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Some Properties of Leonardo Numbers

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Abstract

In this paper, we consider the Leonardo numbers which is defined by Catarino and Borges. Using Binet's formula of this sequence, we obtain new identities of the Leonardo numbers. Also , we give relations among the Fibonacci, Lucas and Leonardo numbers. Finally, using the matrix representation of Leonardo numbers, we obtain the some identities of Leonardo numbers.

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1. Introduction

Sequences of integers have an important place in the literature. The most famous of these sequences are the Fibonacci and Lucas sequences. Fibonacci sequence is defined by the following recurrence relation for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}, \tag{1.1}$$

with the initial conditions $F_0 = 0$, $F_1 = 1$. Similarly, Lucas sequence is defined by the following recurrence relation for $n \geq 2$,

$$L_n = L_{n-1} + L_{n-2}, \tag{1.2}$$

with the initial conditions $L_0 = 2$, $L_1 = 1$. These sequences corresponds to the sequences A000045 and A000032 of the on-line encyclopedia of integers sequences in [7]. The characteristic equation of recurrences (1.1) and (1.2) is

$$\lambda^2 - \lambda - 1 = 0. \tag{1.3}$$

The Binet's formula of the $\{F_n\}$ and $\{L_n\}$ sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.4}$$

and

$$L_n = \alpha^n + \beta^n, \tag{1.5}$$

where α and β are roots of characteristic equation (1.3).
Also, we give some identities between Fibonacci and Lucas numbers as

$$F_{n-1} + F_{n+1} = L_n, \tag{1.6}$$

$$L_{n-1} + L_{n+1} = 5F_n, \tag{1.7}$$

$$F_n + L_n = 2F_{n+1}, \tag{1.8}$$

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$$F_n - L_n = -2F_{n-1}, \tag{1.9}$$

$$F_{n+m} + (-1)^m F_{n-m} = L_m F_n, \tag{1.10}$$

$$F_{n+m} - (-1)^m F_{n-m} = F_m L_n, \tag{1.11}$$

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n, \tag{1.12}$$

$$L_{n+m} - (-1)^m L_{n-m} = 5F_m F_n, \tag{1.13}$$

$$L_{n+h}L_{n+k} - L_n L_{n+h+k} = 5(-1)^{n+1} F_h F_k, \tag{1.14}$$

$$F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k, \tag{1.15}$$

$$F_m F_n - F_{m+k}F_{n-k} = (-1)^{n-k} F_{m+k-n} F_k, \tag{1.16}$$

$$F_{2m+1}F_{2n+1} = F_{m+n+1}^2 + F_{m-n}^2, \tag{1.17}$$

$$L_{2n}L_{2r} = 5(F_{n+r}^2 + F_{n-r}^2) + 4(-1)^{n+r}, \tag{1.18}$$

$$L_{2h} - 2(-1)^h = 5F_h^2, \tag{1.19}$$

$$F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}, \tag{1.20}$$

$$F_n F_{n+1} F_{n+2} = F_{n+1}^3 + (-1)^{n+1} F_{n+1}, \tag{1.21}$$

$$F_f F_g - F_h F_k = (-1)^{g+1} F_{f-h} F_{f-k}. \tag{1.22}$$

for f, g, h and k integer such that $f + g = h + k$, [5, 8, 10]. More information about Fibonacci and Lucas numbers can be found in [5, 10]. Some of studies related to Fibonacci like numbers can see in [3, 4, 9].

Leonardo numbers are introduced and given some properties by Catarino and Borges in [1]. Leonardo sequence is defined by the following recurrence relation for $n \geq 2$,

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \tag{1.23}$$

with the initial conditions $Le_0 = Le_1 = 1$. Such sequence corresponds to the sequence A001595 in the on-line encyclopedia of integers sequences in [7]. Also, there is an equation following between Leonardo numbers

$$Le_{n+1} = 2Le_n - Le_{n-2},\ n \geq 2. \tag{1.24}$$

The characteristic equation of recurrence (1.24) is

$$\lambda^3 - 2\lambda^2 + 1 = 0. \tag{1.25}$$

The Binet's formula of the $\{Le_n\}$ sequence is

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta} \tag{1.26}$$

where α and β are roots of characteristic equation (1.25).

