İSTATİSTİK: JOURNAL OF THE TURKISH STATISTICAL ASSOCIATION

İSTATİSTİK

Vol. 13, No. 3, December 2021, pp. 120–141

ISSN 1300-4077 | 21 | 3 | 120 | 141

ROBUST OUTER PRODUCT OF GRADIENTS TESTS FOR TESTING SPATIAL DEPENDENCE

Osman Doğan*

Department of Economics,
University of Illinois,
Urbana, U.S.A

Süleyman Taşpınar

Department of Economics,
Queens College, CUNY,
New York, U.S.A.

Bülent Güloğlu

Department of Economics,
Istanbul Technical University,
Istanbul, Turkey

Abstract: In this paper, we suggest outer-product-of-gradient (OPG) variants of the Lagrange multiplier (LM) test statistic for testing spatial dependence under the local parametric misspecification in spatial models. Our OPG statistic for testing the presence of a spatial lag in the disturbance term remains valid irrespective of whether or not there is spatial dependence in the dependent variable. Similarly, our suggested OPG statistic for testing the presence of a spatial lag in the dependent variable is robust to the presence of spatial dependence in the disturbance term. We also suggest the OPG variants that are robust to the presence of an unknown form of heteroskedasticity in the disturbance terms. The computations of all suggested tests only require the least squares estimates from a linear regression model. In a Monte Carlo simulation, we investigate the finite sample properties of our tests and some alternative tests suggested in the literature. The simulation results show that our tests work well in finite samples.

Key words: Spatial autoregressive models, SARAR, Heteroskedasticity, Testing, Robust tests, LM tests, OPG tests, Inference.

1. Introduction

In this paper, we propose outer-product-of-gradient (OPG) variants of Lagrange multiplier (LM) test statistic for testing spatial dependence in a spatial model that has spatial dependence in both the dependent variable and the disturbance term. Our OPG test for testing one type of spatial dependence (the spatial lag in the dependent variable or the spatial lag in the disturbance term) is valid whether or not the other type of spatial dependence is present. We show how such robust OPG tests can be systematically constructed in the quasi maximum likelihood (QML) framework for spatial models that have homoskedastic or heteroskedastic disturbances. Importantly, the computation of suggested tests only requires the ordinary least squares (OLS) estimates from a linear regression model. Thus, our approach provides robust alternative tests that can be easily adopted by applied researchers.

The OPG variants of LM tests for testing spatial dependence are based on the fact that the score type-functions of a spatial model, which can be written in terms of linear and quadratic forms of disturbance terms, form a martingale difference array. This inherent martingale structure is explored in Kelejian and Prucha [13, 14] to develop a central limit theorem (CLT) for spatial processes. Born and Breitung [7] show how the variance of linear and quadratic forms can be

* Corresponding author. E-mail: odogan@illinois.edu

estimated by the OPG method, and suggest simple test statistics for testing the presence of a spatial lag in the spatial auto-regressive (SAR) and spatial error (SE) models.[1] Their test statistics are robust to heteroskedasticity, and are equivalent to the one-directional LM tests derived in [8] and [1] under the homoskedastic case. Following Born and Breitung [7], Baltagi and Yang [4, 5] suggest the standardized OPG variants of one-directional LM tests that correct for both the mean and variance of the standard LM test statistics for improving the finite sample properties of tests. In the context of standard panel data models, there is evidence that standardizing an LM test improves its performance, especially when the asymptotic critical values are used to implement the test [3, 15, 20]. The simulation results in Baltagi and Yang [4, 5] show that the standardized OPG variants for testing spatial dependence can also perform relatively better in finite sample.

The OPG variants suggested in [7, 4, 5] are one directional tests in the sense that they are designed for testing one type of spatial dependence (spatial lag or spatial error dependence) in the absence of the other type of spatial dependence. However, it is well known that the one directional LM test for one type of spatial dependence will be invalid in the presence of other type of spatial dependence, i.e., under the local parametric misspecification [23, 9, 6, 2]. Anselin et al. [2] study systematically the consequences of testing one type of spatial dependence at a time, and use the approach suggested in [6] to develop adjusted one-directional LM test for testing the presence of spatial dependence in the dependent variable (spatial dependence in the disturbance term) in the possible presence of spatial dependence in the disturbance term (spatial dependence in the dependent variable). Though the test suggested in [2] are valid under the local parametric misspecification, they may not be valid under the heteroskedastic case, since the (quasi) likelihood function is misspecified when the disturbance terms of the model are heteroskedastic. Recently, following Born and Breitung [7], Jin and Lee [12] suggest the OPG variants of C(α)-type tests in the ML and generalized method of moments (GMM) settings under both hemoskedastic and heteroskedastic cases. In comparison with our suggested tests, the tests suggested in [12] are computationally intensive as they require estimation of the respective null models by a $\sqrt{n}$-consistent constrained estimator, and therefore they do not share the simplicity of tests based on the OLS estimator.

The rest of this study is organized as follows. In Section 2, we state the spatial model and derive its quasi-likelihood function. In Section 3, we show how the OPG variants of LM test can be systematically derived under both homoskedastic and heteroskedastic cases. We provide test statistics for detecting spatial error and spatial lag dependence in a SARAR (1,1) model. In Section 4, we describe our Monte Carlo design and report the simulation results. In Section 5, we provide an empirical illustration. In Section 6, we conclude. Some technical details and simulation results are collected in an appendix.

2. Model Specification and ML Estimation Approach

We consider the following cross-sectional SARAR$(1,1)$ specification

$$Y = \lambda_0 WY + X\beta_0 + U, \quad U = \rho_0 MU + V, \tag{2.1}$$

where $Y = (y_1, \dots, y_n)^{'}$ is the $n \times 1$ vector of a dependent variable, X_n is the $n \times k_x$ matrix of non-stochastic exogenous variables with a matching parameter vector β_0. W and M are the $n \times n$ spatial weights matrices of known constants with zero diagonal elements. The scalar parameters λ_0 and ρ_0 are called the spatial autoregressive parameters. In (2.1), $U = (u_1, \dots, u_n)^{'}$ is the $n \times 1$ vector of regression disturbance terms and $V = (v_1, \dots, v_n)^{'}$ is the $n \times 1$ vector of disturbance terms which are independent and identically distributed (iid) with mean zero and variance σ_0^2.

[1] On the taxonomy and estimation of spatial models, see [1, 17, 11].

The model is stated with the true parameter vector $\theta_0 = (\beta_0^{'}, \sigma_0^2, \lambda_0, \rho_0)^{'}$ and we use $\theta = (\beta^{'}, \sigma^2, \lambda, \rho,)^{'}$ to denote any other arbitrary value in the parameter space. For notational simplicity, let $S(\lambda) = (I_n - \lambda W)$, $R(\rho) = (I_n - \rho M)$, $S = S(\lambda_0)$ and $R = R(\rho_0)$, where I_n denotes the $n \times n$ identity matrix. Under the assumption that the disturbance terms are iid with mean zero and variance σ_0^2, the quasi log-likelihood function of the model can be written as

$$\begin{aligned}\ln L(\theta) = &-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln|S(\lambda)| + \ln|R(\rho)| \\ &- \frac{1}{2\sigma^2}\left(S(\lambda)Y - X\beta\right)^{'} R^{'}(\rho)R(\rho)\left(S(\lambda)Y - X\beta\right), \end{aligned} \tag{2.2}$$

where $|\cdot|$ denotes the determinant operator. The QMLE is the extremum estimator defined by $\hat{\theta} = \arg\max_{\theta\in\Theta} \ln L(\theta)$. Under some regularity conditions, it can be shown that the QMLE is consistent and has asymptotic normal distribution [16].

3. Robust OPG Tests

In this section, we derive robust OPG tests for testing the null hypotheses $H_0^{\rho} : \rho_0 = 0$ and $H_0^{\rho} : \lambda_0 = 0$ under both homekadastic and heteroskedastic cases. We also determine the asymptotic distribution of tests under the local alternative hypotheses defined as $H_a^{\rho} : \rho_0 = \delta_{\rho}/\sqrt{n}$ and $H_a^{\lambda} : \lambda_0 = \delta_{\lambda}/\sqrt{n}$, where δ_{ρ} and δ_{λ} are non-stochastic bounded constants. In testing $H_0^{\rho} : \rho_0 = 0$, the local alternative hypothesis $H_a^{\lambda} : \lambda_0 = \delta_{\lambda}/\sqrt{n}$ also serves as the local parametric misspecification. Similarly, in testing $H_0^{\lambda} : \lambda_0 = 0$, the local alternative hypothesis $H_a^{\rho} : \rho_0 = \delta_{\rho}/\sqrt{n}$ represents the local parametric misspecification in our setting.

3.1. Tests Under Homoskedasticity

Let $S_a(\theta) = \frac{1}{n}\frac{\partial \ln L(\theta_0)}{\partial a}$, where $a \in \{\beta, \sigma^2, \lambda, \rho\}$. Our robust OPG tests are based on the score functions of $\ln L(\theta)/n$ derived as

$$\begin{aligned} &S_{\beta}(\theta_0) = \frac{1}{n\sigma_0^2}X^{'}R^{'}V, \quad S_{\sigma^2}(\theta_0) = \frac{1}{2n\sigma_0^4}V^{'}V - \frac{1}{2\sigma_0^2}, \\ &S_{\lambda}(\theta_0) = \frac{1}{n\sigma_0^2}V^{'}\bar{G}V - \frac{1}{n}\mathrm{tr}(G) + \frac{1}{n\sigma_0^2}(RGX\beta_0)^{'}V, \quad S_{\rho}(\theta_0) = \frac{1}{n\sigma_0^2}V^{'}HV - \frac{1}{n}\mathrm{tr}(H), \end{aligned} \tag{3.1}$$

where $G = WS^{-1}$, $\bar{G} = RGR^{-1}$ and $H = MR^{-1}$. Let $\gamma = (\beta^{'}, \sigma^2)^{'}$. Then, the score functions can be expressed as a vector of linear quadratic forms in the following way

$$S(\theta_0) = (S_{\gamma}^{'}(\theta_0), S_{\lambda}(\theta_0), S_{\rho}(\theta_0))^{'} = \frac{1}{n}\begin{pmatrix} V^{'}A_1V - \sigma_0^2\mathrm{tr}(A_1) + b_1^{'}V \\ \vdots \\ V^{'}A_{k_x}V - \sigma_0^2\mathrm{tr}(A_{k_x}) + b_{k_x}^{'}V \\ V^{'}A_{k_x+1}V - \sigma_0^2\mathrm{tr}(A_{k_x+1}) + b_{k_x+1}^{'}V \\ V^{'}A_{k_x+2}V - \sigma_0^2\mathrm{tr}(A_{k_x+2}) + b_{k_x+2}^{'}V \\ V^{'}A_{k_x+3}V - \sigma_0^2\mathrm{tr}(A_{k_x+3}) + b_{k_x+3}^{'}V \end{pmatrix}, \tag{3.2}$$

where $\mathrm{tr}(\cdot)$ is the trace operator, $A_1 = \cdots = A_{k_x} = 0$, $b_j = RX_j/\sigma_0^2$ for $j = 1, 2, \ldots, k_x$, X_j is the jth column of X, $A_{k_x+1} = I_n/2\sigma_0^4$, $b_{k_x+1} = 0$, $A_{k_x+2} = \bar{G}/\sigma_0^2$, $b_{k_x+2} = (RGX\beta)/\sigma_0^2$, $A_{k_x+3} = H/\sigma_0^2$ and $b_{k_x+3} = 0$. We denote the (i,j)th element of A_k by $a_{k,ij}$, and the ith element of b_k by b_{ki}, where $k \in \{1, 2, \ldots, k_x + 3\}$. Then, we can express $S(\theta_0)$ as a sum of martingale differences in the following way

$$S(\theta_0) = \sum_{i=1}^{n} S_i(\theta_0), \tag{3.3}$$

where $S_i(\theta_0) = \left(S_{i\gamma}^{'}(\theta_0), S_{i\lambda}(\theta_0), S_{i\rho}(\theta_0)\right)^{'}$ with the kth element $(k \in \{1, 2, \dots, k_x + 3\})$

$$S_{ik}(\theta_0) = \frac{1}{n}\left(a_{k,ii}(v_i^2 - \sigma_0^2) + v_i \sum_{j=1}^{i-1}(a_{k,ij} + a_{k,ji})v_j + b_{ki}v_i\right).$$

To show that $S_i(\theta_0)$ is martingale difference sequence, consider the following $\sigma-$fields $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\mathfrak{F}_i = \sigma(v_1, v_2, \dots, v_i)$ for $1 \le i \le n$. Then, it follows that $\mathfrak{F}_{i-1} \subset \mathfrak{F}_i$ and $\mathbb{E}(S_i(\theta_0)|\mathfrak{F}_{i-1}) = 0$. Thus $\{S_i(\theta_0), \mathfrak{F}_i, 1 \le i \le n\}$ forms a martingale difference array. We can thus express the variance of $\sqrt{n}S(\theta_0)$ in the following way

$$\operatorname{Var}\left(\sqrt{n}S(\theta_0)\right) = K(\theta_0) = \sum_{i=1}^{n} \mathbb{E}\left(nS_i(\theta_0)S_i^{'}(\theta_0)\right) \tag{3.4}$$

Under some regularity conditions, $K(\theta_0)$ can be estimated by $K(\hat{\theta}) = n\sum_{i=1}^{n} S_i(\hat{\theta})S_i^{'}(\hat{\theta})$, where $\hat{\theta}$ is a consistent estimator of θ_0.

