Fuzziness on Hopf Algebraic Structures with Its Application

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Abstract − The objective of writing this manuscript is to apply the concept of fuzzy set on some basic Hopf algebraic structures. In this manuscript, the novel concepts of fuzzy Hopf subalgebra, fuzzy Hopf ideal, and fuzzy H-submodule are proposed. Some properties of these concepts are discussed, and some significant results are also proved in it. The advantages of the proposed work are also studied in it. The application of the proposed work is also discussed in it.

1. Introduction

Heinz Hopf did much work in the field of the algebraic topology of Lie groups [1,2]. Later on, mathematicians worked on it and named it after Heinz Hopf as Hopf algebra (HA) [3-5]. Vast applications of HA are in physics, quantum groups, non-commutative geometry and representation theory etc. [6,7], e.g., a particle move in space-time has HA structures [8]. So, it is necessary to extend these concepts in uncertainty. In 1969, Sweedler [9] wrote the first book on HA. After that, much research was done in this algebra [10,11].

HA is an algebra with dual structure coalgebra and has an endomorphism called antipode. In other words, we can also say that an algebra with the structures of cohomology and homology of a topological group is called HA. In 1939, such algebras were introduced and linked with Lie groups. An example of HA is the Steenrod algebra introduced in the 1960s by Milnor and Moore [12], the cohomology algebra.

Zadeh gave the approach to fuzzy sets (FS) in 1965 [13]. Rosenfeld worked on the notion of fuzzy groups [14]. Basically, in fuzzy sets, the range of membership degree is [0,1], which tells us up to which degree the element belongs to a set. The fuzzy set theory deals with uncertainties. Fuzzy set theory is the door to developing an intelligent system for identification and decision making etc. There are vast applications of fuzzy sets in decision-making, pattern recognition, control theory, and optimization. A fuzzy set is also known as a model that represents uncertainty in the universe.

The thought of fuzzy submodule was given by Zahedi and Ameri [15]. Many authors used the concept of fuzzy submodules in different fields of mathematics and physics [16-18]. The thought of fuzzy subcomodule was offered by Chen and Akram [19] in 2012. Notions of comodule and coalgebra are generalizations and dualizations of module and algebra, respectively.

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The thought of fuzzy algebra was offered by Abdulkhalikov et al. [20] in 1988. The thought of fuzzy subcoalgebra was given by Chen [21] in 2009, and the thought of fuzzy subbialgebra was also given by Chen and Wenxu [22] in 2012. The concept of fuzziness was also applied in some other algebras in [23-25].

The main problem in mathematics containing uncertainty is how to carry out the ordinary concepts to the uncertainty case. The proposed work will help in dealing with uncertainty problems in HA and quantum groups. Our obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, expert systems, medical diagnosis, and engineering.

In this study, we introduce the concepts of fuzzy Hopf subalgebra (FHA), fuzzy Hopf ideal (FHI) and fuzzy H-submodule (FHM). Some basic and essential definitions that are helpful in this paper are recalled in Section 2. The notion of fuzzy Hopf subalgebra is introduced in Section 3. In this section, we also investigate some results and examples about fuzzy Hopf subalgebra. The concept of fuzzy Hopf ideal is introduced in Section 4. In this section, we also investigate some examples and results about fuzzy Hopf ideals. The thought of fuzzy H-submodule is offered in Section 5. In this section, we also investigate some examples and results about fuzzy H-submodule. Section 6 is about the advantages of our article, and Section 7 have application.

2. Preliminaries

In this section, we include some relevant definitions which are helpful for the reader. From now onward, except if stated otherwise, we use H for Hopf algebra, k for field and Q for H-module.

Definition 2.1. [5] A HA $(H, m, u, \Delta, \varepsilon)$ is a bialgebra with antipode S, where S is an inverse map under convolution operation.

Definition 2.2. [5] A subspace J of H is called Hopf subalgebra if it is subalgebra and $S(J) \subseteq J$.

Definition 2.3. [5] A B^{cop}(bialgebra having opposite comultiplication) is a co-opposite HA H^{cop} with antipode $\tilde{S}$.