From (1.4) and (1.26), It is clear that

$$Le_n = 2F_{n+1} - 1. \tag{1.27}$$

The first few terms of Fibonacci, Lucas and Leonardo numbers are as the following table

n	0	1	2	3	4	5	6	7	8	9	10	...
F_n	0	1	1	2	3	5	8	13	21	34	55	...
L_n	2	1	3	4	7	11	18	29	47	76	123	...
Le_n	1	1	3	5	9	15	25	41	67	109	177	...

Also, Catarino and Borges obtain generating function, Cassini, Catalan and d' Ocagne identities of Leonardo numbers in [1]. In [2], the authors have defined incomplete Leonardo numbers and given some properties of incomplete Leonardo numbers. In [6], the author have defined generalized Leonardo numbers which are considered Asveld's extension and Horadam' s generalized sequence. In [11], the authors investigate the two-dimensional recurrences relations of Leonardo numbers from its one-dimensional model.
Motivated above papers, we obtain another identities for Leonardo numbers. Also, we give matrix representation of Leonardo numbers.

2. Main Results

In this section, we define negative Leonardo numbers. Then, we obtain the some identities of Leonardo numbers. Also, we give matrix representation of Leonardo numbers.

Theorem 2.1. *For $n \geq 2$, the following identity holds*

$$Le_{-n} = (-1)^n (Le_{n-2} + 1) - 1, \tag{2.1}$$

where Le_n is nth Leonardo number.

Proof. Using (1.27) and $F_{-n} = (-1)^{n+1} F_n$, we have

$$\begin{aligned} Le_{-n} &= 2F_{-n+1} - 1 \\ &= 2(-1)^n F_{n-1} - 1 \\ &= (-1)^n (Le_{n-2} + 1) - 1. \end{aligned}$$

□

Theorem 2.2. *For $n \geq 1$, the the following identities are true*

$$\begin{aligned} Le_{n-1} + Le_{n+1} &= 2L_{n+1} - 2, \\ Le_n + 2F_n &= Le_{n+1}, \\ Le_n + F_n + L_n &= 2Le_n + 1, \\ Le_{n+1}^2 + Le_n^2 &= 2(Le_{2n+2} - Le_{n+2} + 1), \end{aligned}$$

where F_n, L_n and Le_n are nth Fibonacci, Lucas and Leonardo numbers, respectively.

Proof. Let's prove the last given identity. Using (1.27), we have

$$\begin{aligned} Le_{n+1}^2 + Le_n^2 &= (2F_{n+2} - 1)^2 + (2F_{n+1} - 1)^2 \\ &= 4F_{2n+3} - 4F_{n+3} + 2 \\ &= 2(Le_{2n+2} - Le_{n+2} + 1). \end{aligned}$$

Similarly, we can have other identities. □

Theorem 2.3. *For n is nonnegative integer, then the following identity is true*

$$Le_{n+1}F_{n+1} - Le_nF_n = Le_nF_{n+1} + F_n,$$

where F_n and Le_n are nth Fibonacci and Leonardo numbers, respectively.

Proof. Using (1.4) and (1.26), we have

$$Le_{n+1}F_{n+1} - Le_nF_n = \left(\frac{2\alpha^{n+2} - 2\beta^{n+2} - \alpha + \beta}{\alpha - \beta}\right)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - \left(\frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta}\right)\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right).$$

Considering Binet's formula (1.5), we obtain

$$Le_{n+1}F_{n+1} - Le_nF_n = \frac{2L_{2n+2} + 4(-1)^n - L_n - L_{n-2}}{5}.$$

From (1.7), (1.19) and (1.27), the result is clear. □

Theorem 2.4. *For m,n are nonnegative integers and $m \geq n$, Then*

$$Le_{m+n}^2 - Le_{m-n}^2 = 2\left(2F_{2m+2}F_{2n} - Le_{m+n} + Le_{m-n}\right),$$

where F_n and Le_n are nth Fibonacci and Leonardo numbers, respectively.