To define our suggested robust OPG test, we assume the following assumptions.

Assumption 1. *The innovation terms v_is are iid with mean zero, variance σ_0^2, and $\mathbb{E}\,|v_i|^{4+\eta} < \infty$ for some $\eta > 0$ for all i and n.*

Assumption 2. *(i) Let $\tilde{\theta}$ be a constrained estimator under under the joint null hypothesis $H_0 : \lambda_0 = \rho_0 = 0$. Then, $\sqrt{n}(\tilde{\theta} - \theta_0) = O_p(1)$ holds. (ii) $\sqrt{n}S(\theta_0) \xrightarrow{d} N(0, K(\theta_0))$, where $K(\theta_0)$ is a non-singular matrix. (iii) $-\frac{\partial S(\bar{\theta})}{\partial \theta'} = J(\theta_0) + o_p(1)$, where $\bar{\theta} = \theta_0 + o_p(1)$ and $J(\theta_0) = \mathbb{E}\left(-\frac{\partial S(\theta_0)}{\partial \theta'}\right)$ is a non-singular matrix.*

Assumption 1 requires that the disturbance terms are homoskedastic, but allows for a non-normal distribution. Assumption 2 is a high level assumption as our focus is on testing problem.[2] The first part of the assumption requires that the constrained estimator under the joint null hypothesis $H_0 : \lambda_0 = \rho_0 = 0$ is a $\sqrt{n}$-consistent estimator of θ_0. Note that under $H_0 : \lambda_0 = \rho_0 = 0$, our model reduces to the linear regression model, which can be consistently estimated by the OLS estimator. Assumption 2 (ii) is the CLT condition for the score functions, which can be ensured by the CLT for linear-quadratic form in [13, 14]. Finally, Assumption 2 (iii) shows that the negative hessian evaluated at a consistent estimator converges to the information matrix.

We assume that $K(\theta_0)$ and $J(\theta_0)$ are partitioned into sub-matrices $K_{ab}(\theta_0)$ and $J_{ab}(\theta_0)$ according to the dimensions of a and b, where $a, b \in \{\lambda, \rho, \gamma\}$. Then, it can be shown that

$$\begin{aligned}
&J_{\lambda\lambda}(\theta_0) = \frac{1}{n\sigma_0^2}(RGX\beta_0)^{'}RGX\beta_0 + \frac{1}{n}\operatorname{tr}(\bar{G}^{(s)}\bar{G}), \quad J_{\lambda\rho}(\theta_0) = \frac{1}{n}\operatorname{tr}\left(H^{(s)}\bar{G}\right),\\
&J_{\lambda\gamma}(\theta_0) = \left(\frac{1}{n\sigma_0^2}X^{'}R^{'}RGX\beta_0, \frac{1}{n\sigma_0^2}\operatorname{tr}(G)\right), \quad J_{\rho\rho}(\theta_0) = \frac{1}{n}\operatorname{tr}\left(H^{(s)}H\right),\\
&J_{\rho\gamma}(\theta_0) = \left(0, \frac{1}{n\sigma_0^2}\operatorname{tr}(H)\right), \quad J_{\gamma\gamma}(\theta_0) = \operatorname{Diag}\left(\frac{1}{n\sigma^2}X^{'}R^{'}RX, \frac{1}{2\sigma_0^4}\right),
\end{aligned} \tag{3.5}$$

where $A^{(s)} = A + A^{'}$ for any $n \times n$ matrix A.

Before we introduce our suggested tests, we introduce the following notations. We define $J_{\lambda\cdot\gamma}(\theta) = J_{\lambda\lambda}(\theta) - J_{\lambda\gamma}(\theta)J_{\gamma\gamma}^{-1}(\theta)J_{\gamma\lambda}(\theta)$ and $J_{\lambda\rho\cdot\gamma}(\theta) = J_{\lambda\rho}(\theta) - J_{\lambda\gamma}(\theta)J_{\gamma\gamma}^{-1}(\theta)J_{\gamma\rho}(\theta)$. Similarly, $J_{\rho\cdot\gamma}(\theta) = J_{\rho\rho}(\theta) - J_{\rho\gamma}(\theta)J_{\gamma\gamma}^{-1}(\theta)J_{\gamma\rho}(\theta)$ and $J_{\rho\lambda\cdot\gamma}(\theta) = J_{\rho\lambda}(\theta) - J_{\rho\gamma}(\theta)J_{\gamma\gamma}^{-1}(\theta)J_{\gamma\lambda}(\theta)$. Under Assumption 2, it

[2] The primitive conditions ensuring this assumption are provided in [16]. For the sake of brevity, we do not provide these primitive conditions.

can be shown that the mean value expansions of $\sqrt{n}S_\lambda(\tilde{\theta})$ and $\sqrt{n}S_\gamma(\tilde{\theta})$ around θ_0, when both H_a^λ and H_a^ρ hold, yield the following result (see proof of Theorem 1)

$$\sqrt{n}S_\lambda(\tilde{\theta}) \xrightarrow{d} N\left[J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho,\, B_{\lambda\cdot\gamma}(\theta_0)\right], \tag{3.6}$$

where

$$\begin{aligned} B_{\lambda\cdot\gamma}(\theta_0) = {} & K_{\lambda\lambda}(\theta_0) + J_{\lambda\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)K_{\gamma\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)J_{\lambda\gamma}^{'}(\theta_0) \\ & - K_{\lambda\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)J_{\lambda\gamma}^{'}(\theta_0) - J_{\lambda\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)K_{\gamma\lambda}(\theta_0). \end{aligned} \tag{3.7}$$

Hence, under H_0^λ and H_a^ρ, it follows that $\sqrt{n}S_\lambda(\tilde{\theta}) - J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho \xrightarrow{d} N\left(0,\, B_{\lambda\cdot\gamma}(\theta_0)\right)$. The non-zero asymptotic mean term $J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho$ indicates that the null asymptotic distribution of the OPG test based on the $\sqrt{n}S_\lambda(\tilde{\theta})$ will has a non-central chi-squared distribution in the local presence of ρ_0. We show that δ_ρ is the asymptotic mean of $J_{\rho\cdot\gamma}^{-1}\sqrt{n}S_\rho(\tilde{\theta})$ (see the proof of Theorem 1). This result suggests that we can formulate a OPG test that is valid in the local presence of ρ_0 based on the following adjusted score function.

$$\sqrt{n}S_\lambda^*(\tilde{\theta}) = \sqrt{n}\left(S_\lambda(\tilde{\theta}) - J_{\lambda\rho\cdot\gamma}(\tilde{\theta})J_{\rho\cdot\gamma}^{-1}(\tilde{\theta})S_\rho(\tilde{\theta})\right). \tag{3.8}$$

Then, our suggested robust OPG test is given by

$$LM_\lambda = n\, S_\lambda^{*2}(\tilde{\theta})/D_{\lambda\cdot\gamma}(\tilde{\theta}), \tag{3.9}$$

where

$$\begin{aligned} D_{\lambda\cdot\gamma}(\tilde{\theta}) = {} & B_{\lambda\cdot\gamma}(\tilde{\theta}) + J_{\lambda\rho\cdot\gamma}(\tilde{\theta})J_{\rho\cdot\gamma}^{-1}(\tilde{\theta})B_{\rho\cdot\gamma}(\tilde{\theta})J_{\rho\cdot\gamma}^{-1}(\tilde{\theta})J_{\rho\lambda\cdot\gamma}(\tilde{\theta}) \\ & - J_{\lambda\rho\cdot\gamma}(\tilde{\theta})J_{\rho\cdot\gamma}^{-1}(\tilde{\theta})B_{\rho\lambda\cdot\gamma}(\tilde{\theta}) - B_{\lambda\rho\cdot\gamma}(\tilde{\theta})J_{\rho\cdot\gamma}^{-1}(\tilde{\theta})J_{\rho\lambda\cdot\gamma}(\tilde{\theta}), \end{aligned} \tag{3.10}$$

with

$$\begin{aligned} B_{\lambda\cdot\gamma}(\tilde{\theta}) = {} & K_{\lambda\lambda}(\tilde{\theta}) + J_{\lambda\gamma}(\tilde{\theta})J_{\gamma}^{-1}(\tilde{\theta})K_{\gamma\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\lambda}(\tilde{\theta}) \\ & - K_{\lambda\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\lambda}(\tilde{\theta}) - J_{\lambda\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\lambda}(\tilde{\theta}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} B_{\lambda\rho\cdot\gamma}(\tilde{\theta}) = {} & K_{\lambda\rho}(\tilde{\theta}) - J_{\lambda\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\rho}(\tilde{\theta}) - K_{\lambda\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\rho}(\tilde{\theta}) \\ & + J_{\lambda\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\rho}(\tilde{\theta}), \end{aligned} \tag{3.12}$$

$$\begin{aligned} B_{\rho\cdot\gamma}(\tilde{\theta}) = {} & K_{\rho\rho}(\tilde{\theta}) + J_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\rho}(\tilde{\theta}) \\ & - K_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\rho}(\tilde{\theta}) - J_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\rho}(\tilde{\theta}), \end{aligned} \tag{3.13}$$

$$\begin{aligned} B_{\rho\lambda\cdot\gamma}(\tilde{\theta}) = {} & K_{\rho\lambda}(\tilde{\theta}) - J_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\lambda}(\tilde{\theta}) - K_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\lambda}(\tilde{\theta}) \\ & + J_{\rho\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})K_{\gamma\gamma}(\tilde{\theta})J_{\gamma\gamma}^{-1}(\tilde{\theta})J_{\gamma\lambda}(\tilde{\theta}). \end{aligned} \tag{3.14}$$

THEOREM 1. *Assume that Assumptions 1-2 hold. Then, under H_a^λ, it follows that*

$$LM_\lambda \xrightarrow{d} \chi_1^2(\vartheta_1), \tag{3.15}$$

where

$$\vartheta_1 = \delta_\lambda^2\left(J_{\lambda\cdot\gamma}(\theta_0) - J_{\lambda\rho\cdot\gamma}(\theta_0)J_{\rho\cdot\gamma}^{-1}(\theta_0)J_{\rho\lambda\cdot\gamma}(\theta_0)\right)^2/D_{\lambda\cdot\gamma}(\theta_0). \tag{3.16}$$

PROOF. See Appendix 6.

Theorem 1 indicates that $LM_\lambda \xrightarrow{d} \chi_1^2$ under H_0^λ in the local presence of ρ_0, i.e., LM_λ is a valid test statistic irrespective of whether H_0^ρ or H_a^ρ holds. Though our suggested test statistic has a lengthy expression, its calculation only requires $\tilde{\theta}$. Test statistic simplify significantly when $J_{\lambda\rho\cdot\gamma}(\tilde{\theta})=0$ holds. In that case, we have $LM_\lambda = n\,S_\lambda^2(\tilde{\theta})/D_{\lambda\cdot\gamma}(\tilde{\theta})$, where $D_{\lambda\cdot\gamma}(\tilde{\theta}) = B_{\lambda\cdot\gamma}(\tilde{\theta}) - B_{\lambda\rho\cdot\gamma}(\tilde{\theta})J^{-1}_{\rho\cdot\gamma}(\tilde{\theta})J_{\rho\lambda\cdot\gamma}(\tilde{\theta})$.

We can similarly determine the test statistic for testing H_a^ρ in the local presence of λ_0. The robust OGP test statistic is given by

$$LM_\rho = n\,S_\rho^{*2}(\tilde{\theta})/D_{\rho\cdot\gamma}(\tilde{\theta}), \tag{3.17}$$

where $S_\rho^*(\tilde{\theta}) = \left(S_\rho(\tilde{\theta}) - J_{\rho\lambda\cdot\gamma}(\tilde{\theta})J^{-1}_{\lambda\cdot\gamma}(\tilde{\theta})S_\lambda(\tilde{\theta})\right)$ is the adjusted score function and

$$\begin{aligned} D_{\rho\cdot\gamma}(\tilde{\theta}) = {} & B_{\rho\cdot\gamma}(\tilde{\theta}) + J_{\rho\lambda\cdot\gamma}(\tilde{\theta})J^{-1}_{\lambda\cdot\gamma}(\tilde{\theta})B_{\lambda\cdot\gamma}(\tilde{\theta})J^{-1}_{\lambda\cdot\gamma}(\tilde{\theta})J_{\lambda\rho\cdot\gamma}(\tilde{\theta}) \\ & - J_{\rho\lambda\cdot\gamma}(\tilde{\theta})J^{-1}_{\lambda\cdot\gamma}(\tilde{\theta})B_{\lambda\rho\cdot\gamma}(\tilde{\theta}) - B_{\rho\lambda\cdot\gamma}(\tilde{\theta})J^{-1}_{\lambda\cdot\gamma}(\tilde{\theta})J_{\lambda\rho\cdot\gamma}(\tilde{\theta}). \end{aligned} \tag{3.18}$$

The asymptotic null distribution of LM_ρ is a central chi-squared distribution in the local presence of λ_0 as shown in the following theorem.

