms an open subbase in the space $[0, 1]$. Therefore our aim is to prove that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are m-open sets in X. To prove $f^{-1}([0, a))$ is an m-open set in X. Let $x \in U_t$ for some $t < a$, then by definition $f(x) = inf\{s : s \in \mathcal{Q}$ and $x \in U_s\} = r \leq t < a$. That is, $f(x) < a$. Thus $0 \leq f(x) < a$. Conversely, if $f(x) = 0$, then $x \in U_t$ for all $t \in \mathcal{Q}$, hence $x \in U_t$ for some $t < a$. If $0 < f(x) < a$, by definition of we have $f(x) = \{s : s \in \mathcal{Q}$ and $x \in U_t\} < a$. Since $a < 1$ we get $f(x) = t$ for some $t < a$ and hence $x \in U_t$ for some $t < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_t$ for some $t < a$. Hence, $f^{-1}([0, a)) = \cup\{U_t; t \in \mathcal{Q}$ and $x \in U_t\}$ which is an m-open set of X. To prove $f^{-1}((b, 1])$ is an m-open set in X. We need to prove $X - f^{-1}([0, b])$ is an m-open subset of X. For that we have to prove $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$ to get union of m-open subsets U_t. Let $x \in X$ such that $0 \leq f(x) \leq b$ when $t > b$, It is evident that $f(x) < t$ implies $x \in U_t$ for $t > b$. Conversely, let $x \in U_t$ for all $t > b$, then by definition $f(x) = inf\{t : t \in \psi$ and $x \in U_t\} \leq t$. Since $t > b$, $f(x) \leq b$ for all $t > b$. From the definition of f, it is clear that $f(x) \geq 0$. Therefore, we get $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$. Also, $t > b$ implies that there is $r \in \mathcal{Q}$ such that $t > r > b$. then $m - cl(U_t) \subseteq U_t$. Consequently, we have $\cap\{U_t; t \in \mathcal{Q}$ and $t > b\}$=$\cap\{m\text{-}cl(U_t); r \in \mathcal{Q}$ and $r > b\}$. Therefore, $f^{-1}([0, b]) = \{x : 0 \leq f(x) \leq b\} = \cap\{U_t; t \in \mathcal{Q}$ and $t > b\}$=$\cap\{m\text{-}cl(U_t; r \in \psi$ and $r > b)\}$. Since, $f^{-1}((0, 1]) = f^{-1}(X - ([0, b]))$=$X - f^{-1}([0, b]) = \cup\{X - m\text{-}cl(U_t); r \in \mathcal{Q}$ and $r > b\}$, which is m-open in X. Therefore, $f : X \to [0, 1]$ is m-continuous.

Sufficient Part: Consider a pair of $mI\alpha g$-closed sets A and B such that $A \cap B = \phi$. Referring the sufficient condition, there exists an m-continuous mapping $f : X \to [0, 1]$ satisfying $f(A) = \{0\}$ and $f(B) = \{1\}$. Moreover, $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are disjoint m-open subsets of X. Clearly $A \subset U$ and $B \subset V$. Hence, X is an $mI\alpha g$-normal space. □

5. $mI\alpha g$-Regular Spaces

Definition 5.1. An IMS (X, M, I) is referred to be an $mI\alpha g$-regular space, if for every pair consisting a point $x \in X$ and an m-closed set B such that $x \notin B$ there exists at least one pair of $mI\alpha g$-open sets U and V with $U \cap V = \phi$ satisfying $x \in U$ and $B \subset V$.

Example 5.2. Consider an IMS (X, M, I) with $X = \{a, b, c\}$, $M = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ $M^c = \{X, \phi, \{a, c\}, \{c\}, \{a\}\}$ and the ideal $I = \{\phi, \{b\}\}$. Here $mI\alpha g$ closed sets are the elements of the power set $P(X)$ and X is $mI\alpha g$-regular.

Theorem 5.3. If (X, M, I) is an $mI\alpha g$-regular space, then it is an m-regular space, but the converse of this theorem may not be true.