Proof. Using (1.5) and (1.26) in left hand side (LHS), we have

$$(LHS) = \frac{4}{5}\left(L_{2m+2n+2} - L_{2m-2n+2} - L_{m+n+2} - L_{m+n} + L_{m-n+2} + L_{m-n}\right).$$

Then, considering (1.7), (1.13) and (1.27), we obtain

$$Le_{m+n}^2 - Le_{m-n}^2 = 2\left(2F_{2m+2}F_{2n} - Le_{m+n} + Le_{m-n}\right).$$

□

Theorem 2.5. *For m and r are nonnegative integers and $m \geq r+4$. Then, the following identity holds*

$$Le_{m+r}Le_{m+r-2} + Le_{m-r}Le_{m-r-2} = Le_{m+r-1}^2 + Le_{m-r-1}^2 - Le_{m+r-4} - Le_{m-r-4} + 8(-1)^{m-r} - 2.$$

Proof. Using (1.26) to left hand side (LHS) , we have

$$\begin{aligned}(LHS) &= \left(\frac{2\alpha^{m+r+1} - 2\beta^{m+r+1} - \alpha + \beta}{\alpha - \beta}\right)\left(\frac{2\alpha^{m+r-1} - 2\beta^{m+r-1} - \alpha + \beta}{\alpha - \beta}\right) \\ &\quad + \left(\frac{2\alpha^{m-r+1} - 2\beta^{m-r+1} - \alpha + \beta}{\alpha - \beta}\right)\left(\frac{2\alpha^{m-r-1} - 2\beta^{m-r-1} - \alpha + \beta}{\alpha - \beta}\right).\end{aligned}$$

From (1.5), we have

$$(LHS) = \frac{2}{5}\left(2L_{2m+2r} + 2L_{2m-2r} - 2L_{m+r} - 2L_{m-r} - L_{m+r+2} - L_{m-r+2} - L_{m+r-2} - L_{m-r-2} + 5 + 12(-1)^{m-r}\right).$$

Then, considering (1.6), (1.7) and (1.12), we obtain

$$(LHS) = \frac{1}{5}\left(4L_{2m}L_{2r} - 10L_{m+r} - 10L_{m-r} + 24(-1)^{m-r} + 10\right),$$

where L_n is nth Lucas numbers. If we use identities (1.9), (1.18) and (1.27), the result is clear. □

Now, let's give the equality that gives the product of two even Leonardo numbers.

Theorem 2.6. *For n and m are nonnegative integers, $m \geq n+1$. Then*

$$Le_{2m}Le_{2n} = (Le_{m+n}+1)^2 + (Le_{m-n-1}+1)^2 - Le_{2m} - Le_{2n} - 1,$$

where Le_n is nth Leonardo number.

Proof. Using (1.17) and (1.27), we obtain

$$\begin{aligned}Le_{2m}Le_{2n} &= (2F_{2m+1} - 1)(2F_{2n+1} - 1) \\ &= 4F_{2m+1}F_{2n+1} - Le_{2m} - Le_{2n} - 1 \\ &= 4\left(F_{m+n+1}^2 + F_{m-n}^2\right) - Le_{2m} - Le_{2n} - 1.\end{aligned}$$

From (1.27) again, the result is achieved. □

Theorem 2.7. *For n,r and s are nonnegative integers, then*

$$Le_{n+r}Le_{n+s} - Le_nLe_{n+r+s} = 4(-1)^{n+1}F_rF_s - Le_{n+r} - Le_{n+s} + Le_n + Le_{n+r+s},$$

where F_n and Le_n are nth Fibonacci and Leonardo numbers.

Proof. Using (1.27) to left hand side (LHS), we have

$$\begin{aligned}(LHS) &= (2F_{n+r+1} - 1)(2F_{n+s+1} - 1) - (2F_{n+1} - 1)(2F_{n+r+s+1} - 1) \\ &= 4(F_{n+r+1}F_{n+s+1} - F_{n+1}F_{n+r+s+1}) - 2(F_{n+r+1} + F_{n+s+1} - F_{n+1} - F_{n+r+s+1}).\end{aligned}$$

In the last step, taking $n+1$ instead of n, $h = r$ and $k = s$ in (1.15), we obtain

$$(LHS) = 4(-1)^{n+1}F_rF_s - Le_{n+r} - Le_{n+s} + Le_n + Le_{n+r+s}.$$

□

Theorem 2.8. *For m,n are nonnegative integers and $m\geq 1,\ n\geq m$, then the following identities are true*

$$\begin{aligned} Le_{n+m}+(-1)^m Le_{n-m} &= L_m(Le_n+1)-1-(-1)^m, \\ Le_{n+m}-(-1)^m Le_{n-m} &= L_{n+1}(Le_{m-1}+1)-1+(-1)^m, \end{aligned}$$

where L_n and Le_n are nth Lucas and Leonardo numbers.