Definition 2.4. [5] A subspace I of H is a Hopf ideal if it is a bi-ideal and $SI \subseteq I$

Definition 2.5. [5] A right k-space Q is H-Hopf module of H if

i. Q is a right H-module and right H-comodule, via $\rho: Q \rightarrow Q \otimes H$.

ii. ρ is a right H-module map.

Definition 2.6. [13] An FS $\xi: X \rightarrow [0,1]$ is a function of a non-empty set X.

Definition 2.7. If ξ is an FS of X, then the subset $\xi_{t'} = \{x \in X : \xi(x) \geq t', t' \in [0,1]\}$ is called level subsets.

Definition 2.8. [14] The intersection of two FSs ξ and σ of X is also an FS of X and is defined by

$$(\xi \cap \sigma)(x) = \xi(x) \wedge \sigma(x), \text{ for all } x \in X$$

Definition 2.9. [14] The sum of two FSs ξ and σ of X is defined as follows:

$$(\xi + \sigma)(x) = sup\{\xi(a) \wedge \sigma(b)\}, \text{ for all } x \in X$$

Definition 2.10. [20] An FS ξ of k-vector space V is called fuzzy subspace if for any $v, v' \in V$ and $\alpha, \beta \in k$

$$\xi(\alpha v + \beta v') \geq \xi(v) \wedge \xi(v')$$

Definition 2.11. [22] An FS ξ of a bialgebra B is called fuzzy subbialgebra of B, if for any $b, b' \in B$ and $\alpha, \beta \in k$, the following conditions are satisfied:

i. $\xi(\alpha b + \beta b') \geq \xi(b) \wedge \xi(b')$

ii. $\xi(by) \geq \xi(b) \wedge \xi(bb')$

iii. $\xi(b) \leq \xi(b_{i1}) \wedge \xi(b_{i2})$

Definition 2.12. [16] An FS ξ of a P-module Q is said to be a fuzzy submodule of Q if for any $q, q' \in Q$ and $p \in P$,

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q')$

iii. $\xi(pq) \geq \xi(q)$

Definition 2.13. [19] An FS ξ of a C-comodule Q and a left comodule map $\rho: Q \to C \otimes Q$ is said to be fuzzy subcomodule of Q, where $\rho(q) = \sum_{i=1,n} q_{i0} \otimes q_{i1}$ if for any $q, q' \in Q, c \in C$, and $\alpha, \beta \in k$,

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q')$

iii. $\xi(q) \leq \xi(q_{i0})$, for all i.

3. Fuzzy Hopf Subalgebra

In this section, the notion of FHA is proposed. Some significant results related to this concept are also discussed in it.

Definition 3.1. A FS ξ of H is called FHA, if for any $h, h' \in H$ and $\alpha, \beta \in k$ it satisfies:

i. $\xi(\alpha h + \beta h') \geq \xi(h) \wedge \xi(h')$

ii. $\xi(hh') \geq \xi(h) \wedge \xi(h')$

iii. $\xi(h) \leq \xi(h_{i1}) \wedge \xi(h_{i2})$

iv. $\xi(h') \geq \begin{cases} \sup \{\xi(h)\}, & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$

Example 3.2. Consider the 4-dimensional HA

$$H_4 = \langle 1, h_1, h_2, h_1 h_2 \mid h_1{}^2 = 1, h_2{}^2 = 0, h_2 h_1 = -h_1 h_2 \rangle$$

and $\xi: H_4 \to [0,1]$ defined by

$$\xi(a) = \begin{cases} 0.4, & \text{if } a \in H_4 \backslash \{0\} \\ 0.8, & \text{if } a = 0 \end{cases}$$

Then, ξ becomes an FHA of H_4.

Remark 3.3. There is no difference between FHA and fuzzy co-opposite HA because $\xi(h) \wedge \xi(h') = \xi(h') \wedge \xi(h)$.

Theorem 3.4. Let ξ and σ be FHA of H such that $\xi(0) = \sigma(0)$. Then, $\xi + \sigma$ is also fuzzy FHA of H.