Proof. Obvious, since every m-closed set is an $mI\alpha g$-closed set. □

Example 5.4. In Example 4.2., X is an $mI\alpha g$-regular space, but not a m-regular space, since for the point $x = a \in X$ and an m-closed set $B = \{c\}$, there does not exist m-open sets containing x and B.

Theorem 5.5. In an IMS (X, M, I), if every m-open set is m^*-closed, then the minimal space is an $mI\alpha g$-regular space.

Proof. Suppose every m-open subset of X is m^*-closed, then by Theorem 2.11., every subset of X is an $mI\alpha g$-open set. If B is an m-closed set such that $x \notin B$, then $\{x\}$ and B are the two $mI\alpha g$-open sets such that $\{x\} \cap B = \phi$ and also $x \in \{x\}$ and $B \subseteq B$. Therefore, X is an $mI\alpha g$-regular space. □

Definition 5.6. A subset K of (X, M, I) is termed as an $mI\alpha g$-neighbourhood of $B \subseteq X$, if there exists an $mI\alpha g$-open set U such that $B \subseteq U \subseteq K$.

Definition 5.7. A subset K of (X, M, I) is termed to be an $mI\alpha g$-closed neighbourhood of $B \subseteq X$, if there exists an $mI\alpha g$-closed set U such that $B \subseteq U \subseteq K$.

Theorem 5.8. In an IMS (X, M, I) the following are statements equivalent.

(a) (X, M, I) is an $mI\alpha g$-regular space.

(b) For each m-open se)t U and let $x \in U$, there corresponds an $mI\alpha g$-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subset U$.

(c) For each m-closed set A, $\cap A_i = A$ where A_i are $mI\alpha g$-closed neighbourhoods of A.

(d) For any set A and an m-open set B such that $A \cap B$ contains at least one element, there exists an $mI\alpha g$-open set U such that $A \cap U \neq \phi$ and $m\text{-}cl^*(U) \subseteq B$.

(e) For any non empty set A and an m-closed set B such that A and B are disjoint, there exists atleast a pair of $mI\alpha g$-open sets U, V satisfies $A \cap U \neq \phi$ and $B \subseteq V$.

Proof.
(a) ⇒ (b): Consider an m-open set V and let $x \in V$. Hence $X - V$ is m-closed such that $x \notin (X - V)$. Since X is an $mI\alpha g$-regular space, there exists a pair of $mI\alpha g$-open sets U and W such that $U \cap W = \phi$ satisfying $x \in U$ and $X - V \subseteq W$. Observing Theorem 2.8., $X - V$ is αm closed. Theorem 2.14., infers that $X - V \subseteq m\text{-}int^*(W)$. Therefore, $X - (m\text{-}int^*(W)) \subseteq V$. Hence, $U \cap W = \phi$ implies $U \cap m\text{-}int^*(W) = \phi$ and so $m\text{-}cl^*(U) \subseteq X - (m\text{-}int^*(W))$. Which implies $x \in U \subseteq m\text{-}cl^*(U) \subseteq V$.

(b) ⇒ (c): Let A be an m-closed set and $x \notin A$ then $X - A$ is an m-open set containing x. By hypothesis (b), there exists an $mI\alpha g$-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subseteq (X - A)$. Thus, $A \subseteq X - (m\text{-}cl^*(V)) \subseteq (X - V)$. Since $X - (m\text{-}cl^*(V))$ is $mI\alpha g$-open, we get $X - V$ is $mI\alpha g$-closed neighbourhood of A and $x \notin (X - V)$. This shows that A is the intersection of all $mI\alpha g$ neighbourhood of A.