Proof. From (1.10), (1.11) and (1.27), the proof is clear. □

Theorem 2.9. *For $n,m\geq 1$, the following identity is true*

$$Le_{m+1}Le_{n+1}-Le_{m-1}Le_{n-1}=2Le_{m+n+1}-Le_m-Le_n$$

where Le_n is nth Leonardo numbers.

Proof. If we consider (1.27) in LHS, we obtain

$$\begin{aligned} LHS &= (2F_{m+2}-1)(2F_{n+2}-1)-(2F_m-1)(2F_n-1) \\ &= 2(2F_{m+2}F_{n+2}-2F_mF_n-F_{m+2}+F_m-F_{n+2}+F_n). \end{aligned}$$

Using (1.20) and (1.27), the result is clear. □

Theorem 2.10. *For $m\geq 1$ and $n\geq m+1$, then the following identities are true*

$$\begin{aligned} F_nLe_m-F_mLe_n &= (-1)^m(Le_{n-m-1}+1)-F_n+F_m, \\ F_nLe_m+F_mLe_n &= Le_{n+m-1}+F_nLe_{m-1}-F_m+1, \end{aligned}$$

where F_n and Le_n are nth Fibonacci and Leonardo numbers.

Proof. Let's prove the first given identity. If we use (1.27), then

$$\begin{aligned} F_nLe_m-F_mLe_n &= F_n(2F_{m+1}-1)-F_m(2F_{n+1}-1) \\ &= 2(F_nF_{m+1}-F_mF_{n+1})-F_n+F_m. \end{aligned}$$

Taking $m=n$, $k=1$ and $n=m+1$ in (1.16), the result is clear. The other identity can be obtained in a similar way. □

Theorem 2.11. *For n and k are nonnegative integers, then*

$$Le^2_{n+2k}-Le^2_n=2(2F_{2n+2k+2}F_{2k}-Le_{n+2k}+Le_n),$$

where Le_n and F_n are nth Leonardo and Fibonacci numbers.

Proof. From (1.5) and (1.26), we obtain

$$\begin{aligned} Le^2_{n+2k}-Le^2_n &= \left(\frac{2\alpha^{n+2k+1}-2\beta^{n+2k+1}-\alpha+\beta}{\alpha-\beta}\right)^2-\left(\frac{2\alpha^{n+1}-2\beta^{n+1}-\alpha+\beta}{\alpha-\beta}\right)^2 \\ &= \frac{4}{5}(L_{2n+4k+2}-L_{2n+2}-L_{n+2k+2}-L_{n+2k}+L_{n+2}+L_n), \end{aligned}$$

where L_n is nth Lucas number. Afterwards, taking $2n+2k+2$ instead of n and $m=2k$ in (1.13), also using (1.7) and (1.27), we have

$$Le^2_{n+2k}-Le^2_n=4(F_{2n+2k+2}F_{2k}-F_{n+2k+1}+F_{n+1}).$$

Using (1.27), we have

$$Le^2_{n+2k}-Le^2_n=4F_{2n+2k+2}F_{2k}-2Le_{n+2k}+2Le_n.$$

□

Now, we present identity that gives product of three consecutive Leonardo numbers.

