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EXPONENTIAL STABILITY OF A TIMOSHENKO TYPE THERMOELASTIC SYSTEM WITH GURTIN-PIPKIN THERMAL LAW AND FRICTIONAL DAMPING

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Abstract. In this paper we consider a linear thermoelastic system of Timoshenko type where the heat conduction is given by the linearized law of Gurtin-Pipkin. An existence and uniqueness result is proved by the use of a semigroup approach. We establish an exponential stability result without any assumption on the wave speeds once here we have a fully damped system.

1. Introduction

In the present paper we investigate the well-posedness and the asymptotic behavior of the following Timoshenko type system

$$\begin{cases} \rho_1 u_{tt} = \kappa\left(u_x+\varphi\right)_x & in\ (0,\pi)\times\mathbb{R}_+, \\ \rho_2\varphi_{tt} = b\varphi_{xx} - \kappa\left(u_x+\varphi\right) + \delta\theta - \tau\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \\ c\theta_t = -q_x - \delta\varphi_t & in\ (0,\pi)\times\mathbb{R}_+, \end{cases} \tag{1}$$

where u is the transverse displacement of a beam of length π, φ is the rotation angle of filament, θ is the temperature variation from an equilibrium reference value and q is the heat flux. The coefficients $\rho_1, \rho_2, c, \kappa, \tau$ are positive and present the mass density, the polar moment of inertia of a cross section, the specific heat constant, the shear modulus and the intensity of the frictional damping respectively, $b = EI$ is the product of Young's modulus of elasticity and the moment of inertia of a cross section, β and δ are coupling constants that are different from zero but their signs does not matter in the analysis.

To render the system (1) determined an additional equation relating q and θ is needed. In the classical theory of thermoelasticity the constitutive equation for the

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heat flux is expressed through Fourier's law of heat conduction

$$q = -k\theta_x, \tag{2}$$

where $k > 0$ represents the coefficient of the thermal conductivity of the material.

In 1921, Timoshenko [32] introduced a shear deformation and a rotational inertia into the derivation of the vibrating beam theory. He modelled the transverse vibrations of a beam by the conservative system

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in } (0, L) \times (0, \infty) \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in } (0, L) \times (0, \infty). \end{cases} \tag{3}$$

In the last three decades, the system (3) has been intensively studied for possible damping mechanisms. Muñoz Rivera and Racke [25] introduced a thermal damping by coupling system (3) with the classical heat equation. They proved that the system

$$\begin{cases} \rho_1 \varphi_{tt} = k\left(\varphi_x + \psi\right)_x, \\ \rho_2 \psi_{tt} = b\psi_{xx} - k\left(\varphi_x + \psi\right) + \gamma\theta_x, \\ c\theta_t = \kappa\theta_{xx} - \gamma\psi_{tx}. \end{cases} \tag{4}$$

(of course with some boundary and initial conditions), is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}. \tag{5}$$

If (5) does not hold Guesmia *et al.* [17] established a polynomial decay result provided that the initial data are regular enough.

Almeida Junior *et al.* [1] considered the thermal coupling of the system (3) in shear force

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x + \sigma\theta_x = 0 & in\ (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) - \sigma\theta = 0 & in\ (0, L) \times \mathbb{R}_+, \\ \rho_3 \theta_t - \gamma\theta_{xx} + \sigma\left(\varphi_x + \psi\right)_t = 0 & in\ (0, L) \times \mathbb{R}_+, \end{cases} \tag{6}$$

subjected to either the boundary conditions

$$\varphi(t, 0) = \varphi(t, L) = \psi(t, 0) = \psi(t, L) = \theta(t, 0) = \theta(t, L) = 0, \tag{7}$$

or

$$\varphi(t, 0) = \varphi(t, L) = \psi_x(t, 0) = \psi_x(t, L) = \theta_x(t, 0) = \theta_x(t, L) = 0, \tag{8}$$

and proved that the solution is exponentially stable if and only if

$$\chi = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} = 0. \tag{9}$$

Otherwise, when (9) does not hold, the authors showed that the system is polynomially stable with a rate of decay $t^{-1/4}$ for the boundary conditions (7) and an optimal rate of decay $t^{-1/2}$ for the boundary conditions (8). Recently [18] reached the rate $t^{-1/2}$ for the boundary conditions (7) and

$$\varphi_x(t, 0) = \varphi_x(t, L) = \psi_(t, 0) = \psi(t, L) = \theta_x(t, 0) = \theta_x(t, L) = 0.$$

Alves *et al.* [2] improve the results of [1] for the case of different wave speeds and obtained the same rate of decay $t^{-1/2}$ independently of the boundary conditions. Later, Alves *et al.* [3] extended the results of [1] to the non-homogeneous case with the boundary conditions (7). Precisely, they established an exponential stability provided that the non-homogeneous wave speeds satisfy the condition

$$\frac{\kappa(x)}{\rho_1(x)} = \frac{b(x)}{\rho_2(x)}, \quad x \in I \subset (0, L), \tag{10}$$

in an open subinterval I of $(0, L)$. When (10) does not hold they obtained a polynomial stability result with a rate of decay depending on the regularity of the initial data.

Recently, Jorge-Silva and Racke [19] considered (6) with Cattaneo's law and proved that there is non exponential stability no matter if (9) holds which confirms the result of [10].

We recall that the model using Fourier's law (2) leads to a parabolic equation. Consequently, the heat propagates with an infinite speed, that is, any thermal disturbance produced at some point in the body has an instantaneous effect elsewhere in the body. To overcome this physical paradox, many theories were developed. Green and Naghdi [12–14] expanded three new theories based on an entropy equality rather than the entropy inequality. They called them thermoelasticity of type I, type II and type III respectively. In each of these theories the equation for the heat flux is given by a different constitutive assumption. The constitutive equation for the heat flux in the type III theory is given by

$$q = -f_1\alpha_x - f_2\theta_x,$$

where

$$\alpha = \alpha_0(x) + \int_0^t \theta(x, \tau)\, d\tau$$

is the thermal displacement and f_1, f_2 are two positive constants.

In the framework of the thermoelasticity of type III, Messaoudi and Said-Houari [24] considered the following Timoshenko type system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0 & in\ (0,1) \times (0, +\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0 & in\ (0,1) \times (0, +\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \beta\psi_{ttx} + \kappa\theta_{txx} = 0 & in\ (0,1) \times (0, +\infty), \end{cases}$$

and showed that the solution (φ, ψ, θ) decays exponentially provided that $\frac{K}{\rho_1} = \frac{b}{\rho_2}$. The case of non equal speeds was examined by Messaoudi and Fareh [23]. They established a polynomial rate of decay. Fatori *et al.* [9] show that the optimal rate in this case is $t^{-1/2}$.