Proof. Let $h, h' \in H$ and $\alpha, \beta \in k$. Now, we first have to show that $\xi + \sigma$ is a fuzzy subspace.

Suppose on the contrary

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h) \wedge (\xi + \sigma)(h')$$

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h)$$

and

$$(\xi + \sigma)(\alpha h + \beta h') < (\xi + \sigma)(h')$$

Let $\exists t' \in [0,1]$ such that

$$(\xi + \sigma)(\alpha h + \beta h') < t' < (\xi + \sigma)(h)$$

and

$$(\xi + \sigma)(\alpha h + \beta h') < t' < (\xi + \sigma)(h')$$

Then, $\exists h_1, h_2, h_3, h_4 \in H$ with $\alpha h = h_1 + h_2$

$$h = (\frac{h_1 + h_2}{\alpha}) \text{ and } \beta h' = h_3 + h_4$$

$$h' = (\frac{h_3 + h_4}{\beta}) \text{ such that } \xi((\frac{h_1 + h_2}{\alpha})) > t', \sigma((\frac{h_3 + h_4}{\beta})) > t'$$

Now, we have

$$
\begin{aligned}
(\xi + \sigma)(\alpha h + \beta h') &= \sup_{h + h' = m + n} \{\xi(m) \wedge \sigma(n)\} \\
&\geq \sup_{h + h' = m + n} \left\{\xi(\frac{h_1 + h_2}{\alpha}) \wedge \sigma(\frac{h_3 + h_4}{\beta})\right\} \\
&> t' > (\xi + \sigma)(\alpha h + \beta h')
\end{aligned}
$$

which is contradiction. Therefore,

$$(\xi + \sigma)(\alpha h + \beta h') \geq (\xi + \sigma)(h) \wedge (\xi + \sigma)(h')$$

For all $h, h' \in H$, let $h = h_1 + h_2$ and $h' = h_3 + h_4$. Then, $hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4$. Moreover,

$$
\begin{aligned}
(\xi + \sigma)(hh') &= \sup_{hh' = m + n} \{\xi(m) \wedge \sigma(n)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_1 h_3 + h_1 h_4) \wedge \sigma(h_2 h_3 + h_2 h_4)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_1 h_3) \wedge \xi(h_1 h_4) \wedge \sigma(h_2 h_3) \wedge \sigma(h_2 h_4)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_1) \wedge \xi(h_3) \wedge \xi(h_4) \wedge \sigma(h_2) \wedge \sigma(h_3) \wedge \sigma(h_4)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{(\xi(h_1) \wedge \sigma(h_2)) \wedge (\xi(h_3) \wedge \sigma(h_4)) \wedge (\xi(h_4) \wedge \sigma(h_3))\} \\
&= (\xi + \sigma)(h) \wedge (\xi + \sigma)(h') \wedge (\xi + \sigma)(h') \\
&= (\xi + \sigma)(h) \wedge (\xi + \sigma)(h')
\end{aligned}
$$

Furthermore, let $h = h_1 + h_2 \in H$. Then,

$$\sum_{i=1,n} h_{i1} \otimes h_{i2} = \Delta(h) = \Delta(h_1 + h_2) = \Delta(h_1) + \Delta(h_2)$$

Here, $h_1 = \sum_{s=1,n} h_{1(s1)} \otimes h_{1(s2)}$ and $h_2 = \sum_{t=1,n} h_{2(t1)} \otimes h_{2(t2)}$.