Theorem 2.12. *For $n\geq 0$, the following identity holds*

$$Le_nLe_{n+1}Le_{n+2}=(Le_{n+1}+1)\left(Le^2_{n+1}+2L_{n+1}+4(-1)^n-1\right)-Le^2_{n+2},$$

where Le_n and L_n are nth Leonardo and Lucas numbers

Proof. Firstly, let's find the value of Le_nLe_{n+1} expression. Using Binet's formulas of Leonardo and Lucas numbers, we have

$$\begin{aligned} Le_nLe_{n+1} &= \left(\frac{2\alpha^{n+1}-2\beta^{n+1}-\alpha+\beta}{\alpha-\beta}\right)\left(\frac{2\alpha^{n+2}-2\beta^{n+2}-\alpha+\beta}{\alpha-\beta}\right), \\ &= \frac{4\left(L_{2n+3}+(-1)^n\right)-2\left(L_{n+2}+L_n+L_{n+3}+L_{n+1}\right)+5}{5}. \end{aligned}$$

Taking $n+2$ instead of n and $m=n+1$ in (1.13), we obtain

$$Le_nLe_{n+1}=4F_{n+2}F_{n+1}-Le_{n+2},$$

where F_n is nth Fibonacci number. Then

$$Le_nLe_{n+1}Le_{n+2}=\left(4F_{n+2}F_{n+1}-Le_{n+2}\right)Le_{n+2}.$$

From (1.27), we have

$$Le_nLe_{n+1}Le_{n+2}=8F_{n+1}F_{n+2}F_{n+3}-4F_{n+2}F_{n+1}-Le^2_{n+2}.$$

Lastly, taking $n+1$ instead of n in (1.21) and (1.6), (1.27), the result is clear. □

Theorem 2.13. *For m,k and s are nonnegative integers , $m\geq k$ and $m\geq s$, then*

$$Le_{m+k}Le_{m-k}-Le_{m+s}Le_{m-s}=4\left(-1\right)^{m+1}\left(\left(-1\right)^sF_s^2-\left(-1\right)^kF_k^2\right)+Le_{m+s}+Le_{m-s}-Le_{m+k}-Le_{m-k}$$

where Le_n and F_n are nth Leonardo and Fibonacci numbers.

Proof. Using (1.26) to left hand side (LHS) , we have

$$\begin{aligned} LHS &= \left(\frac{2\alpha^{m+k+1}-2\beta^{m+k+1}-\alpha+\beta}{\alpha-\beta}\right)\left(\frac{2\alpha^{m-k+1}-2\beta^{m-k+1}-\alpha+\beta}{\alpha-\beta}\right) \\ &\quad -\left(\frac{2\alpha^{m+s+1}-2\beta^{m+s+1}-\alpha+\beta}{\alpha-\beta}\right)\left(\frac{2\alpha^{m-s+1}-2\beta^{m-s+1}-\alpha+\beta}{\alpha-\beta}\right). \end{aligned}$$

If we consider (1.5), (1.7) and (1.19), we obtain

$$Le_{m+k}Le_{m-k}-Le_{m+s}Le_{m-s}=4\left(-1\right)^{m+1}\left(\left(-1\right)^sF_s^2-\left(-1\right)^kF_k^2\right)+2F_{m+s+1}+2F_{m-s+1}-2F_{m+k+1}-2F_{m-k+1}.$$

From (1.27), the result is clear. □

Theorem 2.14. *For $k+m=s+t$, the following identity holds*

$$Le_kLe_m-Le_sLe_t=4\left(-1\right)^mF_{k-s}F_{k-t}-Le_k-Le_m+Le_s+Le_t,$$

where F_n and Le_n are nth Fibonacci and Leonardo numbers.

Proof. Using (1.26), we have

$$Le_kLe_m-Le_sLe_t=\left(\frac{2\alpha^{k+1}-2\beta^{k+1}-\alpha+\beta}{\alpha-\beta}\right)\left(\frac{2\alpha^{m+1}-2\beta^{m+1}-\alpha+\beta}{\alpha-\beta}\right)-\left(\frac{2\alpha^{s+1}-2\beta^{s+1}-\alpha+\beta}{\alpha-\beta}\right)\left(\frac{2\alpha^{t+1}-2\beta^{t+1}-\alpha+\beta}{\alpha-\beta}\right).$$

Considering (1.5), (1.7) and (1.13), we obtain

$$Le_kLe_m-Le_sLe_t=4\left(F_{k+1}F_{m+1}-F_{s+1}F_{t+1}\right)-2F_{k+1}-2F_{m+1}+2F_{s+1}+2F_{t+1}.$$