Santos and Almeida Júnior [30] extended the results of [23,24] to the Timoshenko system with thermoelastic effect acting on a shear force

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \sigma\theta_{tx} = 0 & in\ (0,L)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) - \sigma\theta_t = 0 & in\ (0,L)\times(0,+\infty), \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \sigma\left(\varphi_x + \psi\right)_t - \gamma\theta_{txx} = 0 & in\ (0,L)\times(0,+\infty). \end{cases}$$

The second theory proposed to overcome the paradox of infinite speed was developed by Lord and Shulman [21]. They suggested to replace Fourier's law (2) by Cattaneo's one

$$\tau_0 q_t + q + k\theta_x = 0,$$

where the positive constant τ_0 represents the time lag in the response of the heat flux to the temperature gradient and is referred to as the thermal relaxation time. According to this theory, the system becomes fully hyperbolic, as a result the heat propagates with a finite speed and is viewed as a wave-like propagation rather than a diffusion phenomenon. A wave-like thermal disturbance is referred to as a second sound (where the first sound being the usual sound) and a nonclassical theory predicting the occurrence of such disturbances are known as thermoelasticity with finite wave speeds or second sound thermoelasticity.

Fernández Sare and Racke [10] considered the following Timoshenko type system with second sound thermoelasticity

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases} \tag{11}$$

and proved that the solution of (11) is no longer exponentially stable even if $\frac{\rho_1}{k} = \frac{\rho_2}{b}$. However, the incorporation of the frictional damping $\mu\varphi_t$ into the first equation of (11) produces an exponential stability independently of the wave speeds [22].

Santos *et al.* [31] introduced the stability number

$$\chi_0 = \left(\tau - \frac{\rho_1}{\rho_3\kappa}\right)\left(\rho_2 - \frac{b\rho_1}{\kappa}\right) - \frac{\tau\rho_1\delta^2}{\kappa\rho_3},$$

and proved that the solution of (11) is exponentially stable provided that $\chi_0 = 0$.

It is worth noting that the type III thermoelasticity and the second sound thermoelasticity are unable to describe the memory effect which reigns in some materials, particularly at a low temperature. This fact leads to the look for a more general constitutive assumption relating the heat flux to the thermal memory. Gurtin and Pipkin [16] assumed that the heat flux depends on the integrated history of the temperature gradient, and established a general nonlinear theory for which thermal disturbances propagate with a finite speed. In accordance with this theory, the linearized constitutive equation for q is given by

$$q = -\int_{-\infty}^{t} k\left(t-s\right)\theta_x\left(x,s\right)ds, \tag{12}$$

where $k(s)$ is the heat conductivity relaxation kernel. The presence of the convolution term (12) renders the Timoshenko system coupled with the heat equation into a fully hyperbolic system, which allows the heat to propagate with a finite speed and admits to describe the memory effect of the heat conduction.

In the context of Gurtin-Pipkin theory Pata and Vuk [26] studied the linear thermoelastic system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - \theta_x(x,t), \\ \theta_t\,(x,t) = -u_{tx}\,(x,t) - q_x\,(x,t)\,, \end{cases}$$

where the heat flux q is given by (12). They proved, under some assumptions on $\mu\,(s) = -k'\,(s)$, that the solution of the system decays exponentially. Fatori and Muñoz Rivera [8] considered the system

$$\begin{cases} u_{tt} - au_{xx} + \alpha\theta_x = 0 \; in \; (0,L) \times \mathbb{R}_+ \\ \theta_t - k * \theta_{xx} + \alpha u_{xt} = 0 \; in \; (0,L) \times \mathbb{R}_+, \end{cases}$$

where

$$(k * \theta_{xx})\,(t) = \int_0^t k\,(t-\tau)\,\theta_{xx}\,(\tau)\,d\tau,$$

and established an exponential decay result provided that the kernel k is positive definite and decays exponentially.

Concerning Timoshenko systems coupled with the heat equation in the framework of Gurtin-Pipkin's theory, Dell'Oro an Pata [7] analyzed the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa\,(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa\,(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^\infty g\,(s)\,\theta_{xx}\,(t-s)\,ds + \delta\psi_{tx} = 0, \end{cases} \tag{13}$$

and proved that the semigroup associated with the solution of the system (13) is exponentially stable if and only if

$$\chi_g = \left[\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g\,(0)}\right]\left[\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right] - \frac{\beta}{g\,(0)}\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0.$$

Closely related to Timoshenko's beam theory, Raposo [29] investigated the laminated Timoshenko system

$$\begin{cases} \rho_1 u_{tt} - \kappa\,(u_x - \psi)_x + \alpha u_t = 0 & in\;\,(0,L) \times \mathbb{R}_+, \\ \rho_2(s-\psi)_{tt} - b(s-\psi)_{xx} + \kappa\,(\psi - u_x) + \beta(s-\psi)_t = 0 & in\;\,(0,L) \times \mathbb{R}_+, \\ \rho_2 s_{tt} - bs_{xx} + 3\kappa(\psi - u_x) + 4\delta s + 4\gamma s_t = 0 & in\;\,(0,L) \times \mathbb{R}_+, \end{cases} \tag{14}$$

and obtained an exponential stability result. Regarding the damping by the heat conduction, Liu and Zhao [20] showed that the laminated beam coupled with the heat equation modelled via Fourier's law of the heat conduction is exponentially stable provided that the wave speeds are equal. Apalara [4] obtained the same result by coupling the laminated beam with the heat equation moddeled via Cattaneo's law, provided that the equal wave speeds is replaced by a relation between the coefficients of the system. Choucha *et al.* [5] added a distributed delay and proved the exponential and the polynomial stability for the equal and the non-equal wave speeds respectively. They also kept the same results in the presence of a viscoelastic damping and a distributed delay [6].

In view of the aforementioned studies we can summarized the stability results for Timoshenko systems coupled with thermal effects as follows:

i) A fully damped Timoshenko system with parabolic thermal effects is exponentially stable regardless any restriction on the wave speeds.
ii) A Timoshenko system damped only by thermal effects is exponentially stable if and only if the coefficients of the system satisfy a stability condition (equal wave speeds, in the case of the classical parabolic heat equation).

To the best of my knowledge there is no results concerning the fully damped Timoshenko system with hyperbolic thermal dissipation. One can expected that this leads to an exponential stability. In the present paper we give a positive answer to this concern.

It should be noted here, that replacing the parabolic heat conduction by a hyperbolic type one is not obviously profitable, first, because the system becomes fully hyperbolic and therefore it loses the exponential decay reached with one dissipation when (5) holds, (see [10, 28]), secondly, because the dissipative effects due to the hyperbolic type heat conduction are generally weaker than those induced by Fourier's law.

In the present paper we consider the fully damped case of (13) and prove the exponential stability of the solution without any condition. The importance of our result manifested from the fact that the case of equal speeds is purely mathematical, since it is physically never satisfied [15]. Therefore, the stability result obtained without any restriction on the coefficients is more realistic than that obtained with a stability condition.

Note that the presence of the convolution term in the constitutive equation for q renders the family operators mapping the initial value $(u_0, u_1, \varphi_0, \varphi_1, \theta_0)$ into the solution (u, φ, θ) not match the semigroup properties. This is due to the fact that the solution value of θ at time t depends on the whole function up to time t.