(c) $\Rightarrow$ (d): Assume a non empty set A and an m-closed set B such that $A \cap B \neq \phi$. Consider an element x of $A \cap B$ then, $X - B$ is m-closed and $x \notin (X - B)$. Observing the hypothesis (c), there exists an $mI\alpha g$-closed neighbourhood V of $X - B$ such that, $x \notin V$. Let $(X - V) \subseteq G \subseteq V$ and G be an $mI\alpha g$-open then, $U = (X - V)$ is an $mI\alpha g$-open set satisfying $x \in U$ and $A \cap U \neq \phi$. Further, $X - G$ is $mI\alpha g$-closed and $m\text{-}cl^*(U) = m\text{-}cl^*(X - V) \subseteq m\text{-}cl^*(X - G) \subseteq B$.

(d) $\Rightarrow$ (e): Consider a non empty set A and m-closed set B, $A \cap B$ contains no element. $X - B$ is an m-open set and so $A \cap (X - B) \neq \phi$. Observing hypothesis (d), there exists an $mI\alpha g$-open set U such that the sets A and U contains at least one common element. Also, $U \subseteq m\text{-}cl^*(U) \subseteq (X - B)$. Assume that $V = X - (m\text{-}cl^*(U))$. Then, U and V are $mI\alpha g$-open sets satisfying $B \subseteq X - (m\text{-}cl^*(U) = V \subseteq (X - U))$ which implies (e).

(e) $\Rightarrow$ (a): Let A be an m-closed set and $x \notin A$. Let the set $B = \{x\}$. Then there exist disjoint $mI\alpha g$-open sets U and V such that $\{x\} \cap U = B \cap U \neq \phi$ and $A \subseteq V$. Thus, $x \in U$. □

Definition 5.9. A subset K of (X, M, I) is referred as an mg-neighbourhood of set $B \subseteq X$, if there exists an mg-open set U such that $B \subseteq U \subseteq K$.

Definition 5.10. A subset K of (X, M, I) is referred as an mg-closed neighbourhood of set $B \subseteq X$, if there exists an mg-closed set U such that $B \subseteq U \subset K$.

Corollary 5.11. Let (X, M, I) be an IMS such that $I = \{\phi\}$. Then, the following statements on mg-regular spaces are equivalent.

(a) (X, M, I) is an mg-regular space.

(b) Let U be an m-open set containing x, then there exists an mg-open set V satisfying $x \in V \subseteq m\text{-}cl^*(V) \subseteq U$.

(c) For any m-closed set A, $\cap A_i = A$ where A_i are mg-closed neighbourhoods of A.

(d) For any set A and an m-open set B, $A \cap B$ is non empty, then there exists an mg-open set U such that $A \cap U \neq \phi$ and $m\text{-}cl^*(U) \subseteq B$.

(e) For any non empty set A and an m-closed set B such that A and B are disjoint, then there exists disjoint mg-open sets U, V satisfies $A \cap U$ is non empty and $B \subseteq V$

Proof. When $I = \{\phi\}$, observing Theorem 2.10., we have inferred that every $mI\alpha g$-open set is mg-open set. Apply this result in Theorem 5.8., the proof follows. □

If $I = \{\phi\}$ in Theorem 2.18., then we have the following Corollary.

Corollary 5.12. If (X, M, I) is an $m\text{-}T_1$ space with $I = \{\phi\}$ then the statements given below are equivalent.

(a) (X, M, I) is an m-regular space.

(b) Consider an m-open set V and let $x \in X$, there exists an $mI\alpha g$-open set U of X such that $x \in U \subseteq m\text{-}cl(U) \subseteq V$.

Proof. Observing Theorem 2.19., every m-closed set is an $mI\alpha g$-closed set, the proof is obvious by Theorem 2.18. □

6. Conclusion

In this work, we discussed about two separations called $mI\alpha g$-normal and $mI\alpha g$-regular spaces in ideal minimal spaces. The famous separation lemma called Urysohn's has been proved under $mI\alpha g$-normal spaces. Few equivalent statements on $mI\alpha g$-normal and $mI\alpha g$-regular spaces were established. In future, this work will be extended to discuss about Tietze extension theorem and Hausdorff spaces in ideal minimal spaces.

Conflicts of Interest

The authors declare no conflict of interest.

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