Since

$$
\begin{aligned}
(\xi + \sigma)(h) &= \sup_{h = h_1 + h_2} \{\xi(h_1) \wedge \sigma(h_2)\} \\
&\leq \sup_{h = h_1 + h_2} \{(\xi(h_{1(s1)}) \wedge \xi(h_{1(s2)})) \wedge (\sigma(h_{2(t1)}) \wedge \sigma(h_{2(t2)}))\} \\
&= \sup_{h = h_1 + h_2} \{\xi(h_{1(s1)}) \wedge \sigma(0) \wedge \xi(0) \wedge \sigma(h_{2(t1)})\} \wedge \sup_{h = h_1 + h_2} \{\xi(h_{1(s2)}) \wedge \sigma(0) \wedge \xi(0) \wedge \sigma(h_{2(t2)})\}
\end{aligned}
$$

$$\leq \big((\xi+\sigma)(h_{1(s1)}) \wedge (\xi+\sigma)(h_{2(t1)})\big) \wedge \big((\xi+\sigma)(h_{1(s2)}) \wedge (\xi+\sigma)(h_{2(t2)})\big)$$

$$\leq (\xi+\sigma)(h_{i1}) \wedge (\xi+\sigma)(h_{i2})$$

$$(\xi+\sigma)(h') \geq \begin{cases} \sup\limits_{h \in S^{-1}(h')} \left\{ \sup\limits_{h=h_1+h_2} \xi(h^1) \wedge \sigma(h^2) \right\}, & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

and

$$(\xi+\sigma)(h') \geq \begin{cases} \sup\limits_{h \in S^{-1}(h')} (\xi+\sigma)(h), & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

Then $\xi + \sigma$ is FHA.

Theorem 3.5. Let ξ and σ be two FHA of H then $\xi \cap \sigma$ is also FHA.

Theorem 3.6. Let $\{\xi_i : i \in N\}$ be any collection of an FHA of H then $\cap \xi_i$ is also FHA.

Theorem 3.7. An FS ξ of H is an FHA iff the level sets $\xi_{t'}$ are Hopf subalgebras of H.

Proof. Assume that ξ is an FHA. Since,

$$\xi(h) \geq 0 = t', \forall h \in H$$

$$\xi_{t'} \neq \phi$$

Let $h, h' \in \xi_{t'}$, $\xi(h) \geq t'$, and $\xi(h') \geq t'$. Since

$$\xi(\alpha h + \beta h') \geq \xi(h) \wedge \xi(h') \geq t' \wedge t' = t'$$

then $\alpha h + \beta h' \in \xi_{t'}$. Similarly, since

$$\xi(hh') \geq \xi(h) \wedge \xi(h') \geq t' \wedge t' = t'$$

then $hh' \in \xi_{t'}$.

Let $h \in H$ such that $\xi(h) = t' \Rightarrow h \in \xi_{t'}$ and $\Delta(h) = \sum_{i=1,n} \Sigma\, h_{i0} \otimes h_{i1} \Rightarrow \xi(h_{i0}) \geq t'$ and $\xi(h_{i1}) \geq t'$. Then,

$$\xi(h_{i0}) \wedge \xi(h_{i1}) \geq t' \wedge t' = t' = \xi(h)$$

Let $h \in H$ such that $\xi(h) = t' \Rightarrow h \in \xi_{t'}$. If $h' \in S(H)$, then,

$$\xi(h') \geq \sup\limits_{h \in S^{-1}(h')} \xi(h) = t'$$

$$\Rightarrow h' \in \xi_{t'}$$

Hence, $\xi_{t'}$ is Hopf subalgebra for all t'.

Conversely, assume that all $\xi_{t'}$ are Hopf subalgebras.

Let $h, h' \in \xi_{t'}$ and $\alpha, \beta \in k$. We may assume that

$$\xi(h') \geq \xi(h) = t'$$

Since $h, h' \in \xi_{t'}$, then $\alpha h + \beta h' \in \xi_{t'}$. Thus,

$$\xi(\alpha h + \beta h') \geq t' = \xi(h) \wedge \xi(h')$$

Since $h, h' \in \xi_{t'}$, then $hh' \in \xi_{t'}$. Thus,

$$\xi(hh') \geq t' = \xi(h) \wedge \xi(h')$$

Let $h \in \xi_{t'}$ and $\Delta(h) = \sum_{i=1,n} h_{i0} \otimes h_{i1}$ where $h_{i0}, h_{i1} \in \xi_{t'}$. Therefore,

$$\xi(h_{i0}) \wedge \xi(h_{i1}) \geq t' \wedge t' = t' = \xi(h)$$

Let $h' \in S(H)$ and $\xi(h') \geq t'$. Since $h' \in \xi_{t'}$ and

$$\xi(h') \geq t' = \sup_{h \in S^{-1}(h')} \xi(h)$$

Then, ξ is FHA.