In order to overcome this difficulty we introduce the new variables

$$\theta^t(x,s) = \theta(x, t-s),\ s \geq 0,$$

and

$$\eta(x,s) = \eta^t(x,s) = \int_0^s \theta^t(x,\tau)d\tau,\ s \geq 0,$$

which denote the past history and the summed past history of θ up to t, respectively.

Clearly $\eta^t(x,s)$ satisfies the boundary conditions

$$\eta(0,s) = \eta(\pi,s) = 0.$$

Moreover, we assume that $k(\infty) = 0$ and $\eta(x, 0) = \lim_{s \longrightarrow 0^+} \eta^t(x, s) = 0$, then

$$q = -\int_{-\infty}^{t} k(t - s)\,\theta_x(x, s)\,ds = \int_0^{\infty} k'(s)\,\eta_x^t(x, s)\,ds.$$

Further, we have

$$\eta_t(x, s) = \theta - \eta_s(x, s). \tag{15}$$

Setting $\mu(s) = -k'(s)$, the system (1) and equations (12), (15) become

$$\begin{cases} \rho_1 u_{tt} = \kappa(u_{xx} + \varphi_x) - \beta\theta_x & in\ (0, \pi) \times \mathbb{R}_+, \\ \rho_2 \varphi_{tt} = b\varphi_{xx} - \kappa(u_x + \varphi) + \delta\theta - \tau\varphi_t & in\ (0, \pi) \times \mathbb{R}_+, \\ c\theta_t = \displaystyle\int_0^{\infty} \mu(s)\,\eta_{xx}^t(s)\,ds - \beta u_{xt} - \delta\varphi_t & in\ (0, \pi) \times \mathbb{R}_+, \\ \eta_t^t(s) = \theta - \eta_s^t(s) & in\ (0, \pi) \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{16}$$

The system (16) is complemented with the boundary conditions

$$\begin{array}{c} u(0, t) = u(\pi, t) = \varphi_x(0, t) = \varphi_x(\pi, t) = \theta(0, t) = \theta(\pi, t) = 0, \\ \eta(0, s) = \eta(\pi, s) = 0, \forall t \in \mathbb{R}^+,\ \eta(x, 0) = 0,\ \forall x \in (0, \pi), \end{array} \tag{17}$$

and the initial data

$$\begin{array}{c} u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ \varphi(x, 0) = \varphi_0(x), \\ \varphi_t(x, 0) = \varphi_1(x),\ \theta(x, 0) = \theta_0(x),\ \eta^0(x, s) = \eta_0(x, s). \end{array} \tag{18}$$

Regarding the memory kernel μ, we assume the following set of hypotheses:

(h1) $\mu \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,

(h2) $\mu(s) \geq 0$, $\mu'(s) \leq 0\ \ \forall s \geq 0$,

(h3) $\int_0^{\infty} \mu(s)\,ds = k_0 > 0$,

(h4) there exists $\xi > 0$, such that $\mu'(s) \leq -\xi\mu(s)$, $\forall s \geq 0$.

The rest of the paper is organized as follows: in Section 2, we introduce some functional preliminaries. Section 3 is devoted to the proof of an existence and uniqueness result. In Section 4, we state and prove our stability result.

2. Functional Setting

Let $A = -D^2$ be the operator defined over $L^2(0, \pi)$. It is well known that the operator A with the Dirichlet boundary conditions is a self-adjoint and positive operator with domain $D(A) = H^2 \cap H_0^1$. Thus, it is possible to define the powers A^α of A for $\alpha \in \mathbb{R}$, and the Hilbert space $V_\alpha = D\left(A^{\alpha/2}\right)$ endowed with the inner product

$$\langle u, v\rangle_\alpha = \left\langle A^{\alpha/2}u, A^{\alpha/2}v\right\rangle$$

and the associated norm denoted by $\|u\|_\alpha$. In particular, $V_0 = L^2$, $V_{-1} = H^{-1}$, $V_1 = H_0^1$ and

$$\left\langle A^{1/2}u, A^{1/2}v\right\rangle = \langle Du, Dv\rangle,\ \forall u, v \in H_0^1.$$

For $\alpha_1 > \alpha_2$ the injection $V_{\alpha_1} \hookrightarrow V_{\alpha_2}$ is continuous.

Furthermore, we introduce the weighted Hilbert space

$$\mathcal{M}_1{=}L^2_\mu\left((0,+\infty);H^1_0(0,\pi)\right)$$

with the inner product

$$\langle\eta,\zeta\rangle_{\mathcal{M}_1}=\int_0^\infty \mu(s)\left\langle\eta(s),\zeta(s)\right\rangle_1 ds$$

and the norm

$$\|\eta\|^2_{\mathcal{M}_1}=\int_0^\infty \mu(s)\|D\eta(s)\|^2 ds.$$

We shall also need to define the spaces

$$\mathcal{M}_0{=}L^2_\mu\left((0,+\infty);L^2(0,\pi)\right)$$

and

$$\begin{aligned}\mathcal{K}{=}&H^1_\mu\left((0,+\infty);H^1_0(0,\pi)\right)\\ &=\{\eta/\eta,\eta_s\in\mathcal{M}_1\}.\end{aligned}$$

The following lemma will be useful in the proof of our main result.

Lemma 1. *Let $v\in L^2(0,\pi)$ be given and*

$$\overline{v}=\frac{1}{\pi}\int_0^\pi v(x)\,dx$$

the mean value of v. Then,

$$\|Dv\|_{-1}=\|v-\overline{v}\|. \tag{19}$$

Proof. We have

$$\|Dv\|_{-1}=\sup_{\|D\psi\|=1}|\langle Dv,\psi\rangle|=\sup_{\|D\psi\|=1}|\langle v,D\psi\rangle|\le\|v\|.$$

Let $\psi(x)=\frac{1}{\|v\|}\int_0^x v(y)\,dy$, then $\|D\psi\|=1$ and

$$|\langle Dv,\psi\rangle|=\|v\|\le\|Dv\|_{-1}.$$

Therefore,

$$\|Dv\|_{-1}=\|v\|.$$

Suppose that $\overline{v}=0$, then

$$\|Dv\|_{-1}=\|v-\overline{v}\|.$$

If $\overline{v}\neq 0$, then

$$\|Dv\|_{-1}=\|D(v-\overline{v})\|_{-1}=\|v-\overline{v}\|.$$

□

3. Well Posedness

In this section we prove that the problem determined by (16)-(18) has a unique solution. The main tools of the proof are the Lumer-Phillips and the Lax-Milgram theorems. First we need to rewrite the problem in the semigroups setting.