Taking $f=k+1, g=m+1, h=s+1$ and $k=t+1$ in (1.22), we have

$$Le_kLe_m-Le_sLe_t=4\left(-1\right)^mF_{k-s}F_{k-t}-Le_k-Le_m+Le_s+Le_t.$$

□

Now, we give the matrix representation of Leonardo numbers. Afterwards, we have several identities by the matrix Q. The matrix Q associated with Leonardo numbers is defined by

$$Q=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

By easy induction, we can see that

$$Q^n=\frac{1}{2}\begin{pmatrix} Le_{n+2}-1 & Le_{n+1}-1 & Le_n-1 \\ 1-Le_n & 1-Le_{n-1} & 1-Le_{n-2} \\ 1-Le_{n+1} & 1-Le_n & 1-Le_{n-1} \end{pmatrix}. \tag{2.2}$$

Theorem 2.15. *Let $n \geq 1$ be an integer. The following equality holds*

$$\begin{pmatrix} Le_{n+3} & Le_{n+2} & Le_{n+1} \\ Le_{n+2} & Le_{n+1} & Le_{n} \\ Le_{n+1} & Le_{n} & Le_{n-1} \end{pmatrix} = \begin{pmatrix} Le_{3} & Le_{2} & Le_{1} \\ Le_{2} & Le_{1} & Le_{0} \\ Le_{1} & Le_{0} & Le_{-1} \end{pmatrix} Q^n. \tag{2.3}$$

Proof. For the proof, we use induction method on n. The equality hold for $n = 1$. Now suppose that the equality is true for $n > 1$. Then we can verify for $n+1$ as follows

$$\begin{aligned} \begin{pmatrix} Le_{3} & Le_{2} & Le_{1} \\ Le_{2} & Le_{1} & Le_{0} \\ Le_{1} & Le_{0} & Le_{-1} \end{pmatrix} Q^{n+1} &= \begin{pmatrix} Le_{3} & Le_{2} & Le_{1} \\ Le_{2} & Le_{1} & Le_{0} \\ Le_{1} & Le_{0} & Le_{-1} \end{pmatrix} Q^n Q \\ &= \begin{pmatrix} Le_{n+3} & Le_{n+2} & Le_{n+1} \\ Le_{n+2} & Le_{n+1} & Le_{n} \\ Le_{n+1} & Le_{n} & Le_{n-1} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} Le_{n+4} & Le_{n+3} & Le_{n+2} \\ Le_{n+3} & Le_{n+2} & Le_{n+1} \\ Le_{n+2} & Le_{n+1} & Le_{n} \end{pmatrix}. \end{aligned}$$

□

Using the matrix Q, we can obtain some interesting properties of Leonardo numbers. Now, let's give the several identities of Leonardo numbers.

Corollary 2.16. *For $n,m \geq 1$, the following identity is true*

$$Le_n Le_{m-1} + Le_{n-1} Le_m = Le_{m+1} Le_{n+1} - 2Le_{m+n} - 1. \tag{2.4}$$

Proof. From the identity $Q^{m+n} = Q^n Q^m$ and matrix equality, the result is clear. □

Taking $m = n$ in (2.4), we obtain following identity

$$Le_{2n} = \frac{1}{2}\left(Le_{n+1}^2 - 2Le_{n-1}Le_n - 1\right).$$

Using $n+1$ instead of m in (2.4), we have another identity as

$$Le_{2n+1} = \frac{1}{2}\left(2Le_{n+1}\left(Le_n + 1\right) - Le_n^2 - 1\right).$$

From (2.3), we have

$$\begin{vmatrix} Le_{n+3} & Le_{n+2} & Le_{n+1} \\ Le_{n+2} & Le_{n+1} & Le_{n} \\ Le_{n+1} & Le_{n} & Le_{n-1} \end{vmatrix} = \begin{vmatrix} Le_{3} & Le_{2} & Le_{1} \\ Le_{2} & Le_{1} & Le_{0} \\ Le_{1} & Le_{0} & Le_{-1} \end{vmatrix} |Q^n|.$$

Therefore, we have Cassini's identity for Leonardo numbers as

$$Le_n^2 - Le_{n-1}Le_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n.$$

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