THEOREM 2. *Assume that Assumptions 1-2 hold. Then, under H_a^ρ, it follows that*

$$LM_\rho \xrightarrow{d} \chi_1^2(\vartheta_2), \tag{3.19}$$

where

$$\vartheta_2 = \delta_\rho^2\left(J_{\rho\cdot\gamma}(\theta_0) - J_{\rho\lambda\cdot\gamma}(\theta_0)J^{-1}_{\lambda\cdot\gamma}(\theta_0)J_{\lambda\rho\cdot\gamma}(\theta_0)\right)^2/D_{\rho\cdot\gamma}(\theta_0). \tag{3.20}$$

PROOF. See Appendix 6.

Theorem 2 indicates that the asymptotic null distribution of LM_ρ is χ_1^2, i.e., $LM_\rho \xrightarrow{d} \chi_1^2$ under H_0^ρ.

REMARK 1. In terms of our notation, the OPG variants of LM test suggested in [7] for H_0^ρ can be expressed as

$$LM_\rho^s = nS_\rho^2(\tilde{\theta})/K_{\rho\rho}(\tilde{\theta}) \tag{3.21}$$

Under the null hypothesis, this test statistic is asymptotically equivalent to the LM test statistic suggested in [8] and the standardized OPG-based LM test suggested in [5, Theorem 2.2]. Here, we show that this test statistic is invalid under the local parametric misspecification of the form $\lambda_0 = \delta_\lambda/\sqrt{n}$. Using the asymptotic argument given in the proof of Theorem 1, it can be shown that the asymptotic distribution of $\sqrt{n}S_\rho(\tilde{\theta})$ under H_a^λ and H_a^ρ is given by

$$\sqrt{n}S_\rho(\tilde{\theta}) \xrightarrow{d} N\left[J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda,\, B_{\rho\cdot\gamma}(\theta_0)\right], \tag{3.22}$$

where

$$\begin{aligned} B_{\rho\cdot\gamma}(\theta_0) = {} & K_{\rho\rho}(\theta_0) + J_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J'_{\lambda\gamma}(\theta_0) \\ & - K_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J'_{\rho\gamma}(\theta_0) - J_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\rho}(\theta_0). \end{aligned} \tag{3.23}$$

Our results in (3.5) indicate that $J_{\rho\gamma}(\theta_0)=0$ under the joint null hypothesis $H_0: \lambda_0 = \rho_0 = 0$. Moreover, our Assumption 2 ensures that a consistent estimator of $B_{\rho\cdot\gamma}(\theta_0)$ is $B_{\rho\cdot\gamma}(\tilde{\theta}) = K_{\rho\rho}(\tilde{\theta})$. Then, under H_a^λ and H_a^ρ , using Theorem 8.6 of White [24] on the asymptotic distribution of quadratic forms, we obtain

$$LM_\rho^s \xrightarrow{d} \chi_1^2(\varphi_1), \tag{3.24}$$

where $\varphi_1 = \left(J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda\right)^2 / K_{\rho\rho}(\theta_0)$. In the local presence of λ_0, the result in (3.24) indicates that the asymptotic null distribution of LM^s_ρ is $\chi^2_1(\varphi_2)$, where $\varphi_2 = \left(J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda\right)^2 / K_{\rho\rho}(\theta_0)$. That is, in the presence of local parametric misspecification of the form $\lambda_0 = \delta_\lambda/\sqrt{n}$, LM^a_ρ will over reject H^ρ_0 due to the non-centrality parameter φ_2.

Remark 2. For testing H^λ_0, our result in (3.6) suggests the following non-robust test statistic

$$LM^s_\lambda = n\,S^2_\lambda(\tilde{\theta})/B_{\lambda\cdot\gamma}(\tilde{\theta}), \tag{3.25}$$

In contrast to LM^s_ρ, our result on the elements of 3.5 indicates that $B_{\lambda\cdot\gamma}(\tilde{\theta})$ is not equal to $K_{\lambda\lambda}(\tilde{\theta})$ since $J_{\lambda\rho}(\tilde{\theta}) = \frac{1}{n}\mathrm{tr}\left(M^{(s)}W\right) = O(1)$. Then, under H^λ_a and H^ρ_a, it follows that

$$LM^s_\lambda \xrightarrow{d} \chi^2_1(\varphi_3), \tag{3.26}$$

where $\varphi_3 = \left(J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho\right)^2 / B_{\lambda\cdot\gamma}(\theta_0)$ is the non-centrality parameter. Thus, in the local presence of ρ_0, the asymptotic null distribution of LM_λ is a non-central chi-squared distribution with the non-centrality parameter of $\varphi_4 = \left(J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho\right)^2 / B_{\lambda\cdot\gamma}(\theta_0)$. The test statistic suggested in [7] for testing H^λ_0 is different from the one in (3.26). Their test statistic is equivalent to the LM test statistics derived in [1] for the SAR model $Y = \lambda_0 WY + X\beta_0 + U$ [7, Proposition 4.1]. Since the LM statistic in [1] for the SAR model is invalid in the local presence of ρ_0 [2], it follows that the OPG variant in [7] for testing H^λ_0 is also invalid in the local presence of ρ_0.

3.2. Tests Under Heteroskedasticity

In this section, we assume that the disturbance terms v_i's are independent, but heteroskedastic with variances σ^2_i. In this case, it is known that the score function derived from the quasi likelihood function in (2.2) do not have zero expected values [18, 10]. This result indicates that the QMLE defined by $\hat{\theta} = \arg\max_{\theta\in\Theta} \ln L(\theta)$ may not be a consistent estimator of θ_0. Liu and Yang [19] and Yang [25] modify the score functions such that they have zero means, and define a consistent estimator as a root of adjusted score functions. We follow the same approach to formulate score-based OPG tests under heteroskedastic case. Since the modified scores can be considered as proper moment functions, our suggested tests can also be called the m-tests [24].

Let $\theta_0 = (\beta_0^{'}, \lambda_0, \rho_0)^{'}$. We use $\mathrm{Diag}(A_k)$ to denote the diagonal matrix whose diagonal elements are the (i,i)th elements $a_{k,ii}$'s, i.e., $\mathrm{Diag}(A_k) = \mathrm{Diag}(a_{k,11}, a_{k,22}, \ldots, a_{k,nn})$. Following [19], we consider the following modified score functions that have zero means.

$$C_\beta(\theta_0) = \frac{1}{n}X^{'}R^{'}V, \quad C_\lambda(\theta_0) = \frac{1}{n}V^{'}\left(\bar{G} - \mathrm{Diag}(\bar{G})\right)V + \frac{1}{n}(RGX\beta_0)^{'}V, \tag{3.27}$$

$$C_\rho(\theta_0) = \frac{1}{n}V^{'}\left(H - \mathrm{Diag}(H)\right)V, \tag{3.28}$$

where $V = R(SY - X\beta_0)$. In terms of linear quadratic forms, these score functions can be expressed as

$$C(\theta_0) = (C^{'}_\beta(\theta_0), C_\lambda(\theta_0), C_\rho(\theta_0))^{'} = \frac{1}{n}\begin{pmatrix} V^{'}A_1V + b^{'}_1V \\ \vdots \\ V^{'}A_{k_x}V + b^{'}_{k_x}V \\ V^{'}A_{k_x+1}V + b^{'}_{k_x+1}V \\ V^{'}A_{k_x+2}V + b^{'}_{k_x+2}V \end{pmatrix}, \tag{3.29}$$

where $A_1 = \cdots = A_{k_x} = 0$, $b_j = RX_j$ for $j = 1, 2, \ldots, k_x$, $A_{k_x+1} = \bar{G} - \mathrm{Diag}(\bar{G})$, $b_{k_x+1} = RGX\beta$, $A_{k_x+2} = H - \mathrm{Diag}(H)$ and $b_{k_x+2} = 0$. As before, we can express $S(\theta_0)$ as a sum of martingale differences in the following way

$$C(\theta_0) = \sum_{i=1}^{n} C_i(\theta_0), \tag{3.30}$$

where $C_i(\theta_0)=\left(C'_{i\beta}(\theta_0),C_{i\lambda}(\theta_0),C_{i\rho}(\theta_0)\right)'$ with the kth element ($k\in\{1,2,\dots,k_x+2\}$), $C_{ik}(\theta_0)=v_i\sum_{j=1}^{i-1}(a_{k,ij}+a_{k,ji})v_j+b_{ki}v_i$. As before, the variance of $\sqrt{n}C(\theta_0)$ takes the following form

$$\mathrm{Var}\left(\sqrt{n}C(\theta_0)\right)=K(\theta_0)=n\sum_{i=1}^{n}\mathbb{E}\left(C_i(\theta_0)C'_i(\theta_0)\right). \tag{3.31}$$

Under some regularity condition, $K(\theta_0)$ can be estimated by $K(\hat{\theta})=n\sum_{i=1}^{n}C_i(\hat{\theta})C'_i(\hat{\theta})$, where $\hat{\theta}$ is a consistent estimator of θ_0.

To define our suggested robust OGP test, we assume the following assumptions.

ASSUMPTION 3. *The innovation terms v_is are independent with mean zero, variance σ_i^2 and $\mathbb{E}\left|v_i\right|^{4+\eta}<\infty$ for some $\eta>0$ for all i and n.*

ASSUMPTION 4. *(i) Let $\tilde{\theta}$ be a constrained estimator under under the joint null hypothesis $H_0:\lambda_0=\rho_0=0$. Then, $\sqrt{n}(\tilde{\theta}-\theta_0)=O_p(1)$ holds. (ii) $\sqrt{n}C(\theta_0)\xrightarrow{d}N(0,K(\theta_0))$, where $K(\theta_0)$ is a non-singular matrix. (iii) $-\frac{\partial C(\bar{\theta})}{\partial\theta'}=J(\theta_0)+o_p(1)$, where $\bar{\theta}=\theta_0+o_p(1)$ and $J(\theta_0)=\mathbb{E}\left(-\frac{\partial C(\theta_0)}{\partial\theta'}\right)$ is a non-singular matrix.*

Assumption 3 specifies heteroskedastic disturbance terms [14, 18]. As in the homoskedastic case, Assumption 4 is a high level assumption and provides conditions that are counterparts of those stated in Assumption 2.[3]

Under Assumption 3, the elements of $J(\theta_0)$ can be derived as

$$\begin{aligned}
J_{\lambda\lambda}(\theta_0)&=\frac{1}{n}\mathrm{tr}\left(\left(\bar{G}-\mathrm{Diag}(\bar{G})\right)^{(s)}\bar{G}\Sigma\right)+\frac{1}{n}(RGX\beta_0)'RGX\beta_0\\
J_{\lambda\rho}(\theta_0)&=J_{\rho\lambda}(\theta_0)=\frac{1}{n}\mathrm{tr}\left(\left(H-\mathrm{Diag}(H)\right)^{(s)}\bar{G}\Sigma\right),\\
J_{\lambda\beta}(\theta_0)&=J'_{\beta\lambda}(\theta_0)=\frac{1}{n}\beta'_0X'G'R'RX,\\
J_{\rho\rho}(\theta_0)&=\frac{1}{n}\mathrm{tr}\left(\left(H-\mathrm{Diag}(H)\right)^{(s)}H\Sigma\right),\quad J_{\rho\beta}(\theta_0)=0,\quad J_{\beta\rho}(\theta_0)=0,\\
J_{\beta\beta}(\theta_0)&=\frac{1}{n}X'R'RX,
\end{aligned}$$

where $\Sigma=\mathrm{Diag}(\sigma_1^2,\sigma_2^2,\dots,\sigma_n^2)$. These expressions show that we can use a plug-in estimator to estimate $J(\theta_0)$ (here, we can use $\tilde{\Sigma}=\mathrm{Diag}(\hat{v}_1^2,\hat{v}_2^2,\dots,\hat{v}_n^2)$ for Σ.[4])

Under heteroskedastic disturbances, our suggested test statistics for H_0^{λ} and H_0^{ρ} are respectively given by

$$LM_{\lambda}^{h}=n\,C_{\lambda}^{*2}(\tilde{\theta})/D_{\lambda\cdot\beta}(\tilde{\theta}), \tag{3.32}$$
$$LM_{\rho}^{h}=n\,C_{\rho}^{*2}(\tilde{\theta})/D_{\rho\cdot\beta}(\tilde{\theta}) \tag{3.33}$$

where $C_{\lambda}^{*}(\tilde{\theta})=\left(C_{\lambda}(\tilde{\theta})-J_{\lambda\rho\cdot\beta}(\tilde{\theta})J_{\rho\cdot\beta}^{-1}(\tilde{\theta})C_{\rho}(\tilde{\theta})\right)$ and $C_{\rho}^{*}(\tilde{\theta})=\left(C_{\rho}(\tilde{\theta})-J_{\rho\lambda\cdot\beta}(\tilde{\theta})J_{\lambda\cdot\beta}^{-1}(\tilde{\theta})C_{\lambda}(\tilde{\theta})\right)$ are the adjusted score functions and

$$\begin{aligned}
D_{\lambda\cdot\beta}(\tilde{\theta})=B_{\lambda\cdot\beta}(\tilde{\theta})&+J_{\lambda\rho\cdot\beta}(\tilde{\theta})J_{\rho\cdot\beta}^{-1}(\tilde{\theta})B_{\rho\cdot\beta}(\tilde{\theta})J_{\rho\cdot\beta}^{-1}(\tilde{\theta})J_{\rho\lambda\cdot\beta}(\tilde{\theta})\\
&-J_{\lambda\rho\cdot\beta}(\tilde{\theta})J_{\rho\cdot\beta}^{-1}(\tilde{\theta})B_{\rho\lambda\cdot\beta}(\tilde{\theta})-B_{\lambda\rho\cdot\beta}(\tilde{\theta})J_{\rho\cdot\beta}^{-1}(\tilde{\theta})J_{\rho\lambda\cdot\beta}(\tilde{\theta}),
\end{aligned} \tag{3.34}$$

[3] The primitive conditions ensuring this assumption are provided in [14, 18]. For the sake of brevity, we do not repeat these primitive conditions.