Let $\mathcal{H}$ be the Hilbert space

$$\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times L^2 \times \mathcal{M}_1$$

endowed with the inner product

$$\begin{aligned} \langle U, U^* \rangle = \quad & \kappa \int_0^\pi (u_x + \varphi)(u_x^* + \varphi^*)\, dx + \rho_1 \int_0^\pi v v^* dx + b \int_0^\pi \varphi_x \varphi_x^* dx \\ & + \rho_2 \int_0^\pi \phi \phi^* dx + c \int_0^\pi \theta \theta^* dx + \int_0^\infty \int_0^\pi \mu(s)\, \eta_x(s)\, \eta_x^*(s)\, dx ds \end{aligned}$$

and the associated norm

$$\|U\|_{\mathcal{H}}^2 = \kappa \|u_x + \varphi\|^2 + \rho_1 \|v\|^2 + b \|\varphi_x\|^2 + \rho_2 \|\phi\|^2 + c \|\theta\|^2 + \|\eta\|_{\mathcal{M}_1}^2 .$$

We note that by virtue of the inequalities

$$\begin{aligned} u_x^2 &\le 2(u_x + \varphi)^2 + 2\varphi^2, \\ (u_x + \varphi)^2 &\le 2u_x^2 + 2\varphi^2, \end{aligned}$$

the above norm in $\mathcal{H}$ is equivalent to the usual norm. Therefore, we use either of the norms indifferently.

To rewrite the system (16) in the semigroup setting we introduce the new variables $v = u_t$ and $\phi = \varphi_t$, then the system (16) becomes

$$\begin{cases} u_t = v \\ v_t = \frac{\kappa}{\rho_1}(u_{xx}(x,t) + \varphi_x(x,t)) - \frac{\beta}{\rho_1}\theta_x(x,t) \\ \varphi_t = \phi \\ \phi_t = \frac{b}{\rho_2}\varphi_{xx}(x,t) - \frac{\kappa}{\rho_2}(u_x(x,t) + \varphi(x,t)) + \frac{\delta}{\rho_2}\theta(x,t) - \frac{\tau}{\rho_2}\phi(x,t) \\ \theta_t(x,t) = \frac{1}{c}\int_0^\infty \mu(s)\, \eta_{xx}^t(x,s)\, ds - \frac{\beta}{c} v_x(x,t) - \frac{\delta}{c}\phi(x,t) \\ \eta_t^t(x,s) = \theta(x,t) - \eta_t^t(x,s) \end{cases}$$

and the problem (16)–(18) rewritten

$$\begin{cases} \dfrac{d}{dt} U = \mathcal{A} U, \ t > 0, \\ \quad U(0) = U_0, \end{cases} \tag{20}$$

where, $\mathcal{A}$ is the operator defined by

$$\mathcal{A}U=\begin{pmatrix} v \\ \dfrac{\kappa}{\rho_1}u_{xx}+\dfrac{\kappa}{\rho_1}\varphi_x-\dfrac{\beta}{\rho_1}\theta_x \\ \phi \\ \dfrac{b}{\rho_2}\varphi_{xx}-\dfrac{\kappa}{\rho_2}u_x-\dfrac{\kappa}{\rho_2}\varphi+\dfrac{\delta}{\rho_2}\theta-\dfrac{\tau}{\rho_2}\phi \\ \dfrac{1}{c}\displaystyle\int_0^{\infty}\mu(s)\,\eta_{xx}(s)\,ds-\dfrac{\beta}{c}v_x-\dfrac{\delta}{c}\phi \\ \theta-\eta_s \end{pmatrix}$$

with domain

$$D(\mathcal{A}):=\left\{\begin{array}{c} U\in\mathcal{H};u,\varphi\in H^2,v,\theta\in H_0^1,\phi\in H_*^1,\eta\in H_\mu^1\left((0,+\infty);H_0^1\right), \\ \displaystyle\int_0^{\infty}\mu(s)\,\eta_{xx}(s)\,ds\in L^2,\eta(0)=0 \end{array}\right\}.$$

Before stating the main result of this section let us recall the following theorems.

Theorem 1. (Lumer-Phillips) *[27,33] Let $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ be a densely defined operator. Then $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ if and only if*

i) $\mathcal{A}$ is dissipative;

ii) there exists a constant $\lambda>0$ such that $\lambda I-\mathcal{A}$ is onto.

Theorem 2. *[33] Let $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ be the infinitesimal generator of a C_0-semigroup $\{S(t);t\geq 0\}$. Then, for each $\xi\in D(\mathcal{A})$ and each $t\geq 0$, we have $S(t)\xi\in D(\mathcal{A})$ and the mapping*

$$\begin{aligned} S:[0,+\infty[&\longrightarrow\mathcal{H}\\ t&\longrightarrow S(t)\xi \end{aligned}$$

is of class C^1 on $[0,+\infty[$ and satisfies

$$\frac{d}{dt}\left(S(t)\,\xi\right)=\mathcal{A}S(t)\,\xi=S(t)\,\mathcal{A}\xi.$$

Our main result reads as follows:

Theorem 3. *Suppose that μ satisfies the hypotheses (h1)-(h4), then for any $U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0,\eta_0)^T\in\mathcal{H}$ the problem (20) has a unique solution $U\in C\left((0,+\infty);\mathcal{H}\right)$. Moreover, if $U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0,\eta_0)^T\in D(\mathcal{A})$ then the solution U satisfies*

$$U\in C\left((0,+\infty);D(\mathcal{A})\right)\cap C^1\left((0,+\infty);\mathcal{H}\right).$$

Proof. First, we prove that $\mathcal{A}$ is dissipative. Indeed, for every $U\in D(\mathcal{A})$ we have

$$\begin{aligned}\langle\mathcal{A}U,U\rangle=\kappa\int_0^{\pi}(v_x+\phi)(u_x+\varphi)\,dx+\int_0^{\pi}(\kappa u_{xx}+\kappa\varphi_x-\beta\theta_x)\,vdx+b\int_0^{\pi}\phi_x\varphi_x dx\\ +\int_0^{\pi}(b\varphi_{xx}-\kappa u_x-\kappa\varphi+\delta\theta-\tau\phi)\,\phi dx\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\pi \left(\int_0^\infty \mu(s)\,\eta_{xx}^t(s)\,ds-\beta v_x-\delta\phi\right)\theta dx\\
&+\int_0^\pi\int_0^\infty \mu(s)\left(\theta_x-\eta_{xs}^t\right)\eta_x(s)\,dsdx,\\
&=-\tau\int_0^\pi \phi^2dx-\frac{1}{2}\int_0^\infty \mu(s)\,\frac{d}{ds}\left\|\eta_x(s)\right\|^2ds.
\end{aligned}
$$

For the second term in the right-hand side, we have

$$\int_0^\infty \mu(s)\frac{d}{ds}\|\eta_x(s)\|^2ds=\mu(s)\|\eta_x(s)\|^2\Big|_0^\infty-\int_0^\infty \mu'(s)\|\eta_x(s)\|^2ds.$$

Since $\mu(s)\left\|\eta_x(s)\right\|^2$ and $\mu(s)\left\|\eta_{xs}(s)\right\|^2$ belong to $L^1(\mathbb{R}^+)$ and $\eta_x(0)=0$, hence

$$
\begin{aligned}
\lim_{s\to 0}\mu(s)\|\eta_x(s)\|^2&=\lim_{s\to 0}\mu(s)\left\|\int_0^s \eta_{xs}(\tau)d\tau\right\|^2,\\
&\le \limsup_{s\to 0}\left(\int_0^s \mu(s)^{1/2}\left\|\eta_{xs}(\tau)\right\|d\tau\right)^2.
\end{aligned}
$$