4. Fuzzy Hopf Ideal

In this section, the concept of fuzzy Hopf Ideal is proposed. Some significant results of the fuzzy Hopf ideal are also studied in it.

Definition 4.1. A fuzzy subspace ξ of H is called fuzzy Hopf left (right) ideal if for any $h, h' \in H$

i. $\xi(hh') \geq \xi(h'), (\xi(hh') \geq \xi(h))$

ii. $\xi(h) \leq \xi(h_{i2}), \ (\xi(h) \leq \xi(h_{i1}))$

iii. $\xi(h') \geq \begin{cases} \sup\limits_{h \in S^{-1}(h')} \xi(h), & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$

Example 4.2. Consider $\xi: H_4 \to [0,1]$ defined by

$$\xi(a) = \begin{cases} 0.5, & \text{if } a \in H_4 \backslash \{0,1\} \\ 0.3, & \text{if } a = 1 \\ 0.8, & \text{if } a = 0 \end{cases}$$

Then, ξ becomes a fuzzy Hopf left ideal of H_4.

Remark 4.3. If ξ is both right and left fuzzy Hopf ideal of H then ξ is called an FHI of H.

Theorem 4.4. Let ξ and σ be two fuzzy Hopf left (right) ideals of H such that $\xi(0) = \sigma(0)$. Then, $\xi + \sigma$ is also a fuzzy Hopf left (right) ideal of H.

Proof. $\xi + \sigma$ is a fuzzy subspace. Let $h = h_1 + h_2, \ h' = h_3 + h_4$. Then,

$$hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4$$

Moreover,

$$
\begin{aligned}
(\xi + \sigma)(hh') &= \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_1 h_3 + h_1 h_4) \wedge \sigma(h_2 h_3 + h_2 h_4)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_1 h_3) \wedge \xi(h_1 h_4) \wedge \sigma(h_2 h_3) \wedge \sigma(h_2 h_4)\} \\
&\geq \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_3) \wedge \xi(h_4) \wedge \sigma(h_3) \wedge \sigma(h_4)\} \\
&= \sup_{hh' = h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4} \{\xi(h_3) \wedge \sigma(h_4) \wedge \sigma(h_3) \wedge \xi(h_4)\} \\
&= (\xi + \sigma)(h') \wedge (\xi + \sigma)(h') \\
&= (\xi + \sigma)(h')
\end{aligned}
$$

Let $h = h_1 + h_2 \in H$. Then,

$$\sum_{i=1,n} h_{i1} \otimes h_{i2} = \Delta(h) = \Delta(h_1 + h_2) = \Delta(h_1) + \Delta(h_2)$$

where, $h_1 = \sum_{s=1,n} h_{1(s1)} \otimes h_{1(s2)}$ and $h_2 = \sum_{t=1,n} h_{2(t1)} \otimes h_{2(t2)}$ such that

$$(\xi + \sigma)(h) = \sup_{h=h_1+h_2} \{\xi(h_1) \wedge \sigma(h_2)\}$$

$$\leq \sup_{h=h_1+h_2} \{\xi(h_{1(s2)}) \wedge \sigma(h_{2(t2)})\}$$

$$= \sup_{h=h_1+h_2} \{\xi(h_{1(s2)}) \wedge \sigma(0) \wedge \xi(0) \wedge \sigma(h_{2(t2)})\}$$

$$\leq (\xi + \sigma)(h_{1(s2)}) \wedge (\xi + \sigma)(h_{2(t2)})$$

$$\leq (\xi + \sigma)(h_{i2})$$

$$(\xi + \sigma)(h') \geq \begin{cases} \sup_{h \in S^{-1}(h')} \left\{ \sup_{h=h_1+h_2} \xi(h_1) \wedge \sigma(h_2) \right\}, & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

and

$$(\xi + \sigma)(h') \geq \begin{cases} \sup_{h \in S^{-1}(h')} (\xi + \sigma)(h), & \text{if } h' \in S(H) \\ 0, & \text{if } h' \notin S(H) \end{cases}$$

Hence, $\xi + \sigma$ is fuzzy Hopf left ideal.