[4] The the asymptotic argument is provided in [18].

$$\begin{aligned}D_{\rho\cdot\beta}(\tilde{\theta}) &= B_{\rho\cdot\beta}(\tilde{\theta}) + J_{\rho\lambda\cdot\beta}(\tilde{\theta})J^{-1}_{\lambda\cdot\beta}(\tilde{\theta})B_{\lambda\cdot\beta}(\tilde{\theta})J^{-1}_{\lambda\cdot\beta}(\tilde{\theta})J_{\lambda\rho\cdot\beta}(\tilde{\theta})\\ &- J_{\rho\lambda\cdot\beta}(\tilde{\theta})J^{-1}_{\lambda\cdot\beta}(\tilde{\theta})B_{\lambda\rho\cdot\beta}(\tilde{\theta}) - B_{\rho\lambda\cdot\beta}(\tilde{\theta})J^{-1}_{\lambda\cdot\beta}(\tilde{\theta})J_{\lambda\rho\cdot\beta}(\tilde{\theta}),\end{aligned} \tag{3.35}$$

with

$$\begin{aligned}B_{\lambda\cdot\beta}(\tilde{\theta}) &= K_{\lambda\lambda}(\tilde{\theta}) + J_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\lambda}(\tilde{\theta})\\ &- K_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\lambda}(\tilde{\theta}) - J_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\lambda}(\tilde{\theta}),\end{aligned} \tag{3.36}$$

$$\begin{aligned}B_{\lambda\rho\cdot\beta}(\tilde{\theta}) &= K_{\lambda\rho}(\tilde{\theta}) - J_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\rho}(\tilde{\theta}) - K_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\rho}(\tilde{\theta})\\ &+ J_{\lambda\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\rho}(\tilde{\theta}),\end{aligned} \tag{3.37}$$

$$\begin{aligned}B_{\rho\cdot\beta}(\tilde{\theta}) &= K_{\rho\rho}(\tilde{\theta}) + J_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\rho}(\tilde{\theta})\\ &- K_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\rho}(\tilde{\theta}) - J_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\rho}(\tilde{\theta}),\end{aligned} \tag{3.38}$$

$$\begin{aligned}B_{\rho\lambda\cdot\beta}(\tilde{\theta}) &= K_{\rho\lambda}(\tilde{\theta}) - J_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\lambda}(\tilde{\theta}) - K_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\lambda}(\tilde{\theta})\\ &+ J_{\rho\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})K_{\beta\beta}(\tilde{\theta})J^{-1}_{\beta\beta}(\tilde{\theta})J_{\beta\lambda}(\tilde{\theta}).\end{aligned} \tag{3.39}$$

The next theorem provides the asymptotic distributions of these tests.

THEOREM 3. *Assume that Assumptions 3-4 hold.*

1. *Under H^{λ}_a, it follows that*

$$LM^h_{\lambda} \xrightarrow{d} \chi^2_1(\vartheta_3), \tag{3.40}$$

where

$$\vartheta_3 = \delta^2_{\lambda}\left(J_{\lambda\cdot\beta}(\theta_0) - J_{\lambda\rho\cdot\beta}(\theta_0)J^{-1}_{\rho\cdot\beta}(\theta_0)J_{\rho\lambda\cdot\beta}(\theta_0)\right)^2/D_{\lambda\cdot\beta}(\theta_0). \tag{3.41}$$

2. *Under H^{ρ}_a, it follows that*

$$LM^h_{\rho} \xrightarrow{d} \chi^2_1(\vartheta_4), \tag{3.42}$$

where

$$\vartheta_4 = \delta^2_{\rho}\left(J_{\rho\cdot\beta}(\theta_0) - J_{\rho\lambda\cdot\beta}(\theta_0)J^{-1}_{\lambda\cdot\beta}(\theta_0)J_{\lambda\rho\cdot\beta}(\theta_0)\right)^2/D_{\rho\cdot\beta}(\theta_0). \tag{3.43}$$

PROOF. See Appendix 6.
Theorem 3 shows that LM_{λ} and LM_{λ} are valid test statistics in the presence of parametric misspecification. That is, $LM^h_{\lambda} \xrightarrow{d} \chi^2_1$ under H^{λ}_0, and $LM^h_{\rho} \xrightarrow{d} \chi^2_1$ under H^{ρ}_0.

REMARK 3. The non-robust versions stated in Remarks 1 and 2 take the following forms

$$LM^c_{\rho} = n\,C^2_{\rho}(\tilde{\theta})/B_{\rho\cdot\gamma}(\tilde{\theta}), \tag{3.44}$$
$$LM^c_{\lambda} = n\,C^2_{\lambda}(\tilde{\theta})/B_{\lambda\cdot\gamma}(\tilde{\theta}), \tag{3.45}$$

These variants are invalid under local parametric misspecification. Under the heteroskedastic case, we also have $J_{\rho\beta}(\tilde{\theta}) = 0$, implying that $B_{\rho\cdot\gamma}(\tilde{\theta}) = K_{\rho\rho}(\tilde{\theta})$.Thus, LM^c_{ρ} can also be formulated with $K_{\rho\rho}(\tilde{\theta})$.

4. Monte Carlo Simulations

4.1. Design

In order to study the finite sample properties of the suggested tests, we design a Monte Carlo study. For the specification in (2.1), we consider two regressors $X = (X_1, X_2)$ with the parameter vector $(\beta_{01}, \beta_{02})^{'} = (1\,, 1)^{'}$. For X_1 and X_2, we use the U.S. county-level data set of [21] on the 1980 presidential election: X_1 is the standardized value of log income per-capita and X_2 is the standardized value of log homeownership. This data set describes 3107 U.S. counties, and we use the first n observations in our Monte Carlo study. We consider the three specifications for the weights matrix considered by [22]: (i) the 49×49 contiguity based weights matrix generated for 48 US states and the District of Columbia, (ii) the 98×98 contiguity based weights matrix corresponding to five nearest neighbors of each of the 98 census tracts in Toledo, Ohio, and (iii) the 361×361 contiguity based weights matrix corresponding to whether the school districts are in the same county in Iowa in 2009. In our simulation, we set $W = M$. The spatial autoregressive parameters, λ_0 and ρ_0, take values from $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.

In generating disturbance terms, we consider two cases: (i) a homoskedastic case and (ii) a heteroskedastic case based on a skedastic function. In the homoskedastic case, we generate disturbance terms that have standard normal and chi-squared distributions: (i) $v_i \sim N(0,1)$ and (ii) $v_i \sim (\chi^2_2 - 2)/2$. In the skedastic case, we generate the disturbance terms according to $v_i = \sigma_i \xi_i$, where $\sigma^2_i = \exp(0.1 + 0.35 \cdot X_{i,1})$, and the innovation term ξ_i is generated according to (i) $\xi_i \sim N(0,1)$, and (ii) $v_i \sim (\chi^2_2 - 2)/2$. We set the nominal size set to 0.05 and the number of repetitions is 1000.

In our simulation, we will also consider the test statistics suggested in [7] and [4]. More specifically, we consider the tests in the equations (3.2) and (3.4) of [7], and the tests in the equations (16) and (17) of [4]. Moreover, we also consider the robust test statistics that are derived in the LM framework in [2] under the assumption that the disturbance terms are homoskedastic.

4.2. Simulation Results

The simulation results are presented in Tables 1 through 6 for the normal distribution case.[5] In these tables, (i) LM_λ is the test statistic in Theorem 1, (ii) LM_ρ is the test statistic in Theorem 2, (iii) LM^h_λ and LM^h_ρ are the test statistics given in Theorem 3, (iv) LM^B_ρ and LM^B_λ are the test statistics suggested in [7], (v) LM^Z_λ and LM^Z_ρ are the tests statistics suggested in [4], and (vi) LM^A_λ and LM^A_ρ are the test statistics suggested in [2]. The salient features of the results in Tables 1-6 are as follows.

1. The first row in each table shows the empirical size of the tests when there is no parametric misspecification. Our suggested tests, i.e., LM_ρ, LM_λ, LM^h_ρ and LM^h_λ, and those suggested in [2], i.e., LM^A_ρ and LM^A_λ, report size values that are close to the nominal value of 0.05 under both homoskedasticity and heteroskedasticity. The test statistics LM^B_ρ, LM^Z_ρ, LM^B_λ and LM^Z_λ are over-sized, especially LM^B_ρ and LM^Z_ρ.

2. The empirical size properties of tests for testing H^λ_0 under parametric misspecification can be examined through the first panel of each table. In the local presence of ρ_0, our suggested tests and those suggested in [2] perform better than other tests. As expected, LM^Z_λ and LM^B_λ are not robust to the local presence of ρ_0.

3. The empirical size properties of tests for testing H^ρ_0 under parametric misspecification can be examined under the cases where $\rho_0 = 0$ and $\lambda_0 \neq 0$. Both LM^h_ρ and LM^A_ρ perform relatively better than other tests in all cases. LM^Z_ρ and LM^B_ρ are over-sized in all cases as these tests are not robust to local parametric misspecification.

[5] We do not present the simulation results for the chi-squared distribution case, since they are similar to the normal distribution case. These results are available from the authors upon request.

4. The simulation results in Tables 1-6 indicates that heteroskedasticity specified in the form of a skedastic function may not affect the performance of the robust tests suggested in [2].

5. The empirical power properties of test statistics for testing H_0^λ can be examined when λ_0 deviates from zero. When λ_0 is near zero, LM_λ^B and LM_λ^Z report relatively large empirical powers under both homoskedastic and heteroskedastic cases. As λ_0 gets larger, our suggested tests and those in [2] get larger empirical powers, and perform similar to LM_λ^B and LM_λ^Z. Similarly, the empirical power properties of test statistics for testing H_0^ρ can be examined when ρ_0 deviates from zero. Here, we also observe a similar pattern. That is, LM_λ^B and LM_λ^Z perform better than other tests under both homoskedastic and heteroskedastic cases only when ρ_0 is near to zero.