The Cauchy-Schwarz inequality, leads to

$$\lim_{s\to 0}\mu(s)\|\eta_x(s)\|^2\le \limsup_{s\to 0} s\int_0^s \mu(\tau)\left\|\eta_{xs}(\tau)\right\|^2d\tau=0.$$

Therefore,

$$\int_0^\infty \mu(s)\frac{d}{ds}\|\eta_x(s)\|^2ds=\lim_{s\to\infty}\mu(s)\|\eta_x(s)\|^2-\int_0^\infty \mu'(s)\|\eta_x(s)\|^2ds.$$

The left-hand side of the last equation is bounded, and from (h2) both terms on the right-hand side are positive. Then, the limit in the right hand side exists and is finite, and therefore equals zero. Thus,

$$\langle \mathcal{A}U,U\rangle=-\tau\int_0^\pi \phi^2dx+\frac{1}{2}\int_0^\infty \mu'(s)\|\eta_x(s)\|^2ds\le 0,$$

which proves the dissipativeness of $\mathcal{A}$. Next, we show that $\mathcal{A}$ is maximal. Let $U^*=(u^*,v^*,\varphi^*,\phi^*,\theta^*,\eta^*)^T\in\mathcal{H}$, and find $U=(u,v,\varphi,\phi,\theta,\eta)^T\in D(\mathcal{A})$ such that

$$(I-\mathcal{A})\,U=U^*, \tag{21}$$

which reads in components

$$u-v=u^*, \tag{22}$$

$$\rho v-\kappa u_{xx}-\kappa\varphi_x+\beta\theta_x=\rho_1 v^*, \tag{23}$$

$$\varphi-\phi=\varphi^*, \tag{24}$$

$$(\rho_2+\tau)\,\phi-b\varphi_{xx}+\kappa u_x+\kappa\varphi-\delta\theta=\rho_2\phi^*, \tag{25}$$

$$c\theta-\int_0^\infty \mu(s)\,\eta_{xx}^t(s)\,ds+\beta v_x+\delta\phi=c\theta^*, \tag{26}$$

$$\eta-\theta+\eta_s=\eta^*. \tag{27}$$

Solving equation (27) gives

$$\eta(s) = \left(1 - e^{-s}\right)\theta + \int_0^s e^{y-s}\eta^*(y)\,dy. \tag{28}$$

Substituting (22), (24) and (28) into (23), (25) and (26) we get

$$\begin{cases} \kappa u_{xx} + \kappa\varphi_x - \beta\theta_x - \rho_1 u = -\rho_1(u^* + v^*), \\ b\varphi_{xx} - \kappa u_x - (\kappa + \rho_2 + \tau)\varphi + \delta\theta = -(\rho_2 + \tau)\varphi^* - \rho_2\phi^*, \\ c_\mu\theta_{xx} - c\theta - \beta u_x - \delta\varphi = -(c\theta^* + \beta u^* + \delta\varphi^*) - \displaystyle\int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_{xx}(y)\,dy\right)ds \end{cases} \tag{29}$$

where,

$$c_\mu = \int_0^\infty \mu(s)\left(1 - e^{-s}\right)ds$$

is a positive constant. The last term in the right-hand side of the third equation of (29) belongs to H^{-1}. Indeed, let $\psi \in H_0^1$ such that $\|\psi_x\| \leq 1$, then

$$\begin{aligned} \left|\left\langle \int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_{xx}(y)\,dy\right)ds, \psi\right\rangle\right| &= \left|\left\langle \int_0^\infty \mu(s)\left(\int_0^s e^{y-s}\eta^*_x(y)\,dy\right)ds, \psi_x\right\rangle\right| \\ &\leq \int_0^\infty \mu(s)e^{-s}\left(\int_0^s e^y\|\eta^*_x(y)\|\,dy\right)ds \\ &= \int_0^\infty e^y\|\eta^*_x(y)\|\int_y^\infty \mu(s)e^{-s}dsdy \\ &\leq \int_0^\infty \mu(y)e^y\|\eta^*_x(y)\|\int_y^\infty e^{-s}dsdy \\ &= \int_0^\infty \mu(y)\|\eta^*_x(y)\|\,dy < \infty. \end{aligned}$$

At this point we multiply the equations $(29)_1$,$(29)_2$ and $(29)_3$ by $\widetilde{u}, \widetilde{\varphi}$ and $\widetilde{\theta}$ respectively, integrating over $(0, \pi)$ and summing up, we obtain

$$B\left(U, \widetilde{U}\right) = L\left(\widetilde{U}\right), \tag{30}$$

where

$$\begin{aligned} B\left(U, \widetilde{U}\right) := {} & \kappa\int_0^\pi u_x\widetilde{u}_x dx - \kappa\int_0^\pi \varphi_x\widetilde{u}dx + \beta\int_0^\pi \theta_x\widetilde{u}dx + \rho_1\int_0^\pi u\widetilde{u}dx \\ & + b\int_0^\pi \varphi_x\widetilde{\varphi}_x dx + \kappa\int_0^\pi u_x\widetilde{\varphi}dx + (\kappa + \rho_2 + \tau)\int_0^\pi \varphi\widetilde{\varphi}dx \\ & - \delta\int_0^\pi \theta\widetilde{\varphi}dx + c_\mu\int_0^\pi \theta_x\widetilde{\theta}_x dx + c\int_0^\pi \theta\widetilde{\theta}dx + \beta\int_0^\pi u_x\widetilde{\theta}dx + \delta\int_0^\pi \varphi\widetilde{\theta}dx, \end{aligned}$$

and

$$L\left(\widetilde{U}\right) := \rho_1 \int_0^\pi \left(u^* + v^*\right) \widetilde{u} dx + \left(\rho_2 + \tau\right) \int_0^\pi \varphi^* \widetilde{\varphi} dx - \rho_2 \int_0^\pi \phi^* \widetilde{\varphi} dx$$

$$+ \int_0^\pi \left(c\theta^* + \beta u^* + \delta \varphi^*\right) \widetilde{\theta} dx + \int_0^\pi \widetilde{\theta} \int_0^\infty \mu(s) \left(\int_0^s e^{y-s} \eta^*_{xx}(y)\, dy\right) ds dx.$$

Clearly, $B(\cdot,\cdot)$ is a bounded bilinear form over $\mathcal{W} = H_0^1 \times H_*^1 \times H_0^1$ and L is a bounded linear form. Furthermore, we have

$$\begin{aligned} B(U,U) = \quad & \kappa \int_0^\pi u_x^2 dx - \kappa \int_0^\pi \varphi_x u dx + \beta \int_0^\pi \theta_x u dx + \rho_1 \int_0^\pi u^2 dx + b \int_0^\pi \varphi_x^2 dx \\ & + \kappa \int_0^\pi u_x \varphi dx + (\kappa + \rho_2 + \tau) \int_0^\pi \varphi^2 dx - \delta \int_0^\pi \theta \varphi dx + c_\mu \int_0^\pi \theta_x^2 dx \\ & + c \int_0^\pi \theta^2 dx + \beta \int_0^\pi u_x \theta dx + \delta \int_0^\pi \varphi \theta dx, \end{aligned}$$

$$\begin{aligned} B(U,U) = \kappa \int_0^\pi \left(u_x + \varphi\right)^2 dx + \rho_1 \int_0^\pi u^2 dx + b \int_0^\pi \varphi_x^2 dx \\ + \left(\rho_2 + \tau\right) \int_0^\pi \varphi^2 dx + c_\mu \int_0^\pi \theta_x^2 dx + c \int_0^\pi \theta^2 dx. \end{aligned}$$