Theorem 4.5. Let ξ and σ be two fuzzy Hopf left (right) ideals then $\xi \cap \sigma$ is also a fuzzy Hopf left (right) ideal.

Theorem 4.6. Let $\{\xi_i : i \in N\}$ be a collection of fuzzy Hopf left (right) ideals then $\cap \xi_i$ is also a fuzzy Hopf left (right) ideal.

5. Fuzzy H-Submodule

In this section, the concept of fuzzy H-Submodule is proposed. Some significant results of fuzzy H-Submodule are also studied in it.

Definition 5.1. A FS ξ of Q is right FHM if for any $q, q' \in Q$, $h \in H$ and $\alpha, \beta \in k$

i. $\xi(0) = 1$

ii. $\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q')$

iii. $\xi(qh) \geq \xi(q)$

iv. $\xi(q) \leq \xi(q_{i0})$

v. $\xi(q') \geq \begin{cases} \sup_{q \in \rho^{-1}(q')} \xi(q), & \text{if } q' \in \rho(Q) \\ 0, & \text{if } q' \notin \rho(Q) \end{cases}$

(vi). $\xi(\rho(q)h) \geq \xi(\rho(q))$

Example 5.2. Consider $\xi : H_4 \to [0,1]$ defined by

$$\xi(a) = \begin{cases} 0.4, & \text{if } a \in H_4 \backslash \{0\} \\ 1, & \text{if } a = 0 \end{cases}$$

Then, ξ becomes an FHM of H_4.

Theorem 5.3. Let ξ and σ be two FHM of Q such that $\xi(0) = \sigma(0)$. Then, $\xi + \sigma$ is also an FHM of Q.

PROOF. Let $q, q' \in Q$ and $\alpha, \beta \in k$. Now, we have to show that $\xi + \sigma$ is a fuzzy subspace. On the contrary, suppose that

$$(\xi + \sigma)(\alpha q + \beta q') \; < \; (\xi + \sigma)(q) \wedge (\xi + \sigma)(q'),$$
$$(\xi + \sigma)(\alpha q + \beta q') \; < \; (\xi + \sigma)(q)$$

and

$$(\xi + \sigma)(\alpha q + \beta q') < (\xi + \sigma)(q')$$

If $\exists t' \in [0,1]$ such that

$$(\xi + \sigma)(\alpha q + \beta q') < t' < (\xi + \sigma)(q)$$

and

$$(\xi + \sigma)(\alpha q + \beta q') < t' < (\xi + \sigma)(q')$$

If $\exists\, q_1, q_2, q_3, q_4 \in Q$ with $\alpha q = q_1 + q_2$, then

$$q = ((q_1 + q_2)/\alpha) \text{ and } \beta q' = q_3 + q_4$$
$$q' = (\tfrac{q_3 + q_4}{\beta}) \text{ such that } \xi((\tfrac{q_1 + q_2}{\alpha})) > t', \sigma((\tfrac{q_3 + q_4}{\beta})) > t'$$

Now, we have

$$(\xi + \sigma)(\alpha q + \beta q') \;\; = \sup_{q + q' = n + n'} \{\xi(n) \wedge \sigma(n')\}$$

$$\geq \sup_{q + q' = \frac{q_1 + q_2}{\alpha} + \frac{q_3 + q_4}{\beta}} \left\{ \xi((\tfrac{q_1 + q_2}{\alpha})) \wedge \sigma((\tfrac{q_3 + q_4}{\beta})) \right\}$$

$$> t' > (\xi + \sigma)(\alpha q + \beta q')$$

which is a contradiction. Therefore,

$$(\xi + \sigma)(\alpha q + \beta q') \geq (\xi + \sigma)(q) \wedge (\xi + \sigma)(q')$$