TABLE 1. Empirical size and power of tests: Homoskedastic disturbances and $n = 49$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.049	0.044	0.168	0.171	0.051	0.050	0.052	0.075	0.081	0.050
0.0	0.1	0.048	0.061	0.190	0.211	0.051	0.057	0.060	0.092	0.096	0.057
0.0	0.2	0.056	0.113	0.312	0.342	0.091	0.060	0.059	0.137	0.149	0.060
0.0	0.3	0.055	0.197	0.460	0.486	0.150	0.058	0.057	0.205	0.215	0.058
0.0	0.4	0.063	0.335	0.656	0.688	0.258	0.082	0.086	0.337	0.349	0.082
0.0	0.5	0.079	0.496	0.805	0.835	0.397	0.100	0.108	0.525	0.541	0.100
0.0	0.6	0.133	0.655	0.915	0.928	0.531	0.124	0.137	0.705	0.717	0.124
0.1	0.0	0.075	0.051	0.185	0.207	0.052	0.098	0.104	0.157	0.162	0.098
0.1	0.1	0.061	0.068	0.261	0.302	0.054	0.095	0.105	0.206	0.220	0.095
0.1	0.2	0.057	0.126	0.477	0.513	0.086	0.093	0.102	0.340	0.354	0.093
0.1	0.3	0.063	0.211	0.638	0.670	0.152	0.116	0.121	0.457	0.478	0.116
0.1	0.4	0.064	0.342	0.780	0.806	0.242	0.134	0.146	0.607	0.630	0.134
0.1	0.5	0.105	0.483	0.891	0.909	0.356	0.174	0.184	0.772	0.783	0.174
0.1	0.6	0.192	0.644	0.957	0.966	0.493	0.175	0.185	0.858	0.872	0.175
0.2	0.0	0.119	0.048	0.295	0.331	0.042	0.238	0.248	0.389	0.405	0.238
0.2	0.1	0.108	0.079	0.430	0.488	0.056	0.251	0.278	0.502	0.517	0.251
0.2	0.2	0.084	0.123	0.611	0.661	0.083	0.262	0.277	0.628	0.645	0.262
0.2	0.3	0.090	0.209	0.760	0.796	0.139	0.264	0.278	0.706	0.718	0.264
0.2	0.4	0.100	0.328	0.872	0.893	0.226	0.270	0.287	0.815	0.822	0.270
0.2	0.5	0.167	0.485	0.937	0.949	0.330	0.264	0.281	0.893	0.898	0.264
0.2	0.6	0.304	0.622	0.973	0.983	0.444	0.276	0.297	0.937	0.942	0.276
0.3	0.0	0.230	0.053	0.451	0.523	0.048	0.505	0.533	0.733	0.745	0.505
0.3	0.1	0.191	0.067	0.597	0.657	0.045	0.488	0.509	0.776	0.787	0.488
0.3	0.2	0.148	0.124	0.760	0.798	0.075	0.471	0.498	0.852	0.858	0.471
0.3	0.3	0.167	0.192	0.852	0.886	0.124	0.478	0.492	0.904	0.908	0.478
0.3	0.4	0.220	0.302	0.918	0.946	0.185	0.451	0.475	0.943	0.949	0.451
0.3	0.5	0.306	0.456	0.972	0.983	0.300	0.425	0.447	0.961	0.964	0.425
0.3	0.6	0.469	0.570	0.986	0.991	0.386	0.401	0.421	0.983	0.984	0.401
0.4	0.0	0.343	0.049	0.632	0.695	0.041	0.781	0.801	0.928	0.932	0.781
0.4	0.1	0.324	0.061	0.758	0.826	0.043	0.782	0.801	0.957	0.959	0.782
0.4	0.2	0.297	0.117	0.855	0.898	0.060	0.716	0.749	0.965	0.970	0.716
0.4	0.3	0.308	0.189	0.922	0.950	0.112	0.673	0.698	0.981	0.982	0.673
0.4	0.4	0.388	0.282	0.961	0.986	0.160	0.644	0.658	0.987	0.988	0.644
0.4	0.5	0.487	0.409	0.985	0.992	0.234	0.604	0.627	0.988	0.989	0.604
0.4	0.6	0.657	0.509	0.994	0.998	0.329	0.534	0.553	0.996	0.997	0.534
0.5	0.0	0.509	0.043	0.756	0.843	0.038	0.932	0.945	0.992	0.993	0.932
0.5	0.1	0.495	0.050	0.853	0.904	0.029	0.921	0.931	0.996	0.996	0.921
0.5	0.2	0.475	0.088	0.905	0.946	0.049	0.892	0.903	0.994	0.994	0.892
0.5	0.3	0.510	0.138	0.951	0.977	0.074	0.851	0.866	0.997	0.997	0.851
0.5	0.4	0.600	0.217	0.972	0.989	0.110	0.808	0.822	0.996	0.997	0.808
0.5	0.5	0.680	0.333	0.989	0.994	0.185	0.737	0.760	0.997	0.998	0.737
0.5	0.6	0.830	0.423	0.992	0.997	0.243	0.663	0.682	0.998	0.999	0.663
0.6	0.0	0.712	0.034	0.873	0.943	0.029	0.989	0.992	1.000	1.000	0.989
0.6	0.1	0.723	0.038	0.929	0.974	0.022	0.980	0.985	1.000	1.000	0.980
0.6	0.2	0.741	0.060	0.949	0.982	0.033	0.963	0.965	1.000	1.000	0.963
0.6	0.3	0.753	0.104	0.975	0.993	0.045	0.939	0.952	1.000	1.000	0.939
0.6	0.4	0.803	0.172	0.989	0.995	0.076	0.912	0.922	1.000	1.000	0.912
0.6	0.5	0.872	0.248	0.989	0.997	0.113	0.849	0.869	1.000	1.000	0.849
0.6	0.6	0.934	0.312	0.995	0.999	0.155	0.781	0.801	1.000	1.000	0.781

TABLE 2. Empirical size and power of tests: Heteroskedastic disturbances and $n = 49$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.045	0.046	0.176	0.188	0.046	0.051	0.061	0.074	0.079	0.051
0.0	0.1	0.056	0.066	0.211	0.223	0.052	0.053	0.071	0.089	0.093	0.053
0.0	0.2	0.058	0.121	0.338	0.349	0.090	0.059	0.072	0.129	0.134	0.059
0.0	0.3	0.058	0.213	0.485	0.496	0.150	0.054	0.065	0.193	0.196	0.054
0.0	0.4	0.082	0.350	0.673	0.698	0.257	0.076	0.092	0.320	0.328	0.076
0.0	0.5	0.090	0.519	0.828	0.835	0.411	0.093	0.119	0.500	0.506	0.093
0.0	0.6	0.116	0.671	0.926	0.934	0.531	0.114	0.137	0.673	0.685	0.114
0.1	0.0	0.067	0.052	0.206	0.219	0.044	0.102	0.117	0.150	0.153	0.102
0.1	0.1	0.052	0.074	0.296	0.313	0.053	0.096	0.120	0.202	0.207	0.096
0.1	0.2	0.053	0.134	0.491	0.510	0.087	0.095	0.111	0.325	0.334	0.095
0.1	0.3	0.058	0.225	0.666	0.688	0.161	0.113	0.131	0.448	0.453	0.113
0.1	0.4	0.064	0.356	0.797	0.817	0.247	0.128	0.154	0.589	0.599	0.128
0.1	0.5	0.089	0.502	0.903	0.916	0.356	0.168	0.194	0.757	0.761	0.168
0.1	0.6	0.147	0.653	0.964	0.969	0.500	0.162	0.184	0.847	0.856	0.162
0.2	0.0	0.108	0.045	0.324	0.341	0.043	0.229	0.276	0.386	0.392	0.229
0.2	0.1	0.098	0.091	0.476	0.502	0.057	0.258	0.297	0.508	0.514	0.258
0.2	0.2	0.075	0.137	0.639	0.664	0.083	0.260	0.303	0.612	0.624	0.260
0.2	0.3	0.075	0.215	0.789	0.807	0.137	0.255	0.291	0.701	0.705	0.255
0.2	0.4	0.087	0.341	0.883	0.898	0.229	0.271	0.303	0.802	0.807	0.271
0.2	0.5	0.135	0.501	0.950	0.960	0.322	0.267	0.299	0.887	0.891	0.267
0.2	0.6	0.247	0.646	0.979	0.984	0.441	0.271	0.302	0.935	0.940	0.271
0.3	0.0	0.210	0.053	0.489	0.523	0.045	0.509	0.568	0.730	0.736	0.509
0.3	0.1	0.168	0.070	0.635	0.670	0.044	0.494	0.545	0.778	0.783	0.494
0.3	0.2	0.113	0.128	0.787	0.807	0.067	0.482	0.531	0.851	0.851	0.482
0.3	0.3	0.128	0.203	0.864	0.890	0.126	0.476	0.526	0.904	0.905	0.476
0.3	0.4	0.167	0.315	0.936	0.951	0.187	0.462	0.503	0.942	0.943	0.462
0.3	0.5	0.232	0.465	0.975	0.981	0.303	0.426	0.463	0.962	0.963	0.426
0.3	0.6	0.396	0.585	0.986	0.988	0.384	0.406	0.439	0.982	0.983	0.406
0.4	0.0	0.313	0.050	0.661	0.700	0.041	0.779	0.821	0.933	0.935	0.779
0.4	0.1	0.284	0.060	0.789	0.828	0.039	0.791	0.824	0.964	0.964	0.791
0.4	0.2	0.259	0.114	0.865	0.899	0.059	0.728	0.773	0.963	0.967	0.728
0.4	0.3	0.254	0.191	0.928	0.947	0.108	0.688	0.730	0.980	0.980	0.688
0.4	0.4	0.339	0.293	0.973	0.985	0.158	0.662	0.700	0.987	0.988	0.662
0.4	0.5	0.399	0.420	0.987	0.991	0.234	0.617	0.657	0.989	0.990	0.617
0.4	0.6	0.594	0.520	0.996	0.998	0.323	0.545	0.582	0.996	0.998	0.545
0.5	0.0	0.475	0.038	0.790	0.845	0.036	0.943	0.958	0.995	0.995	0.943
0.5	0.1	0.446	0.048	0.879	0.914	0.028	0.927	0.939	0.996	0.997	0.927
0.5	0.2	0.417	0.089	0.919	0.949	0.041	0.906	0.924	0.994	0.995	0.906
0.5	0.3	0.444	0.142	0.962	0.981	0.067	0.864	0.888	0.997	0.997	0.864
0.5	0.4	0.525	0.213	0.981	0.991	0.110	0.830	0.854	0.998	0.998	0.830
0.5	0.5	0.613	0.340	0.991	0.995	0.178	0.762	0.785	0.998	0.999	0.762
0.5	0.6	0.769	0.430	0.996	0.998	0.234	0.680	0.711	0.999	0.999	0.680
0.6	0.0	0.667	0.031	0.899	0.948	0.025	0.992	0.994	1.000	1.000	0.992
0.6	0.1	0.668	0.035	0.941	0.979	0.021	0.984	0.991	1.000	1.000	0.984
0.6	0.2	0.686	0.059	0.960	0.984	0.028	0.971	0.976	1.000	1.000	0.971
0.6	0.3	0.700	0.100	0.979	0.992	0.041	0.951	0.963	1.000	1.000	0.951
0.6	0.4	0.729	0.166	0.993	0.996	0.070	0.922	0.935	1.000	1.000	0.922
0.6	0.5	0.828	0.251	0.990	0.997	0.104	0.868	0.884	1.000	1.000	0.868
0.6	0.6	0.904	0.312	0.997	1.000	0.160	0.791	0.822	1.000	1.000	0.791

TABLE 3. Empirical size and power of tests: Homoskedastic disturbances and $n = 98$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.052	0.046	0.158	0.153	0.049	0.058	0.053	0.077	0.074	0.058
0.0	0.1	0.061	0.086	0.208	0.226	0.064	0.060	0.062	0.099	0.104	0.060
0.0	0.2	0.071	0.172	0.379	0.412	0.137	0.061	0.063	0.172	0.179	0.061
0.0	0.3	0.115	0.325	0.622	0.649	0.285	0.099	0.100	0.310	0.321	0.099
0.0	0.4	0.129	0.561	0.844	0.862	0.500	0.101	0.102	0.506	0.523	0.101
0.0	0.5	0.141	0.745	0.957	0.961	0.694	0.143	0.149	0.733	0.743	0.143
0.0	0.6	0.184	0.908	0.990	0.992	0.877	0.208	0.214	0.890	0.896	0.208
0.1	0.0	0.073	0.046	0.198	0.223	0.050	0.119	0.130	0.194	0.205	0.119
0.1	0.1	0.075	0.099	0.380	0.410	0.082	0.142	0.146	0.316	0.331	0.142
0.1	0.2	0.068	0.205	0.614	0.647	0.173	0.139	0.148	0.459	0.476	0.139
0.1	0.3	0.085	0.363	0.814	0.836	0.300	0.183	0.196	0.623	0.637	0.183
0.1	0.4	0.106	0.611	0.925	0.936	0.539	0.183	0.188	0.778	0.786	0.183
0.1	0.5	0.129	0.789	0.985	0.986	0.716	0.225	0.238	0.896	0.903	0.225
0.1	0.6	0.213	0.919	0.998	0.998	0.883	0.270	0.273	0.973	0.975	0.270
0.2	0.0	0.164	0.065	0.406	0.450	0.056	0.374	0.389	0.569	0.586	0.374
0.2	0.1	0.119	0.118	0.630	0.665	0.095	0.378	0.400	0.700	0.709	0.378
0.2	0.2	0.103	0.240	0.821	0.842	0.201	0.365	0.388	0.804	0.813	0.365
0.2	0.3	0.092	0.435	0.927	0.940	0.371	0.378	0.384	0.870	0.875	0.378
0.2	0.4	0.118	0.642	0.981	0.984	0.567	0.365	0.388	0.935	0.942	0.365
0.2	0.5	0.167	0.822	0.996	0.997	0.767	0.385	0.395	0.979	0.982	0.385
0.2	0.6	0.300	0.932	1.000	1.000	0.897	0.383	0.390	0.989	0.991	0.383
0.3	0.0	0.258	0.070	0.696	0.739	0.062	0.718	0.732	0.893	0.898	0.718
0.3	0.1	0.197	0.131	0.858	0.876	0.102	0.716	0.734	0.944	0.946	0.716
0.3	0.2	0.149	0.281	0.931	0.943	0.212	0.670	0.688	0.958	0.963	0.670
0.3	0.3	0.147	0.492	0.973	0.981	0.412	0.635	0.655	0.978	0.980	0.635
0.3	0.4	0.174	0.693	0.995	0.998	0.606	0.611	0.620	0.988	0.990	0.611
0.3	0.5	0.257	0.848	1.000	1.000	0.786	0.583	0.588	0.997	0.998	0.583
0.3	0.6	0.449	0.956	1.000	1.000	0.917	0.544	0.556	0.998	0.998	0.544
0.4	0.0	0.339	0.095	0.925	0.945	0.066	0.933	0.943	0.993	0.994	0.933
0.4	0.1	0.302	0.185	0.971	0.982	0.141	0.917	0.928	0.995	0.996	0.917
0.4	0.2	0.245	0.371	0.992	0.996	0.284	0.892	0.904	0.996	0.996	0.892
0.4	0.3	0.246	0.524	0.996	0.998	0.428	0.863	0.871	0.998	0.998	0.863
0.4	0.4	0.305	0.735	1.000	1.000	0.655	0.803	0.807	0.998	0.998	0.803
0.4	0.5	0.426	0.874	1.000	1.000	0.799	0.763	0.770	1.000	1.000	0.763
0.4	0.6	0.620	0.955	1.000	1.000	0.914	0.679	0.691	1.000	1.000	0.679
0.5	0.0	0.480	0.135	0.993	0.996	0.102	0.993	0.994	1.000	1.000	0.993
0.5	0.1	0.450	0.274	0.996	0.997	0.188	0.990	0.993	1.000	1.000	0.990
0.5	0.2	0.427	0.438	0.999	1.000	0.341	0.980	0.981	1.000	1.000	0.980
0.5	0.3	0.443	0.613	0.999	0.999	0.516	0.966	0.969	1.000	1.000	0.966
0.5	0.4	0.495	0.773	1.000	1.000	0.665	0.936	0.938	1.000	1.000	0.936
0.5	0.5	0.623	0.881	1.000	1.000	0.802	0.886	0.890	1.000	1.000	0.886
0.5	0.6	0.814	0.951	1.000	1.000	0.896	0.804	0.809	1.000	1.000	0.804
0.6	0.0	0.690	0.216	1.000	1.000	0.150	1.000	1.000	1.000	1.000	1.000
0.6	0.1	0.670	0.319	1.000	1.000	0.225	1.000	1.000	1.000	1.000	1.000
0.6	0.2	0.674	0.482	1.000	1.000	0.349	0.997	0.998	1.000	1.000	0.997
0.6	0.3	0.724	0.623	1.000	1.000	0.498	0.994	0.995	1.000	1.000	0.994
0.6	0.4	0.762	0.785	1.000	1.000	0.666	0.970	0.973	1.000	1.000	0.970
0.6	0.5	0.852	0.869	1.000	1.000	0.772	0.945	0.951	1.000	1.000	0.945
0.6	0.6	0.935	0.934	1.000	1.000	0.852	0.886	0.887	1.000	1.000	0.886