Therefore, there exists a positive constant α such that

$$B(U,U) \geq \alpha \left\|U\right\|^2.$$

Thus, $B(\cdot,\cdot)$ is coercive and by means of the Lax-Milgram theorem, the problem (30) has a unique solution

$$(u, \varphi, \theta) \in \mathcal{W}.$$

Moreover, taking $\left(\widetilde{u}, \widetilde{\varphi}, \widetilde{\theta}\right) = (\widetilde{u}, 0, 0)$ in (30) we get

$$\kappa \int_0^\pi u_x \widetilde{u}_x dx = \int_0^\pi \left(\kappa \varphi_x - \beta \theta_x - \rho_1 u + \rho_1 \left(u^* + v^*\right)\right) \widetilde{u} dx, \; \forall \widetilde{u} \in H_0^1. \tag{31}$$

Using standard arguments of elliptic equations we infer that

$$u \in H^2(0,\pi) \cap H_0^1(0,\pi),$$

with

$$\kappa u_{xx} = -\kappa \varphi_x + \beta \theta_x + \rho_1 u - \rho_1 \left(u^* + v^*\right),$$

which solves $(29)_1$. Similarly, by choosing $\left(\widetilde{u}, \widetilde{\varphi}, \widetilde{\theta}\right) = (0, \widetilde{\varphi}, 0)$, we obtain

$$b \int_0^\pi \varphi_x \widetilde{\varphi}_x dx = -\int_0^\pi \left(\kappa \left(u_x + \varphi\right) + \left(\rho_2 + \tau\right)\left(\varphi - \varphi^*\right) - \delta \theta - \rho_2 \phi^*\right) \widetilde{\varphi} dx, \; \forall \widetilde{\varphi} \in H_*^1.$$

Let $\Psi \in H_0^1(0,\pi)$ and set

$$\widetilde{\Psi}(x,t) = \Psi(x,t) - \int_0^\pi \Psi(x,t)\,dx.$$

Clearly $\widetilde{\Psi} \in H_*^1(0,\pi)$. Plugging $\widetilde{\Psi}$ in (31) and recalling that

$$\kappa(u_x + \varphi) + (\rho_2 + \tau)(\varphi - \varphi^*) - \delta\theta - \rho_2\phi^* \in L_*^2(0,\pi),$$

we arrive at

$$b\int_0^\pi \varphi_x \Psi_x dx = \int_0^\pi (\kappa(u_x+\varphi) + (\rho_2+\tau)(\varphi-\varphi^*) - \delta\theta - \rho_2\phi^*)\,\Psi dx,\ \forall \Psi \in H_0^1(0,\pi).$$

Thus, by virtue of the theory of elliptic equations, $\varphi \in H^2(0,\pi) \cap H_*^1(0,\pi)$ with

$$\varphi_{xx} = \frac{-1}{b}(\kappa(u_x+\varphi) + (\rho_2+\tau)(\varphi-\varphi^*) - \delta\theta - \rho_2\phi^*).$$

Then, φ solves $(29)_2$.

Substituting u, φ, θ just obtained in (22), (24) and (28), we infer that

$$v \in H_0^1(0,\pi), \phi \in H_*^1(0,\pi) \quad \text{and} \quad \eta \in H_\mu^1\left((0,+\infty); H_0^1(0,\pi)\right).$$

Moreover, (26) implies that

$$\int_0^\infty \mu(s)\,\eta_{xx}^t(s)\,ds \in L^2(0,\pi).$$

Finally we have

$$\eta_s(s) = e^{-s}\theta + \eta^*(s) - \int_0^s e^{y-s}\eta^*(y)\,dy \in \mathcal{M}_0$$

and $\eta(0) = 0$, which proves that the solution U of (21) belongs to $D(\mathcal{A})$. Hence, Lumer-Phillips theorem ensures that the problem (20) has a unique solution $U(x,t) = e^{\mathcal{A}t}U_0(x)$. This completes the proof of Theorem 3. □

4. Asymptotic Behavior

In this section we establish an exponential rate of decay for the solution of the system (16)-(18). The following Lemma gives a sufficient condition for a C_0−semigroup in order to be exponentially stable.

Lemma 2. *[11] Let $S(t)$ be a contraction semigroup on $\mathcal{H}$, and let $\mathcal{A}$ be its infinitesimal generator. If the operator $i\beta I - \mathcal{A}$ is bounded below as $\beta \in \mathbb{R}$, that is there exists $\lambda > 0$ such that*

$$\inf_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})U\| \geq \lambda\|U\|,\ \forall U \in D(\mathcal{A}),$$

then $S(t)$ is exponentially stable.

The main result of this paper reads as follows:

Theorem 4. *Assume that the memory kernel μ satisfies the hypotheses (h1)–(h5). Then the semigroup $S(t) = e^{\mathcal{A}t}$ associated to the problem (16)-(18) is exponentially stable.*

Proof. The proof will be done by a contradiction argument. Suppose that the assertion is false. Then there exist a sequence $(\lambda_n) \subset \mathbb{R}$ and a sequence $(U_n) \subset D(\mathcal{A})$, of unit norm

$$\kappa \|Du_n + \varphi_n\|^2 + \rho_1 \|v_n\|^2 + b \|D\varphi_n\|^2 + \rho_2 \|\phi_n\|^2 + c \|\theta_n\|^2 + \int_0^\infty \mu(s) \|D\eta_n(s)\|^2 ds = 1,$$

such that

$$\lim_{n \longrightarrow \infty} \|(i\lambda_n I - \mathcal{A}) U_n\| = 0,$$

which reads in components as

$$i\lambda_n u_n - v_n \longrightarrow 0 \ \ in \ \ H_0^1, \tag{32}$$

$$i\rho_1 \lambda_n v_n - \kappa D^2 u_n - \kappa D\varphi_n + \beta D\theta_n \longrightarrow 0 \ \ in \ \ L^2, \tag{33}$$

$$i\lambda_n \varphi_n - \phi_n \longrightarrow 0 \ \ in \ \ H_*^1, \tag{34}$$

$$i\rho_2 \lambda_n \phi_n - bD^2\varphi_n + \kappa Du_n + \kappa\varphi_n + \tau\phi_n - \delta\theta_n \longrightarrow 0 \ \ in \ \ L_*^2, \tag{35}$$

$$ic\lambda_n \theta_n - \int_0^\infty \mu(s) D^2\eta_n(s) ds + \beta Dv_n + \delta\phi_n \longrightarrow 0 \ \ in \ \ L^2, \tag{36}$$

$$i\lambda_n \eta_n - \theta_n + D_s\eta_n \longrightarrow 0 \ \ in \ \ \mathcal{M}_1. \tag{37}$$