Now, let $q \in Q$ and $h \in H$ such that

$$(\xi + \sigma)(qh) \;\; = \sup_{qh = q_1 + q_2} \{\xi(q_1) \wedge \sigma(q_2)\}$$

$$\geq \sup_{q = \frac{q_1}{h} + \frac{q_2}{h}} \left\{ \xi(\tfrac{q_1}{h}) \wedge \sigma(\tfrac{q_2}{h}) \right\}$$

$$= (\xi + \sigma)(q)$$

Let $q = q_1 + q_2 \in Q$. Then,

$$\sum_{i=1,n} q_{i0} \otimes q_{i1} \;\; = \rho(q) \;\; = \rho(q_1 + q_2)$$

$$= \rho(q_1) + \rho(q_2)$$

$$= \sum_{s=1,n} q_{1(s0)} \otimes q_{1(s1)} + \sum_{t=1,n} q_{2(t0)} \otimes q_{2(t1)}$$

Moreover,

$$
\begin{aligned}
(\xi + \sigma)(q) \quad &= \quad \sup_{q=q_1+q_2} \{\xi(q_1) \wedge \sigma(q_2)\} \\
&\leq \quad \sup_{q=q_1+q_2} \{\xi(q_{1(s0)}) \wedge \sigma(q_{2(t0)})\} \\
&= \quad \sup_{q=q_1+q_2} \{\xi(q_{1(s0)}) \wedge \sigma(0) \wedge \xi(0) \wedge \sigma(q_{2(t0)})\} \\
&= \quad \sup_{q=q_1+q_2} \{\xi(q_{1(s0)}) \wedge \sigma(0)\} \wedge \sup_{q=q_1+q_2} \xi(0) \wedge \sigma(q_{2(t0)}) \\
&\leq \quad (\xi + \sigma)(q_{1(s0)}) \wedge (\xi + \sigma)(q_{2(t0)}) \\
&\leq \quad (\xi + \sigma)(q_{i0})
\end{aligned}
$$

and

$$
(\xi + \sigma)(0) = \sup_{0=q_1+q_2} \{\xi(q_1) \wedge \sigma(q_2)\} \geq \xi(0) \wedge \sigma(0) = 1 \wedge 1 = 1
$$

Thus,

$$
\begin{aligned}
(\xi + \sigma)(\rho(q)h) \quad &= \quad \sup_{\rho(q)h=q_1+q_2} \{\xi(q_1) \wedge \sigma(q_2)\} \\
&\geq \quad \sup_{\rho(q)=\frac{q_1}{h}+\frac{q_2}{h}} \left\{\xi(\frac{q_1}{h}) \wedge \sigma(\frac{q_2}{h})\right\} \\
&= \quad (\xi + \sigma)(\rho(q))
\end{aligned}
$$

Hence, $\xi + \sigma$ is FHM of Q.

Theorem 5.4. Let ξ and σ be two FHMs of Q then $\xi \cap \sigma$ is also an FHM.

Theorem 5.5. Let $\{\xi_i : i \in N\}$ be a collection of FHMs of Q. Then $\cap \xi_i$ is also an FHM.

Theorem 5.6. An FS ξ of Q is an FHM iff the level sets $\xi_{t'}$ are $H-$submodules of Q.

PROOF. Assume that ξ is an FHM. As $\xi(0) = 1$. Therefore, $\xi_t \neq \varphi$. Let $q, q' \in \xi_{t'}$. Then,

$$
\xi(q) \geq t' \text{ and } \xi(q') \geq t'
$$

Since

$$
\xi(\alpha q + \beta q') \geq \xi(q) \wedge \xi(q') \geq t' \wedge t' = t'
$$

then $\alpha q + \beta q' \in \xi_{t'}$. Similarly, since $\xi(qh) \geq \xi(q) = t'$ then $qh \in \xi_{t'}$. Moreover, since $\xi(q_{i0}) \geq \xi(q) \geq t'$ then $x_{i0} \in \xi_{t'}$. Let $\rho(x) \in \xi_{t'}$ and $h \in H$. Since

$$
\xi(\rho(q)h) \geq \xi(\rho(q)) \geq t'
$$

then $\rho(q)h \in \xi_{t'}$. Therefore, each $\xi_{t'}$ is $H-$submodule of Q.