TABLE 4. Empirical size and power of tests: Heteroskedastic disturbances and $n = 98$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.054	0.057	0.158	0.161	0.052	0.058	0.086	0.073	0.071	0.058
0.0	0.1	0.063	0.090	0.220	0.230	0.062	0.063	0.080	0.092	0.092	0.063
0.0	0.2	0.073	0.182	0.391	0.403	0.131	0.061	0.081	0.163	0.168	0.061
0.0	0.3	0.120	0.331	0.630	0.645	0.283	0.096	0.122	0.284	0.288	0.096
0.0	0.4	0.148	0.566	0.847	0.858	0.487	0.093	0.127	0.460	0.469	0.093
0.0	0.5	0.160	0.758	0.960	0.964	0.686	0.134	0.169	0.691	0.701	0.134
0.0	0.6	0.183	0.911	0.990	0.990	0.872	0.186	0.223	0.867	0.871	0.186
0.1	0.0	0.071	0.054	0.207	0.215	0.050	0.117	0.154	0.190	0.193	0.117
0.1	0.1	0.072	0.102	0.382	0.402	0.079	0.129	0.173	0.301	0.306	0.129
0.1	0.2	0.072	0.220	0.624	0.635	0.172	0.132	0.169	0.419	0.425	0.132
0.1	0.3	0.093	0.362	0.818	0.831	0.293	0.183	0.227	0.591	0.601	0.183
0.1	0.4	0.113	0.612	0.930	0.934	0.532	0.164	0.209	0.749	0.753	0.164
0.1	0.5	0.128	0.778	0.982	0.983	0.709	0.222	0.255	0.882	0.885	0.222
0.1	0.6	0.171	0.923	0.998	0.998	0.879	0.254	0.287	0.965	0.966	0.254
0.2	0.0	0.143	0.066	0.421	0.442	0.057	0.350	0.413	0.539	0.546	0.350
0.2	0.1	0.110	0.124	0.642	0.659	0.098	0.357	0.433	0.670	0.675	0.357
0.2	0.2	0.089	0.244	0.823	0.836	0.198	0.344	0.401	0.770	0.777	0.344
0.2	0.3	0.087	0.430	0.927	0.936	0.364	0.362	0.412	0.849	0.852	0.362
0.2	0.4	0.107	0.640	0.981	0.983	0.557	0.352	0.405	0.919	0.920	0.352
0.2	0.5	0.136	0.814	0.997	0.997	0.758	0.370	0.414	0.971	0.974	0.370
0.2	0.6	0.233	0.930	1.000	1.000	0.895	0.368	0.404	0.988	0.988	0.368
0.3	0.0	0.224	0.071	0.707	0.735	0.060	0.683	0.741	0.875	0.879	0.683
0.3	0.1	0.163	0.134	0.855	0.869	0.102	0.693	0.754	0.934	0.936	0.693
0.3	0.2	0.125	0.283	0.939	0.948	0.220	0.652	0.707	0.946	0.949	0.652
0.3	0.3	0.129	0.486	0.973	0.978	0.410	0.611	0.669	0.975	0.975	0.611
0.3	0.4	0.130	0.691	0.995	0.996	0.597	0.600	0.644	0.984	0.985	0.600
0.3	0.5	0.204	0.847	1.000	1.000	0.785	0.565	0.598	0.994	0.993	0.565
0.3	0.6	0.369	0.950	1.000	1.000	0.915	0.534	0.564	0.998	0.998	0.534
0.4	0.0	0.302	0.101	0.937	0.947	0.074	0.925	0.943	0.991	0.992	0.925
0.4	0.1	0.240	0.184	0.977	0.982	0.150	0.907	0.931	0.994	0.994	0.907
0.4	0.2	0.197	0.381	0.995	0.996	0.306	0.886	0.914	0.995	0.995	0.886
0.4	0.3	0.200	0.521	0.997	0.998	0.441	0.857	0.881	0.998	0.998	0.857
0.4	0.4	0.239	0.732	1.000	1.000	0.663	0.799	0.828	0.998	0.998	0.799
0.4	0.5	0.342	0.865	1.000	1.000	0.803	0.765	0.782	0.999	1.000	0.765
0.4	0.6	0.526	0.951	1.000	1.000	0.912	0.685	0.711	1.000	1.000	0.685
0.5	0.0	0.417	0.137	0.994	0.996	0.108	0.991	0.995	1.000	1.000	0.991
0.5	0.1	0.385	0.267	0.996	0.997	0.198	0.990	0.995	1.000	1.000	0.990
0.5	0.2	0.353	0.429	0.999	0.999	0.356	0.977	0.985	1.000	1.000	0.977
0.5	0.3	0.364	0.608	1.000	1.000	0.523	0.968	0.974	1.000	1.000	0.968
0.5	0.4	0.420	0.769	1.000	1.000	0.669	0.935	0.944	1.000	1.000	0.935
0.5	0.5	0.540	0.877	1.000	1.000	0.803	0.890	0.904	1.000	1.000	0.890
0.5	0.6	0.745	0.948	1.000	1.000	0.896	0.807	0.826	1.000	1.000	0.807
0.6	0.0	0.646	0.212	1.000	1.000	0.178	1.000	1.000	1.000	1.000	1.000
0.6	0.1	0.606	0.314	1.000	1.000	0.247	1.000	1.000	1.000	1.000	1.000
0.6	0.2	0.599	0.479	1.000	1.000	0.372	0.998	0.999	1.000	1.000	0.998
0.6	0.3	0.643	0.620	1.000	1.000	0.526	0.995	0.998	1.000	1.000	0.995
0.6	0.4	0.688	0.773	1.000	1.000	0.672	0.975	0.979	1.000	1.000	0.975
0.6	0.5	0.793	0.861	1.000	1.000	0.769	0.955	0.960	1.000	1.000	0.955
0.6	0.6	0.895	0.927	1.000	1.000	0.844	0.897	0.904	1.000	1.000	0.897

Table 5. Empirical size and power of tests: Homoskedastic disturbances and $n = 361$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.059	0.043	0.173	0.172	0.043	0.061	0.061	0.092	0.091	0.061
0.0	0.1	0.060	0.148	0.417	0.416	0.135	0.061	0.059	0.211	0.211	0.061
0.0	0.2	0.071	0.456	0.847	0.848	0.424	0.069	0.072	0.518	0.518	0.069
0.0	0.3	0.098	0.780	0.985	0.985	0.753	0.096	0.102	0.846	0.847	0.096
0.0	0.4	0.144	0.957	1.000	1.000	0.944	0.110	0.108	0.979	0.979	0.110
0.0	0.5	0.311	0.996	1.000	1.000	0.995	0.145	0.144	0.999	0.999	0.145
0.0	0.6	0.707	1.000	1.000	1.000	1.000	0.189	0.188	1.000	1.000	0.189
0.1	0.0	0.240	0.055	0.431	0.431	0.054	0.252	0.259	0.495	0.491	0.252
0.1	0.1	0.238	0.171	0.860	0.860	0.148	0.249	0.257	0.794	0.794	0.249
0.1	0.2	0.278	0.497	0.984	0.984	0.472	0.254	0.253	0.951	0.951	0.254
0.1	0.3	0.390	0.815	0.999	0.999	0.788	0.288	0.291	0.994	0.994	0.288
0.1	0.4	0.533	0.970	1.000	1.000	0.960	0.279	0.283	1.000	1.000	0.279
0.1	0.5	0.791	0.997	1.000	1.000	0.995	0.294	0.293	1.000	1.000	0.294
0.1	0.6	0.963	1.000	1.000	1.000	1.000	0.311	0.311	1.000	1.000	0.311
0.2	0.0	0.693	0.061	0.873	0.874	0.054	0.702	0.708	0.958	0.959	0.702
0.2	0.1	0.699	0.206	0.991	0.991	0.182	0.697	0.704	0.997	0.997	0.697
0.2	0.2	0.731	0.543	1.000	1.000	0.509	0.653	0.658	0.999	0.999	0.653
0.2	0.3	0.829	0.851	1.000	1.000	0.819	0.656	0.664	1.000	1.000	0.656
0.2	0.4	0.908	0.976	1.000	1.000	0.966	0.597	0.608	1.000	1.000	0.597
0.2	0.5	0.982	0.999	1.000	1.000	0.996	0.585	0.594	1.000	1.000	0.585
0.2	0.6	0.997	1.000	1.000	1.000	1.000	0.522	0.523	1.000	1.000	0.522
0.3	0.0	0.961	0.073	0.997	0.997	0.059	0.960	0.960	1.000	1.000	0.960
0.3	0.1	0.962	0.259	1.000	1.000	0.231	0.943	0.946	1.000	1.000	0.943
0.3	0.2	0.977	0.607	1.000	1.000	0.550	0.932	0.941	1.000	1.000	0.932
0.3	0.3	0.989	0.868	1.000	1.000	0.839	0.916	0.919	1.000	1.000	0.916
0.3	0.4	0.996	0.986	1.000	1.000	0.967	0.852	0.855	1.000	1.000	0.852
0.3	0.5	1.000	1.000	1.000	1.000	0.998	0.818	0.825	1.000	1.000	0.818
0.3	0.6	1.000	1.000	1.000	1.000	1.000	0.760	0.765	1.000	1.000	0.760
0.4	0.0	1.000	0.093	1.000	1.000	0.076	0.998	0.999	1.000	1.000	0.998
0.4	0.1	1.000	0.337	1.000	1.000	0.281	0.999	1.000	1.000	1.000	0.999
0.4	0.2	1.000	0.673	1.000	1.000	0.590	0.997	0.998	1.000	1.000	0.997
0.4	0.3	1.000	0.898	1.000	1.000	0.842	0.986	0.988	1.000	1.000	0.986
0.4	0.4	1.000	0.988	1.000	1.000	0.968	0.968	0.971	1.000	1.000	0.968
0.4	0.5	1.000	0.999	1.000	1.000	0.992	0.949	0.954	1.000	1.000	0.949
0.4	0.6	1.000	1.000	1.000	1.000	0.999	0.875	0.883	1.000	1.000	0.875
0.5	0.0	1.000	0.167	1.000	1.000	0.120	1.000	1.000	1.000	1.000	1.000
0.5	0.1	1.000	0.419	1.000	1.000	0.315	1.000	1.000	1.000	1.000	1.000
0.5	0.2	1.000	0.725	1.000	1.000	0.613	0.999	1.000	1.000	1.000	0.999
0.5	0.3	1.000	0.927	1.000	1.000	0.852	0.999	0.999	1.000	1.000	0.999
0.5	0.4	1.000	0.983	1.000	1.000	0.957	0.996	0.995	1.000	1.000	0.996
0.5	0.5	1.000	0.997	1.000	1.000	0.984	0.983	0.984	1.000	1.000	0.983
0.5	0.6	1.000	1.000	1.000	1.000	0.995	0.962	0.966	1.000	1.000	0.962
0.6	0.0	1.000	0.232	1.000	1.000	0.150	1.000	1.000	1.000	1.000	1.000
0.6	0.1	1.000	0.484	1.000	1.000	0.339	1.000	1.000	1.000	1.000	1.000
0.6	0.2	1.000	0.727	1.000	1.000	0.566	1.000	1.000	1.000	1.000	1.000
0.6	0.3	1.000	0.895	1.000	1.000	0.783	1.000	1.000	1.000	1.000	1.000
0.6	0.4	1.000	0.971	1.000	1.000	0.917	0.999	0.999	1.000	1.000	0.999
0.6	0.5	1.000	0.991	1.000	1.000	0.952	0.998	0.998	1.000	1.000	0.998
0.6	0.6	1.000	0.997	1.000	1.000	0.976	0.985	0.985	1.000	1.000	0.985