Note that since the norm in $\mathcal{H}$ is equivalent to the usual norm, then there exists $\gamma > 0$ such that for any $U \in D(\mathcal{A})$ of unit norm, we have

$$\|Du_n\|^2 + \|v_n\|^2 + \|\varphi_n\|^2 + \|D\varphi_n\|^2 + \|\phi_n\|^2 + \|\theta_n\|^2 + \int_0^\infty \mu(s) \|D\eta_n(s)\|^2 ds = \gamma. \tag{38}$$

First we have

$$\mathrm{Re} \langle (i\lambda_n I - \mathcal{A}) U_n, U_n \rangle = \tau \int_0^\pi \phi_n^2 dx - \frac{1}{2} \int_0^\infty \mu'(s) \|D\eta_n(s)\|^2 ds \longrightarrow 0.$$

Thus,

$$\|\phi_n\| \longrightarrow 0 \tag{39}$$

and

$$\|\eta_n\|_{\mathcal{M}_1}^2 \leq -\frac{1}{\xi} \int_0^\infty \mu'(s) \|D\eta_n(s)\|^2 ds \longrightarrow 0. \tag{40}$$

Moreover, from (34) we have

$$\varphi_n \sim \frac{1}{\lambda_n} \phi_n \longrightarrow 0 \ in \ L^2. \tag{41}$$

The injection $L^2 \hookrightarrow H^{-1}$ is continuous, hence (33) holds in H^{-1} instead of L^2 and

$$i\rho_1 \lambda_n v_n \sim \kappa D^2 u_n + \kappa D\varphi_n - \beta D\theta_n \ \ in \ \ H^{-1}.$$

On the other hand we have

$$\left\|\kappa\left(D^2u_n+D\varphi_n\right)-\beta D\theta_n\right\|_{-1}=\sup_{\|D\psi\|\le 1}\left|\left\langle\kappa\left(D^2u_n+D\varphi_n\right)-\beta D\theta_n,\psi\right\rangle\right|,$$
$$\le\|\kappa\left(Du_n+\varphi_n\right)-\beta\theta_n\|\,.\sup_{\|D\psi\|\le 1}\|D\psi\|,$$
$$\le\kappa\left\|Du_n+\varphi_n\right\|+|\beta|\left\|\theta_n\right\|\le\sqrt{2}.$$

Therefore,

$$|\lambda_n|\,\|v_n\|_{-1}\le C_1, \tag{42}$$

for a positive constant C_1 independent of $n\in\mathbb{N}$.
Similarly, we get

$$\left\|\int_0^\infty\mu\left(s\right)D^2\eta_n\left(s\right)ds\right\|_{-1}\le\int_0^\infty\mu\left(s\right)\left\|D\eta_n\left(s\right)\right\|ds,$$
$$\le\sqrt{\int_0^\infty\mu\left(s\right)ds}\left(\int_0^\infty\mu\left(s\right)\int_0^\pi\left|D\eta_n\right|^2\left(s\right)dxds\right)^{1/2},$$

then

$$\left\|\int_0^\infty\mu\left(s\right)D^2\eta_n\left(s\right)ds\right\|_{-1}\le\sqrt{\int_0^\infty\mu\left(s\right)ds}\,\|\eta_n\|_{\mathcal{M}_1}\longrightarrow 0.$$

Note that (36) holds with H^{-1} instead of L^2, hence

$$\|ic\lambda_n\theta_n+\beta Dv_n\|_{-1}\longrightarrow 0\,. \tag{43}$$

Since

$$\|Dv_n\|_{-1}=\sup_{\|D\psi\|\le 1}|\langle Dv_n,\psi\rangle|\le\|v_n\|<\infty,$$

Dv_n is bounded in H^{-1}, then

$$\|c\lambda_n\theta_n\|_{-1}\le C_2,$$

for a positive constant C_2 independent of $n\in\mathbb{N}$.
Next, we need to show that $\|\theta_n\|\longrightarrow 0$. Exploiting the continuous embedding of $\mathcal{M}_1$ into $\mathcal{M}_0$, (37) holds in $\mathcal{M}_0$ instead of $\mathcal{M}_1$. Let (ξ_n) be the sequence $\xi_n=s\theta_n$. Clearly $\xi_n\in\mathcal{M}_0$. Indeed, from (h2), $\mu(s)$ goes to zero exponentially fast, then

$$\int_0^\infty s^2\mu\left(s\right)\int_0^\pi\left|\theta_n\right|^2dxds=\|\theta_n\|^2\int_0^\infty s^2\mu\left(s\right)ds=C_3<\infty.$$

Multiplying (37) by ξ_n in $\mathcal{M}_0$ we get

$$\langle i\lambda_n\eta_n,\xi_n\rangle_0-\langle\theta_n,\xi_n\rangle_0+\langle D_s\eta_n,\xi_n\rangle_0\longrightarrow 0. \tag{44}$$

For the first term we have

$$|\langle i\lambda_n\eta_n,\xi_n\rangle_0|=|\lambda_n|\int_0^\infty s\mu\left(s\right)\int_0^\pi\eta_n\theta_n dxds.$$

Then, using Hölder inequality we get

$$
\begin{aligned}
|\langle i\lambda_n\eta_n,\xi_n\rangle_0| &\le |\lambda_n|\,\|\theta_n\|_{-1}\int_0^\infty s\mu(s)\,\|D\eta_n(s)\|\,ds,\\
&\le |\lambda_n|\,\|\theta_n\|_{-1}\sqrt{\int_0^\infty s^2\mu(s)\,ds\int_0^\infty \mu(s)\,\|D\eta_n(s)\|^2\,ds},\\
&\le C_2\sqrt{C_3}\,\|\eta_n\|_1\longrightarrow 0.
\end{aligned}
$$

From (h4) we infer that $\lim\limits_{s\to+\infty}s^2\mu(s)=0$, then, again (h4) and integration by parts yield

$$
-\int_0^\infty s^2\mu'(s)\,ds = 2\int_0^\infty s\mu(s)\,ds = C_4<\infty.
$$

For the third term of (44) we have,

$$
\begin{aligned}
|\langle D_s\eta_n,\xi_n\rangle_0| &= \left|\int_0^\infty s\mu(s)\frac{d}{ds}\int_0^\pi \eta_n\theta_n dxds\right|,\\
&= \left|\int_0^\infty \mu(s)\int_0^\pi \eta_n\theta_n dxds+\int_0^\infty s\mu'(s)\int_0^\pi \eta_n\theta_n dxds\right|,
\end{aligned}
$$

then,

$$
\begin{aligned}
|\langle D_s\eta_n,\xi_n\rangle_0| &\le \|\theta_n\|\left[\int_0^\infty \mu(s)\,\|\eta_n\|\,ds-\int_0^\infty s\mu'(s)\,\|\eta_n\|\,ds\right],\\
&\le \int_0^\infty \mu(s)\,\|\eta_n\|\,ds-\int_0^\infty s\mu'(s)\,\|\eta_n\|\,ds.
\end{aligned}
$$