Conversely, assume that each $\xi_{t'}$ is a fuzzy H-submodules. Let $q, q' \in Q$ and $\alpha, \beta \in k$. We may assume that $\xi(q') \geq \xi(q) = t'$. Therefore, $q, q' \in \xi_{t'}$ Since, $\xi_{t'}$ is $H-$submodule of Q, then $\alpha q + \beta q' \in \xi_{t'}$. Therefore,

$$
\xi(\alpha q + \beta q') \geq t' = \xi(q) \wedge \xi(q')
$$

Let $q \in Q$ and $h \in H$. Let $\xi(q) = t'$. Therefore, $q \in \xi_{t'}$ and $qh \in \xi_{t'}$. Thus, $\xi(qh) \geq t' = \xi(q)$.

Since $\rho(q) = \sum_{i=1,n} q_{i0} \otimes q_{i1}$ where $q_{i0} \in \xi_{t'}$, then

$$
\xi(q_{i0}) \geq t' = \xi(q)
$$

Moreover,

$$\xi(0) = \sup_{q \in Q}\{\xi(q)\} = 1$$

Let $\rho(q) \in Q$ and $h \in H$, such that $\xi(\rho(q)) = t'$. Then, $\rho(q)h \in \xi_{t'}$

$$\xi(\rho(q)h) \geq t' = \xi(\rho(q))$$

Thus, ξ is an FHM.

6. Advantages

The importance of Hopf algebra in Quantum physics and physics cannot be rejected. Shahn Majid and other physicist and mathematician use Hopf algebra to solve many problems. By using fuzzy theory, we can solve these problems with easier. So, it is crucial to extend these concepts in uncertainty. So, in this manuscript, we proposed fuzziness on Hopf algebraic structures.

7. Application

Discrete gauge theory in the $2 + 1$ dimension arises by breaking a gauge symmetry with gauge group G to discrete subgroup H due to photons becoming massive w, making the guage force ultra-short ranged. Some particles are charged, and some carry flux. Charged particles carry a representation of H, and massive charges are ultra-short ranged. By using fuzziness, we can deal with these particles easily. Here is a simple example,

Example 7.1. Let $H = \{h_1, h_2, h_3, h_4\}$ be a Hopf algebra with antipode S, $k = \mathbb{R}$, $\Delta(h) = h \otimes h$, $\varepsilon(h) = 1$, $S(h) = h^{-1}$,

+	h_1	h_2	h_3	h_4
h_1	h_1	h_1	h_1	h_1
h_2	h_1	h_1	h_1	h_1
h_3	h_1	h_1	h_1	h_1
h_4	h_1	h_1	h_1	h_1

and

$\cdot$	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	h_1	h_4	h_3
h_3	h_3	h_4	h_2	h_1
h_4	h_4	h_3	h_1	h_2

Then, $\xi: H \to [0,1]$ defined by

$$\xi(d) = \begin{cases} 0.6, & \text{if } h = h_1 \\ 0.4, & \text{if } h \in H/\{h_1\} \end{cases}$$

is an FHA. Therefore, h_1 is the short, ranged particle.

Similarly, we can apply this uncertainty in other fields of physics, like to know the energy level of photons in non-ideal lasers etc.

8. Conclusion

This manuscript introduces the concept of fuzzy Hopf subalgebra, fuzzy Hopf ideal, and fuzzy H-submodule. Some properties and significant results are also studied in it. Our obtained results probably can be applied in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, expert systems, medical diagnosis, and engineering. Weak Hopf algebra is a Hopf algebra with a linear map that satisfy some specific conditions. In the future study, we aim to extend this concept in the field of weak Hopf algebra.

Conflict of Interest

The authors declare no conflict of interest.

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