TABLE 6. Empirical size and power of tests: Heteroskedastic disturbances and $n = 361$

λ_0	ρ_0	LM_ρ	LM_ρ^A	LM_ρ^B	LM_ρ^Z	LM_ρ^h	LM_λ	LM_λ^A	LM_λ^B	LM_λ^Z	LM_λ^h
0.0	0.0	0.059	0.044	0.172	0.174	0.049	0.059	0.055	0.099	0.098	0.059
0.0	0.1	0.059	0.139	0.418	0.415	0.131	0.058	0.058	0.224	0.222	0.058
0.0	0.2	0.091	0.419	0.853	0.851	0.398	0.073	0.073	0.558	0.555	0.073
0.0	0.3	0.165	0.742	0.988	0.988	0.732	0.099	0.090	0.873	0.871	0.099
0.0	0.4	0.313	0.939	1.000	1.000	0.929	0.108	0.102	0.986	0.986	0.108
0.0	0.5	0.611	0.993	1.000	1.000	0.991	0.148	0.141	1.000	1.000	0.148
0.0	0.6	0.915	1.000	1.000	1.000	1.000	0.187	0.186	1.000	1.000	0.187
0.1	0.0	0.278	0.051	0.432	0.432	0.054	0.231	0.228	0.483	0.482	0.231
0.1	0.1	0.332	0.163	0.859	0.860	0.151	0.225	0.223	0.804	0.803	0.225
0.1	0.2	0.420	0.462	0.984	0.984	0.450	0.237	0.230	0.958	0.957	0.237
0.1	0.3	0.609	0.782	0.998	0.998	0.771	0.268	0.259	0.995	0.995	0.268
0.1	0.4	0.790	0.953	1.000	1.000	0.950	0.261	0.263	1.000	1.000	0.261
0.1	0.5	0.933	0.994	1.000	1.000	0.992	0.273	0.271	1.000	1.000	0.273
0.1	0.6	0.997	1.000	1.000	1.000	1.000	0.304	0.294	1.000	1.000	0.304
0.2	0.0	0.751	0.054	0.876	0.876	0.053	0.645	0.652	0.956	0.955	0.645
0.2	0.1	0.820	0.192	0.990	0.991	0.185	0.642	0.639	0.994	0.993	0.642
0.2	0.2	0.855	0.500	1.000	1.000	0.483	0.611	0.601	0.999	0.999	0.611
0.2	0.3	0.943	0.812	1.000	1.000	0.790	0.596	0.604	1.000	1.000	0.596
0.2	0.4	0.978	0.964	1.000	1.000	0.961	0.558	0.558	1.000	1.000	0.558
0.2	0.5	0.999	0.998	1.000	1.000	0.994	0.548	0.554	1.000	1.000	0.548
0.2	0.6	1.000	1.000	1.000	1.000	1.000	0.491	0.494	1.000	1.000	0.491
0.3	0.0	0.979	0.061	0.997	0.997	0.057	0.942	0.937	1.000	0.999	0.942
0.3	0.1	0.987	0.234	1.000	1.000	0.218	0.916	0.918	1.000	1.000	0.916
0.3	0.2	0.992	0.548	1.000	1.000	0.520	0.904	0.906	1.000	1.000	0.904
0.3	0.3	0.997	0.839	1.000	1.000	0.815	0.875	0.877	1.000	1.000	0.875
0.3	0.4	0.999	0.972	1.000	1.000	0.958	0.828	0.821	1.000	1.000	0.828
0.3	0.5	1.000	0.999	1.000	1.000	0.997	0.781	0.787	1.000	1.000	0.781
0.3	0.6	1.000	1.000	1.000	1.000	1.000	0.721	0.719	1.000	1.000	0.721
0.4	0.0	1.000	0.086	1.000	1.000	0.075	0.995	0.995	1.000	1.000	0.995
0.4	0.1	1.000	0.291	1.000	1.000	0.260	0.997	0.997	1.000	1.000	0.997
0.4	0.2	1.000	0.619	1.000	1.000	0.560	0.991	0.991	1.000	1.000	0.991
0.4	0.3	1.000	0.856	1.000	1.000	0.823	0.976	0.976	1.000	1.000	0.976
0.4	0.4	1.000	0.975	1.000	1.000	0.956	0.951	0.956	1.000	1.000	0.951
0.4	0.5	1.000	0.994	1.000	1.000	0.987	0.923	0.922	1.000	1.000	0.923
0.4	0.6	1.000	0.999	1.000	1.000	0.999	0.837	0.843	1.000	1.000	0.837
0.5	0.0	1.000	0.136	1.000	1.000	0.112	1.000	1.000	1.000	1.000	1.000
0.5	0.1	1.000	0.358	1.000	1.000	0.288	1.000	1.000	1.000	1.000	1.000
0.5	0.2	1.000	0.659	1.000	1.000	0.582	0.999	0.999	1.000	1.000	0.999
0.5	0.3	1.000	0.884	1.000	1.000	0.829	0.998	0.998	1.000	1.000	0.998
0.5	0.4	1.000	0.975	1.000	1.000	0.945	0.991	0.990	1.000	1.000	0.991
0.5	0.5	1.000	0.993	1.000	1.000	0.978	0.971	0.972	1.000	1.000	0.971
0.5	0.6	1.000	0.999	1.000	1.000	0.994	0.934	0.940	1.000	1.000	0.934
0.6	0.0	1.000	0.172	1.000	1.000	0.132	1.000	1.000	1.000	1.000	1.000
0.6	0.1	1.000	0.407	1.000	1.000	0.318	1.000	1.000	1.000	1.000	1.000
0.6	0.2	1.000	0.657	1.000	1.000	0.534	1.000	1.000	1.000	1.000	1.000
0.6	0.3	1.000	0.846	1.000	1.000	0.753	1.000	1.000	1.000	1.000	1.000
0.6	0.4	1.000	0.953	1.000	1.000	0.898	0.998	0.998	1.000	1.000	0.998
0.6	0.5	1.000	0.985	1.000	1.000	0.945	0.992	0.992	1.000	1.000	0.992
0.6	0.6	1.000	0.992	1.000	1.000	0.968	0.972	0.973	1.000	1.000	0.972

5. An Empirical Illustration

In this section, we illustrate the use of our proposed test statistic. To this end, we use an example from [21] on the US presidential election in 1980. The dataset contains variables on the election results and county characteristics for 3107 US counties. We consider the following regression model

$$\begin{aligned}\ln(\text{PR\_VOTES}) &= \beta_0 + \lambda W \ln(\text{PR\_VOTES}) + \beta_1 \ln(\text{POP}) + \beta_2 \ln(\text{EDUC}) \\ &+ \beta_3 \ln(\text{HOUSE}) + \beta_4 \ln(\text{INC}) + U, \quad U = \rho W U + V. \end{aligned} \tag{5.1}$$

The outcome variable is the natural log of the proportion of votes cast for both candidates in the 1980 presidential election (PR_VOTES). The explanatory variables are the natural log of the population in each county of eighteen years of age or older (POP), the natural log of the population in each county with a 12th grade or higher education (EDUC), number of owner-occupied housing units (HOUSE), and the aggregate income (INC). The spatial weights matrix W is the delaunay contiguity based weights matrix constructed using the latitudes and longitudes of the counties (see [17] for the details).

Table 7. The tests results

LM_ρ	LM_ρ^h	LM_ρ^B	LM_ρ^Z	LM_ρ^A	LM_λ	LM_λ^h	LM_λ^B	LM_λ^Z	LM_λ^A
198.766	5.598	27.740	27.767	530.488	19.036	0.168	25.301	25.279	26.831

Our goal is to test the presence of λ and ρ in (5.1). The test statistics' values are presented in Table 7. In this table, (i) LM_λ is the test statistic in Theorem 1, (ii) LM_ρ is the test statistic in Theorem 2, (iii) LM_λ^h and LM_ρ^h are the test statistics given in Theorem 3, (iv) LM_ρ^B and LM_λ^B are the test statistics suggested in [7], (v) LM_λ^Z and LM_ρ^Z are the tests statistics suggested in [4], and (vi) LM_λ^A and LM_ρ^A are the test statistics suggested in [2].

We observe that all tests show strong statistical evidence for the existence of spatial dependence in the error term.[6] The lowest value is observed for LM_ρ^h which still rejects the null hypothesis of no spatial dependence at the 0.05 significance level. Similarly, all tests, except LM_λ^h, show strong statistical evidence for the existence of spatial dependence in the outcome variable at the 0.05 significance level. The lowest value is observed for LM_λ^h which fails to reject the null hypothesis of no spatial dependence at the conventional significance level of 0.05. Given that an unknown form of heteroskedasticity is likely to be present in the observational cross-sectional dataset, the dichotomy between LM_λ^h and the rest of the test statistics is important, and the empirical modeling accounting for spatial dependence needs to consider estimating the nested null specification as a robustness check.

6. Conclusion

In this paper, we proposed the OPG variants of the LM test statistic for testing spatial dependence in spatial models with homoskedastic disturbance terms and in spatial models with heteroskedastic disturbance terms. Our OPG tests for testing one type of spatial dependence (the spatial lag in the dependent variable or the spatial lag in the disturbance term) are valid whether or not the other type of spatial dependence is present. We showed how such robust OPG tests can be systematically constructed in the quasi maximum likelihood (QML) framework. We derived the asymptotic distributions of the suggested tests under the null and local alternative hypotheses.

[6] Note that our suggested test statistics and those suggested by [2] have asymptotic χ_1^2 distribution. The critical value based on χ_1^2 at the 0.05 significance level is 3.841. The other tests, those suggested by [7] and [4], have asymptotic $N(0,1)$ distribution. The critical value based on $N(0,1)$ at the 0.05 significance level is 1.96.

Our suggested tests are simple to compute, since they only require the OLS estimates from a linear regression model.

In a Monte Carlo study, we investigated the finite sample properties of our tests along with some alternative tests suggested in the literature. For testing the presence of spatial lag term, our simulation results showed that the suggested test statistic (LM_λ and LM_λ^h) has good size and power properties under both homoskedasticity and heteroskedasticity. Our results showed that LM_ρ^h has a satisfactory performance in finite samples. The results also indicated that the robust test statistics suggested in [2] may perform well under local parametric misspecification. Moreover, heteroskedasticity specified in the form of a skedastic function seems to be not affecting the performance of these tests. The simulation results also showed that the test statistics suggested in [7] and [4] can be over-sized under local parametric misspecification.

In future studies, our testing approach can be extended to other variants of spatial models. First, our approach can be easily extended to the cross-sectional spatial models that have higher order spatial lags in the dependent and the disturbance terms. Second, our approach can be used to develop similar tests for testing the presence of spatial dependence in the static and dynamic spatial panel data models. Finally, our testing approach can be considered for the matrix exponential spatial models suggested in the literature. All of these extensions can be explored in future studies.