Using the Cauchy-Schwarz and Poincaré's inequalities we conclude that

$$
\begin{aligned}
\int_0^\infty \mu(s)\,\|\eta_n\|\,ds &\le \sqrt{\int_0^\infty \mu(s)\,ds}\sqrt{\int_0^\infty \mu(s)\,\|\eta_n\|\,ds},\\
&\le \sqrt{\int_0^\infty \mu(s)\,ds}\,\|\eta_n\|_0,\\
&\le C_P\sqrt{\int_0^\infty \mu(s)\,ds}\,\|\eta_n\|_1\longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_0^\infty s\mu'(s)\,\|\eta_n\|\,ds &= \int_0^\infty s\sqrt{-\mu'(s)}\sqrt{-\mu'(s)}\,\|\eta_n\|\,ds\\
&\le \left(-\int_0^\infty s^2\mu'(s)\,ds\right)^{1/2}\left(-\int_0^\infty \mu'(s)\,\|\eta_n\|^2\,ds\right)^{1/2}\\
&\le \left(-C_4C_P\int_0^\infty \mu'(s)\,\|D\eta_n\|^2\,ds\right)^{1/2}\longrightarrow 0.
\end{aligned}
$$

Thus, (44) is reduced to

$$\|\theta_n\|^2 \int_0^\infty s\mu(s)\,ds = \langle \theta_n, \xi_n \rangle_0 \longrightarrow 0,$$

that is,

$$\|\theta_n\|^2 = \frac{2\langle \theta_n, \xi_n \rangle_0}{C_4} \longrightarrow 0. \tag{45}$$

Removing the terms that tend to 0 from (35), then multiplying by φ_n we obtain

$$i\rho_2\lambda_n \langle \phi_n, \varphi_n \rangle + b\,\|D\varphi_n\|^2 + \kappa \langle Du_n, \varphi_n \rangle \longrightarrow 0\,. \tag{46}$$

We point out that

$$\langle Du_n, \varphi_n \rangle \leq \|Du_n\|\,\|\varphi_n\| \longrightarrow 0$$

and

$$i\lambda_n \langle \phi_n, \varphi_n \rangle \sim \|\varphi_n\|^2 \longrightarrow 0.$$

Therefore,

$$\|D\varphi_n\| \longrightarrow 0. \tag{47}$$

Multiplying (32) by $\rho_1 v_n$ and (33) by u_n we get

$$i\rho_1\lambda_n \langle u_n, v_n \rangle - \rho_1 \|v_n\|^2 \longrightarrow 0, \tag{48}$$

and

$$i\rho_1\lambda_n \langle v_n, u_n \rangle + \kappa \|Du_n\|^2 \longrightarrow 0. \tag{49}$$

Adding (48) to the complex conjugate of (49), we get

$$\kappa \|Du_n\|^2 - \rho_1 \|v_n\|^2 \longrightarrow 0. \tag{50}$$

Combining (38), (39), (40), (41), (45),(47), and (50) we obtain

$$\left(1 + \frac{\rho_1}{\kappa}\right) \|v_n\|^2 \to \gamma. \tag{51}$$

We complete the proof by showing that (51) leads to a contradiction.
Since $A^{-1}Dv_n$ is bounded in H_0^1 (recall that $A = -D^2$), from (43) we have

$$\left\langle ic\lambda_n\theta_n + \beta Dv_n, A^{-1}Dv_n \right\rangle = \left\langle ic\lambda_n\theta_n, A^{-1}Dv_n \right\rangle + \beta \|Dv_n\|_{-1}^2 \longrightarrow 0. \tag{52}$$

On the other hand, from (45) we have

$$\begin{aligned} \left|\left\langle ic\lambda_n\theta_n, A^{-1}Dv_n \right\rangle\right| &= \left|\left\langle ic\lambda_n\theta_n, A^{-1/2}v_n \right\rangle\right| \\ &\leq c\,|\lambda_n| \left\|A^{-1/2}v_n\right\| \|\theta_n\| = c\,|\lambda_n|\,\|v_n\|_{-1}\,\|\theta_n\| \\ &\leq cC_1\,\|\theta_n\| \longrightarrow 0. \end{aligned}$$

Thus, (52) leads to

$$\|Dv_n\|_{-1} \longrightarrow 0.$$

From (19) we infer that

$$\|v_n - \overline{v}_n\| = \|Dv_n\|_{-1} \longrightarrow 0.$$

Therefore,

$$\|v_n - \overline{v}_n\|^2 = \|v_n\|^2 - \pi\,|\overline{v}_n|^2 \longrightarrow 0. \tag{53}$$

The comparison of (51) and (53) leads to

$$|\overline{v}_n| \longrightarrow \sqrt{\frac{\kappa\gamma}{\pi\,(\kappa+\rho_1)}}.$$

Thus, there exists a subsequence $(\overline{v}_n)$ that converges to $\overline{v}$, such that

$$|\overline{v}| = \sqrt{\frac{\kappa\gamma}{\pi\,(\kappa+\rho_1)}}. \tag{54}$$

Using (53) again we conclude that there exists a subsequence of (v_n) which converges to $\overline{v}$ in $L^2(0,\pi)$. Exploiting the continuous embedding of $L^2(0,\pi)$ into $H^{-1}(0,\pi)$, one can deduce that

$$v_n \longrightarrow \overline{v}, \qquad \text{in } H^{-1}(0,\pi). \tag{55}$$

At this point we distinguish two cases. Suppose that (λ_n) is unbounded, then we can choose a subsequence (λ_n) such that $|\lambda_n| \longrightarrow \infty$ and from (42) we have

$$v_n \longrightarrow 0 \; in \; H^{-1}(0,\pi).$$

From the uniqueness of the limit we conclude that $\overline{v} = 0$, which is incompatible with (54).

Conversely, assume that (λ_n) is bounded, again, there exists a subsequence (λ_n) that converges to some $\lambda \in \mathbb{R}$. In this case we have

$$\lim_{n\longrightarrow\infty} \|(i\lambda I - \mathcal{A})\,U_n\| = 0,$$

and (32)-(37) hold with λ instead of λ_n. In particular

$$i\lambda u_n - v_n \longrightarrow 0 \; in \; H_0^1(0,\pi).$$

Since (u_n) is bounded in $H_0^1(0,\pi)$, we conclude that there exists $v^* \in H_0^1(0,\pi)$ and a subsequence (v_n) that converges weakly to v^* in $H_0^1(0,\pi)$. From the uniqueness of the limit we infer that $v^* = \overline{v}$, which is in contradiction with $v^* \in H_0^1(0,\pi)$, since $\overline{v}$ is a non-zero constant function, and therefore cannot be in $H_0^1(0,\pi)$. This completes the proof of Theorem 4. □

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