Appendix

In this section, we provide only the proof of Theorem 1. Other theorems can be proved similarly, so we omit their proofs. Consider the mean value expansions of $\sqrt{n}S_\lambda(\tilde\theta)$, $\sqrt{n}S_\rho(\tilde\theta)$ and $\sqrt{n}S_\gamma(\tilde\theta)$ around θ_0 when both H_a^λ and H_a^ρ hold:

$$\sqrt{n}S_\lambda(\tilde\theta) = \sqrt{n}S_\lambda(\theta_0) - \frac{\partial S_\lambda(\bar\theta)}{\partial\lambda}\delta_\lambda - \frac{\partial S_\lambda(\bar\theta)}{\partial\rho}\delta_\rho + \frac{\partial S_\lambda(\bar\theta)}{\partial\gamma'}\sqrt{n}(\tilde\gamma-\gamma_0), \tag{6.1}$$

$$\sqrt{n}S_\rho(\tilde\theta) = \sqrt{n}S_\rho(\theta_0) - \frac{\partial S_\rho(\bar\theta)}{\partial\lambda}\delta_\lambda - \frac{\partial S_\rho(\bar\theta)}{\partial\rho}\delta_\rho + \frac{\partial S_\rho(\bar\theta)}{\partial\gamma'}\sqrt{n}(\tilde\gamma-\gamma_0), \tag{6.2}$$

$$\sqrt{n}S_\gamma(\tilde\theta) = \sqrt{n}S_\gamma(\theta_0) - \frac{\partial S_\gamma(\bar\theta)}{\partial\lambda}\delta_\lambda - \frac{\partial S_\gamma(\bar\theta)}{\partial\rho}\delta_\rho + \frac{\partial S_\gamma(\bar\theta)}{\partial\gamma'}\sqrt{n}(\tilde\gamma-\gamma_0). \tag{6.3}$$

Our Assumption 2 ensures that

$$\sqrt{n}S_\lambda(\tilde\theta) = \sqrt{n}S_\lambda(\theta_0) + J_{\lambda\lambda}(\theta_0)\delta_\lambda + J_{\lambda\rho}(\theta_0)\delta_\rho - J_{\lambda\gamma}(\theta_0)\sqrt{n}(\tilde\gamma-\gamma_0) + o_p(1), \tag{6.4}$$

$$\sqrt{n}S_\rho(\tilde\theta) = \sqrt{n}S_\rho(\theta_0) + J_{\rho\lambda}(\theta_0)\delta_\lambda + J_{\rho\rho}(\theta_0)\delta_\rho - J_{\rho\gamma}(\theta_0)\sqrt{n}(\tilde\gamma-\gamma_0) + o_p(1), \tag{6.5}$$

$$\sqrt{n}S_\gamma(\tilde\theta) = \sqrt{n}S_\gamma(\theta_0) + J_{\gamma\lambda}(\theta_0)\delta_\lambda + J_{\gamma\rho}(\theta_0)\delta_\rho - J_{\gamma\gamma}(\theta_0)\sqrt{n}(\tilde\gamma-\gamma_0) + o_p(1), \tag{6.6}$$

Note that $\sqrt{n}S_\gamma(\tilde\theta) = 0$ holds in (6.6) by definition. Then, solving (6.6) for $\sqrt{n}(\tilde\gamma-\gamma_0)$ and substituting the resulting equation into (6.4) and (6.5), we obtain

$$\sqrt{n}S_\lambda(\tilde\theta) = \left(1,\, -J_{\lambda\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)\right)\begin{pmatrix}\sqrt{n}S_\lambda(\theta_0)\\ \sqrt{n}S_\gamma(\theta_0)\end{pmatrix} + J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho + o_p(1), \tag{6.7}$$

$$\sqrt{n}S_\rho(\tilde\theta) = \left(1,\, -J_{\rho\gamma}(\theta_0)J_{\gamma\gamma}^{-1}(\theta_0)\right)\begin{pmatrix}\sqrt{n}S_\rho(\theta_0)\\ \sqrt{n}S_\gamma(\theta_0)\end{pmatrix} + J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda + o_p(1). \tag{6.8}$$

We first show how the adjusted score function that has a zero asymptotic mean can be derived. We then determine the asymptotic distribution of the adjusted score function. The asymptotic distribution of $\sqrt{n}S_\lambda(\tilde\theta)$ can be determined from (6.7) by using the asymptotic normality of score functions given in Assumption 2. Thus, it follows that

$$\sqrt{n}S_\lambda(\tilde\theta) \xrightarrow{d} N\left[J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho,\, B_{\lambda\cdot\gamma}(\theta_0)\right], \tag{6.9}$$

where

$$B_{\lambda\cdot\gamma}(\theta_0) = K_{\lambda\lambda}(\theta_0) + J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\lambda\gamma}(\theta_0) \\ - K_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\lambda\gamma}(\theta_0) - J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\lambda}(\theta_0). \tag{6.10}$$

Similarly, from (6.8), we obtain

$$\sqrt{n}S_\rho(\tilde{\theta}) \xrightarrow{d} N\left[J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda,\, B_{\rho\cdot\gamma}(\theta_0)\right], \tag{6.11}$$

where

$$B_{\rho\cdot\gamma}(\theta_0) = K_{\rho\rho}(\theta_0) + J_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\rho\gamma}(\theta_0) \\ - K_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\rho\gamma}(\theta_0) - J_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\rho}(\theta_0). \tag{6.12}$$

Under H_0^λ, the result in (6.11) shows that $J^{-1}_{\rho\cdot\gamma}(\theta_0)\sqrt{n}S_\rho(\tilde{\theta}) \xrightarrow{d} N\left[\delta_\rho,\, B_{\rho\cdot\gamma}(\theta_0)\right]$. Then, using (6.9) and this last result, an adjusted score function that has zero asymptotic mean in the local presence of ρ_0 can be derived as

$$\sqrt{n}S^*_\lambda(\tilde{\theta}) = \sqrt{n}\left(S_\lambda(\tilde{\theta}) - J_{\lambda\rho\cdot\gamma}(\tilde{\theta})J^{-1}_{\rho\cdot\gamma}(\tilde{\theta})S_\rho(\tilde{\theta})\right). \tag{6.13}$$

Next we show how to determine the asymptotic distribution of $\sqrt{n}S^*_\lambda(\tilde{\theta})$. For this purpose, we consider (6.7) and (6.8) as a combined system

$$\begin{pmatrix} \sqrt{n}S_\lambda(\tilde{\theta}) \\ \sqrt{n}S_\rho(\tilde{\theta}) \end{pmatrix} = \begin{pmatrix} -J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0) & 1 & 0 \\ -J_{\rho\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0) & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{n}S_\gamma(\theta_0) \\ \sqrt{n}S_\lambda(\theta_0) \\ \sqrt{n}S_\rho(\theta_0) \end{pmatrix} \\ + \begin{pmatrix} J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho \\ J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda \end{pmatrix} + o_p(1). \tag{6.14}$$

The joint asymptotic distribution of $\sqrt{n}S_\lambda(\tilde{\theta})$ and $\sqrt{n}S_\rho(\tilde{\theta})$ can now be determined from (6.14) by using the asymptotic normality of score functions in Assumption 2. Thus, we have

$$\begin{pmatrix} \sqrt{n}S_\lambda(\tilde{\theta}) \\ \sqrt{n}S_\rho(\tilde{\theta}) \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} J_{\lambda\cdot\gamma}(\theta_0)\delta_\lambda + J_{\lambda\rho\cdot\gamma}(\theta_0)\delta_\rho \\ J_{\rho\cdot\gamma}(\theta_0)\delta_\rho + J_{\rho\lambda\cdot\gamma}(\theta_0)\delta_\lambda \end{pmatrix},\, \begin{pmatrix} B_{\lambda\cdot\gamma}(\theta_0) & B_{\lambda\rho\cdot\gamma}(\theta_0) \\ B_{\rho\lambda\cdot\gamma}(\theta_0) & B_{\rho\cdot\gamma}(\theta_0) \end{pmatrix}\right], \tag{6.15}$$

where

$$B_{\lambda\cdot\gamma}(\theta_0) = K_{\lambda\lambda}(\theta_0) + J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma}(\theta_0)K_{\gamma\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\lambda\gamma}(\theta_0) \\ - K_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J^{'}_{\lambda\gamma}(\theta_0) - J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\lambda}(\theta_0). \tag{6.16}$$

$$B_{\lambda\rho\cdot\gamma}(\theta_0) = K_{\lambda\rho}(\theta_0) - J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\rho}(\theta_0) - K_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J_{\gamma\rho}(\theta_0) \\ + J_{\lambda\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)K_{\gamma\gamma}(\theta_0)J^{-1}_{\gamma\gamma}(\theta_0)J_{\gamma\rho}(\theta_0), \tag{6.17}$$

$B_{\rho\cdot\gamma}(\theta_0)$ and $B_{\rho\lambda\cdot\gamma}(\theta_0)$ are defined similarly. Under our assumptions, we have

$$\sqrt{n}S^*_\lambda(\tilde{\theta}) = \left(1,\, -J_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)\right)\begin{pmatrix} \sqrt{n}S_\lambda(\tilde{\theta}) \\ \sqrt{n}S_\rho(\tilde{\theta}) \end{pmatrix} + o_p(1). \tag{6.18}$$

Then, using (6.15), under H_0^λ and H_a^ρ, we obtain

$$\sqrt{n}S^*_\lambda(\tilde{\theta}) \xrightarrow{d} N\left[0,\, D_{\lambda\cdot\gamma}(\theta_0)\right], \tag{6.19}$$

where

$$D_{\lambda\cdot\gamma}(\theta_0) = B_{\lambda\cdot\gamma}(\theta_0) + J_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)B_{\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)J_{\rho\lambda\cdot\gamma}(\theta_0) \\ - J_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)B_{\rho\lambda\cdot\gamma}(\theta_0) - B_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)J_{\rho\lambda\cdot\gamma}(\theta_0). \tag{6.20}$$

Now, consider the asymptotic distribution of $\sqrt{n}S^*_\lambda(\tilde{\theta})$ under H^λ_a and H^ρ_0. Using (6.15) and (6.18), we can derive that

$$\sqrt{n}S^*_\lambda(\tilde{\theta}) \xrightarrow{d} N\left[\left(J_{\lambda\cdot\gamma}(\theta_0) - J_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)J_{\rho\lambda\cdot\gamma}(\theta_0)\right)\delta_\lambda,\, D_{\lambda\cdot\gamma}(\theta_0)\right], \tag{6.21}$$

Thus, $LM_\lambda \overset{A}{\sim} \chi^2_1(\vartheta_1)$ by Theorem 8.6 of White [24] on the asymptotic distribution of quadratic forms, where $\vartheta_1 = \delta^2_\lambda\left(J_{\lambda\cdot\gamma}(\theta_0) - J_{\lambda\rho\cdot\gamma}(\theta_0)J^{-1}_{\rho\cdot\gamma}(\theta_0)J_{\rho\lambda\cdot\gamma}(\theta_0)\right)^2/D_{\lambda\cdot\gamma}(\theta_0)$ is the non-centrality parameter.

References

[1] Anselin, L. (1998). *Spatial Econometrics: Methods and Models*. Springer, New York.

[2] Anselin, L., Bera, A.K., Florax, R. and Yoon, M.J. (1996). Simple diagnostic tests for spatial dependence. *Regional Science and Urban Economics*, 26(1), 77-104.

[3] Baltagi, B.H., Chang, Y.J. and Li, Q. (1992). Monte Carlo results on several new and existing tests for the error component model. *Journal of Econometrics*, 54(1), 95-120.

[4] Baltagi, B.H. and Yang, Z. (2013a). Heteroskedasticity and non-normality robust LM tests for spatial dependence. *Regional Science and Urban Economics*, 43(5), 725-739.

[5] Baltagi, B.H. and Yang, Z. (2013b). Standardized LM tests for spatial error dependence in linear or panel regressions. *The Econometrics Journal*, 16(1), 103-134.

[6] Bera, A.K. and Yoon, M.J. (1993). Specification testing with locally misspecified alternatives. *Econometric Theory*, 9(4), 649-658.

[7] Born, B. and Breitung, J. (2011). Simple regression-based tests for spatial dependence. *The Econometrics Journal*, 14(2), 330-342.

[8] Burridge, P. (1980). On the Cliff-Ord test for spatial correlation. *Journal of the Royal Statistical Society, Series B: Methodological*, 42, 107-108.

[9] Davidson, R. and MacKinnon, J.G. (1987). Implicit alternatives and the local power of test statistics. *Econometrica*, 55(6), 1305-1329.

[10] Doğan, O. (2015). Heteroskedasticity of unknown form in spatial autoregressive models with a moving average disturbance term. *Econometrics*, 3(1), 101-127.

[11] Elhorst, J.P. (2014). *Spatial Econometrics: From Cross-Sectional Data to Spatial Panels*. Springer, Berlin.

[12] Jin, F. and Lee, L. (2018). Outer-product-of-gradients tests for spatial autoregressive models. *Regional Science and Urban Economics*, 72, 35-57.

[13] Kelejian, H.H. and Prucha, I.R. (2001). On the asymptotic distribution of the Moran I test statistic with applications. *Journal of Econometrics*, 104(2), 219-257.

[14] Kelejian, H.H. and Prucha, I.R. (2010). Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances. *Journal of Econometrics*, 157, 53-67.

[15] Koenker, R. (1981). A note on studentizing a test for heteroscedasticity. *Journal of Econometrics*, 17(1), 107-112.

[16] Lee, L. (2004). Asymptotic distributions of quasi-maximum likelihood estimators for spatial autoregressive models. *Econometrica*, 72(6), 1899-1925.

[17] LeSage, L. and Pace, R.K. (2009). *Introduction to Spatial Econometrics*. Chapman and Hall/CRC, London.

[18] Lin, X. and Lee, L. (2010). GMM estimation of spatial autoregressive models with unknown heteroskedasticity. *Journal of Econometrics*, 157(1), 34-52.

[19] Liu, S.F. and Yang, Z. (2014). Modified QML estimation of spatial autoregressive models with unknown heteroskedasticity and nonnormality. *Regional Science and Urban Economics*, 52, 50-70.

[20] Moulton, B.R. and Randolph, W.C. (1989). Alternative tests of the error components model. *Econometrica*, 57(3), 685-693.

[21] Pace, R.K. and Barry, R. (1997). Quick computation of spatial autoregressive estimators. *Geographical Analysis*, 29(3), 232-247.

[22] Qu, X. and Lee, L. (2015). Estimating a spatial autoregressive model with an endogenous spatial weight matrix. *Journal of Econometrics*, 184(2), 209-232.

[23] Saikkonen, P. (1989). Asymptotic relative efficiency of the classical test statistics under misspecification. *Journal of Econometrics*, 42(3), 351-369.

[24] White, H. (1994). *Estimation, Inference and Specification Analysis*. Cambridge University Press, Cambridge.

[25] Yang, Z. (2018). Unified M-estimation of fixed-effects spatial dynamic models with short panels. *Journal of Econometrics*, 205(2), 423-447.